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ON THE SOLVABILITY OF GENERAL LINEAR METHODS FOR DISSIPATIVE DYNAMICAL SYSTEMS*1)

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Abstract

The main purpose of the present paper is to examine the existence and local uniqueness of solutions of the implicit equations arising in the application of a weakly algebraically stable general linear methods to dissipative dynamical systems, and to extend the existing relevant results of Runge-Kutta methods by Humphries and Stuart(1994).

Key words: General linear methods, Dissipative dynamical systems, Weak algebraic stability, Solvability.

1. Introduction

The numerical approximation of dissipative initial value problems on R^m by fixed time-stepping Runge-Kutta methods has been considered by Humphries and Stuart[1], and it was shown that the numerical solution defined by an algebraically stable method has an absorbing set and is hence dissipative for any fixed step-size $h > 0$. In 1996, Xiao[8] extended the corresponding relevant results in [1], and showed that two classes of algebraically stable general linear methods applied to dissipative dynamical systems on R^m are dissipative and possess an absorbing set. But the results in [8] have an implicit assumption that the implicit equations arising in the application of the general linear method to dissipative dynamical systems are soluble.

The main purpose of the present paper is to examine the existence and local uniqueness of solutions of the implicit equations arising in the application of a weakly algebraically stable general linear methods to dissipative dynamical systems, and to extend the existing relevant results of Runge-Kutta methods by Humphries and Stuart[1].

Consider the dissipative initial value problem on R^m(cf.[1])

$$y'(t) = f(y), \quad t\geq 0; \qquad y(0) = y_0 \in R^N, \tag{1.1}$$

where the map $f : R^N \to R^N$ is assumed to be locally Lipschitz and continuous, and satisfies the following condition:

$$< x, f(x) > \leq \alpha - \beta \|x\|^2, \quad \forall x \in R^m, \tag{1.2}$$

where and throughout the following, $\alpha\geq 0$, $\beta > 0$, $< .,. >$ is the standard inner product

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on R^m with the corresponding norm $\|.\|$ denoted by $\|u\|^2 =< u, u >$. By means of the theory of ordinary differential equations, we can know that the problem (1.1) is locally uniquely soluble with the solution $y(t)$.

The problem (1.1)-(1.2) arises in many applications and the class defined by (1.1)-(1.2) contains many-known problems (such as some forms of Cahn-Hilliand equations, the Navier-Stokes equations in two dimensions, the Lorenz equations, etc.). The problem (1.1)-(1.2) defines a dynamical system on R^m and possesses an absorbing set $B = B(0, (\frac{\alpha}{\beta})^{\frac{1}{2}} + \varepsilon)$ (i.e. an open ball with the radius $(\frac{\alpha}{\beta})^{\frac{1}{2}} + \varepsilon$ and the center 0) for any $\varepsilon > 0$ and a global attractor A defined by $A = \omega(B)$, where $\omega(B)$ is the ω-limit set of B (cf.[1]).

Consider the r-value s-stage general linear method(cf.[4,6,7]) applied to (1.1)

$$\begin{cases} Y_i &= h\sum\limits_{j=1}^{s} c_{ij}^{11} f(Y_j) + \sum\limits_{j=1}^{r} c_{ij}^{12} y_j^{(n-1)}, \qquad i = 1, 2, \cdots, s, \\ y_i^{(n)} &= h\sum\limits_{j=1}^{s} c_{ij}^{21} f(Y_j) + \sum\limits_{j=1}^{r} c_{ij}^{22} y_j^{(n-1)}, \qquad i = 1, 2, \cdots, r, \\ y_n &= \sum\limits_{j=1}^{r} \sigma_j y_j^{(n)}, \end{cases} \tag{1.3}$$

where $h > 0$ is the given stepsize, c_{ij}^{IJ} and σ_j are real constants, the vectors Y_i are the internal stages of the current step and are approximations to $y(t_n + \mu_i h)$; the vectors $y_i^{(n)}$ are the external stages which contain all information from the previous step necessary for the computation of the new approximation and are approximations to $H_i(t_n + \nu_i h)$; y_n approximates to $y(t_n + \eta h)$. $t_n = nh$, μ_i, ν_i and η are real constants, each $H_i(t_n+\nu_i h)$ denotes a piece of information about the true solution $y(t)$. Let

$$\begin{cases} y^{(n)} &= (y_1^{(n)^T}, y_2^{(n)^T}, \cdots, y_r^{(n)^T})^T \in R^{Nr}, \\ Y &= (Y_1^T, Y_2^T, \cdots, Y_s^T)^T \in R^{ms}, \\ F(Y) &= (f^T(Y_1), f^T(Y_2), \cdots, f^T(Y_s))^T \in R^{ms}, \\ C_{IJ} &= [c_{ij}^{IJ}], \qquad \widetilde{C}_{IJ} = C_{IJ} \otimes I_m, \qquad I, J = 1, 2, \\ \sigma &= (\sigma_1, \sigma_2, \cdots, \sigma_r)^T \in R^{mr}, \qquad \widetilde{\sigma} = \sigma \otimes I_m, \end{cases}$$

where I_m is an $m\times m$ unit matrix, the symbol $A\otimes B$ denotes Kronecker product of the matrices A and B. Then the method (2.5) can be written in more compact form

$$\begin{cases} Y &= h\widetilde{C}_{11} F(Y) + \widetilde{C}_{12} y^{(n-1)}, \\ y^{(n)} &= h\widetilde{C}_{21} F(Y) + \widetilde{C}_{22} y^{(n-1)}, \\ y_n &= \widetilde{\sigma} y^{(n)}. \end{cases} \tag{1.4}$$

Definition 1.1.(cf.[6,7]) *Let k, p, q be real constants with $k > 0$ and $pq < 1$, $G = [g_{ij}]$ a real positive definite symmetric $r\times r$ matrix, $D = diag(d_1, d_2, \cdots, d_s)$ a real nonnegative definite diagonal $s\times s$ matrix, furthermore, for $l > 0$, $\widetilde{D}$ denotes an $l\times l$ nonnegative definite diagonal matrix. The method (1.3) is said to be (k, p, q)-weakly algebraically stable (about the matrices $G, D, \psi(\widetilde{D})$) if the matrix*

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

is nonnegative definite, where

$$M_{11} = kG - C_{22}^T G C_{22} - pC_{12}^T D C_{12} + \phi(\widetilde{D}), \qquad M_{12} = M_{21}^T = C_{12}^T D - C_{22}^T G C_{21} - pC_{12}^T D C_{11},$$
$$M_{22} = C_{11}^T D + DC_{11} - C_{21}^T G C_{21} - pC_{11}^T D C_{11} - qD,$$

see [6,7] about the meaning of $\phi(\widetilde{D})$.

As an important special case, a (1,0,0)-weakly algebraically stable method is called weakly algebraically stable for short. When $\phi(\widetilde{D}) = 0$, a (k,p,q)-weakly algebraically stable method is said to be (k, p, q)-algebraically stable.

Definition 1.2.(cf.[6]) *The method (1.3) is said to be reducible if there exists a permutation matrix* π *such that*

$$\pi C_{11}\pi^T = