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THE CONVERGENCE OF DURAND-KERNER METHOD FOR SIMULTANEOUSLY FINDING ALL ZEROS OF THE POLYNOMIAL*1)

Dan-fu Han
(*Department of Mathematics, Zhejiang University, Hangzhou 310028, China*)

Abstract

In this paper we propose the two kinds of different criterions and use them to judge the convergence of Durand-Kerner method and to compare the obtained results with other methods.

Key words: Durand-Kerner method, Convergence, Two kinds of Criterions.

1. Introduction

Consider a monic polynomial of degree n

$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_n := \prod_{j=1}^{n} (z_j - \zeta_j).$$

To solve all zeros of this polynomial, the common-used is Durand-Kerner method which has been considered simple and effective. For this reason, there are many works (Docev (1962), kerner (1966), Yamamoto (1990) and Zheng (1982)) to discuss the properties of this method. In this paper, we propose the two kinds of criterion and use them to judge the convergence of Durand-Kerner method or to compare with other methods.

Durand-Kerner method is defined by

$$z_i^{(k+1)} = z_i^{(k)} - \frac{f(z_i^{(k)})}{\prod\limits_{j\neq i}(z_i^{(k)} - z_j^{(k)})}, \quad i = 1, 2, \cdots, n;\ k = 0, 1, \cdots. \tag{1}$$

This iteration is quadratically convergent. If the initial values $z_i^{(0)}$ $(i = 1, \cdots, n)$ are sufficiently close to ζ_i, then it is known that the iterative sequence $\{z_i^{(k)}\}_{k=0}^{\infty}$ is well defined and convergent to the corresponding zero for $i = 1, \cdots, n$.

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Let $r_i^{(k)}=\min_{j\neq i}|z_i^{(k)}-z_j^{(k)}|$, $\rho_i^{(k)}=\min_{j\neq i}|\zeta_i^{(k)}-\zeta_j^{(k)}|$, and

$$\begin{aligned} e_i^{(k)}&=|z_i^{(k)}-\zeta_i^{(k)}|/\rho_i^{(k)},\\ \varepsilon_i^{(k)}&=|z_i^{(k)}-\zeta_i^{(k)}|/r_i^{(k)},\\ \delta_i^{(k)}&=|z_i^{(k+1)}-z_i^{(k)}|/r_i^{(k)}. \end{aligned}$$

Denote $e^{(k)}=(e_1^{(k)},\cdots,e_n^{(k)})^T$, $\varepsilon^{(k)}=(\varepsilon_1^{(k)},\cdots,\varepsilon_n^{(k)})^T$ and $\delta^{(k)}=(\delta_1^{(k)},\cdots,\delta_n^{(k)})^T$. First we give the definitions of two criterions.

Definition 1. $\delta^{(0)}$ *is called the first kind of criterion of Durand-Kerner method.*

Definition 2. $e^{(0)}$ *or* $\varepsilon^{(0)}$ *is called the second kind of criterion of an iteration.*

From the definitions above we see that the first criterion varies for the different iteration, but it does not include the information about the zeros of $f(z)$. And we also see that the second kind one is not dependent on the iterations, so it can be used to compare the convergent conditions with other iteration.

In this paper we mainly discuss the l_p-norm convergent codnitons of $e^{(0)}$, $\varepsilon^{(0)}$ and $\delta^{(0)}$ for Durand-Kerner method.

2. On the Second Kind of Criterion

For the convergence of discussing, we only analyse the iteration one step of (1) and simply denote $e_i^{(0)}$ and $e_i^{(1)}$ by e_i and e_i'. Other notations have the same meaning except the special statements.

From (1) we have

$$|z_i'-\zeta_i|=\Big|\prod_{j\neq i}^{n}\Big(\frac{z_i-\zeta_j}{z_i-z_j}\Big)-1\Big||z_i-\zeta_i|,$$

and

$$\Big|\prod_{j\neq i}^{n}\Big(\frac{z_i-\zeta_j}{z_i-z_j}\Big)\Big|\le\Big|\prod_{j\neq i}^{n}\Big(1+\frac{z_j-\zeta_j}{\zeta_i-\zeta_j-(\zeta_i-z_i)-(z_j-\zeta_j)}\Big)\Big|\le\prod_{j\neq i}^{n}\Big(1+\frac{e_j}{1-e_i-e_j}\Big). \tag{2}$$

Let $1\le p\le\infty$ and $1/p+1/q=1$. By Hölder inequality it holds $e_i+e_j\le 2^{1/q}\|e\|_p$, where $\|e\|_p=\Big(\sum_{i=1}^{n}e_i^p\Big)^{1/p}$. Thus

$$\begin{aligned}\Big|\prod_{j\neq i}^{n}\Big(\frac{z_i-\zeta_j}{z_i-z_j}\Big)\Big|\le&\prod_{j\neq i}^{n}\Big(1+\frac{e_j}{1-2^{1/q}\|e\|_p}\Big)\le\Big(1+\frac{\sum_{j\neq i}^{n}e_j}{(n-1)(1-2^{1/q}\|e\|_p)}\Big)^{n-1}\\ \le&\Big(1+\frac{\|e\|_p}{(n-1)^{1/p}(1-2^{1/q}\|e\|_p)}\Big)^{n-1}. \end{aligned}\tag{3}$$

Hence

$$|z_i'-\zeta_i|\le\Big(\Big(1+\frac{\|e\|_p}{(n-1)^{1/p}(1-2^{1/q}\|e\|_p)}\Big)^{n-1}-1\Big)|z_i-\zeta_i|$$

or

$$\|e'\|_p \le \Big(\Big(1 + \frac{\|e\|_p}{(n-1)^{1/p}(1 - 2^{1/q}\|e\|_p)}\Big)^{n-1} - 1\Big)\|e\|_p, \quad p \ge 1. \tag{4}$$

Therefore if

$$\Big(1 + \frac{\|e\|_p}{(n-1)^{1/p}(1 - 2^{1/q}\|e\|_p)}\Big)^{n-1} < 2$$

or

$$\|e\|_p < \frac{(2^{\frac{1}{n-1}} - 1)(n-1)^{1/p}}{1 + 2^{1/q}(2^{1/n-1} - 1)(n-1)^{1/p}},$$

then $\|e'\|_p < \|e\|_p$. In general we have

Theorem 1. *Let $1 \le p,\ q \le \infty$ satisfying $1/p + 1/q = 1$. For n distinct initial values $z_i^{(0)}$ $(i = 1, \cdots, n)$, if $\|e^{(0)}\|_p < \dfrac{(2^{1/n-1} - 1)(n-1)^{1/p}}{1 + 2^{1/q}(2^{1/n-1} - 1)(n-1)^{1/p}}$, then the iterative sequence $\{z_i^{(k)}\}_{k=0}^{\infty}$ is well defined for $i = 1, \cdots, n$ and satisfies*

$$\|e^{(k+1)}\|_p \le \frac{1 + 2^{1/q}(2^{1/n-1} - 1)(n-1)^{1/p}}{(2^{1/n-1} - 1)(n-1)^{1/p}} \cdot \|e^{(k)}\|_p^2, \quad k = 0, 1, \cdots. \tag{5}$$

For the criterion of $\varepsilon^{(0)}$ we note that facts that

$$\Big|\prod_{j \ne i}^{n} \Big(\frac{z_i - \zeta_j}{z_i - z_j}\Big)\Big| \le \prod_{j \ne i}^{n}(1 + \varepsilon_j) \le \Big(1 + \frac{\|\varepsilon\|_p}{(n-1)^{1/p}}\Big)^{n-1} := A \tag{6}$$

and

$$\frac{|z_i' - z_j'|}{r_i} \ge 1 - \frac{|z_i' - z_i|}{r_i} - \frac{z_j' - z_j|}{r_j} \ge 1 - A(\varepsilon_i + \varepsilon_j) \ge 1 - A2^{1/q}\|\varepsilon\|_p. \tag{7}$$

So by the same analysis to e_i we have

$$\|\varepsilon'\|_p \le \frac{A - 1}{1 - A2^{1/q}\|\varepsilon\|_p} \cdot \|\varepsilon\|_p \tag{8}$$

From (8) and (6) we have that if $(1 + 2^{1/q}\|\varepsilon\|_p)\Big(1 + \dfrac{\|\varepsilon\|_p}{(n-1)^{1/p}}\Big)^{n-1} < 2$, then $\|\varepsilon'\|_p < \|\varepsilon\|_p$.

Lemma 1. *If $0 \le t < t^* := \dfrac{n(2^{1/n} - 1)}{(n-1)^{1/q} + 2^{1/q}}$, then $(1 + w^{1/q}t)\Big(1 + \dfrac{t}{(n-1)^{1/p}}\Big)^{n-1} < 2$.*

From (8) and Lemma 1 we can obtain

Theorem 2. *Let $1 \le p, q \le \infty$ satisfying $1/p + 1/q = 1$. For n distinct initial values $z_i^{(0)}$ $(i = 1, \cdots, n)$, if $\|\varepsilon^{(0)}\|_p < \dfrac{n(2^{1/n} - 1)}{(n-1)^{1/q} + 2^{1/q}}$, then the iterative sequence $\{z_i^{(k)}\}_{k=0}^{\infty}$ is well defined for $i = 1, \cdots, n$ and satisfies*

$$\|\varepsilon^{(k+1)}\|_p \le \frac{(n-1)^{1/q} + 2^{1/q}}{n(2^{1/n} - 1)} \cdot \|\varepsilon^{(k)}\|_p^2, \quad k = 0, 1, \cdots. \tag{9}$$

3. The First Kind of Criterion

Using the same notations and convention above we have

$$|z_i' - z_i| = \Big|\prod_{j\neq i}^{n}\Big(\frac{z_i-\zeta_j}{z_i-z_j}\Big)\Big||z_i-\zeta_i|$$

and

$$\delta_i \geq \Big(\prod_{j\neq i}^{n}(1-\varepsilon_j)\Big)\varepsilon_i \geq \Big(1-\sum_{j\neq i}\varepsilon_j\Big)\varepsilon_i \geq (1-(n-1)^{1/q}\|\varepsilon\|_p)\varepsilon_i$$

or

$$\|\delta\|_p \geq (1-(n-1)^{1/q}\|\varepsilon\|_p)\|\varepsilon\|_p. \tag{10}$$

Lemma 2. *Let t^* be given in Lemma 1. If $\|\delta\|_p \leq t^*(1-(n-1)^{1/q}t^*)$, then $\|\varepsilon\|_p \leq t^*$.*

Proof. From the iteration (1) it obvious that if $\|\delta\|_p \to 0$, then $\|\varepsilon\|_p \to 0$. Thus by (10) we have

$$\|\varepsilon\|_p \leq \frac{2\|\delta\|_p}{1+\sqrt{1-4(n-1)^{1/q}\|\delta\|_p}}. \tag{11}$$

From this the lemma follows.

By this lemma we have

Theorem 3. *Let $1 \leq p,\ q \leq \infty$ satisfying $1/p+1/q=1$. For n distinct initial values $z_i^{(0)}$ $(i=1,\cdots,n)$, if $\|\delta^{(0)}\|_p < t^*(1-(n-1)^{1/q}t^*)$, then the iterative sequence $\{z_i^{(k)}\}_{k=0}^{\infty}$ is well defined and as $k\to\infty$, $z_i^{(k)} \to \zeta_i$ for $i=1,\cdots,n$.*

The proof of Theorem 3 directly follows from Theorem 2 and Lemma 2.

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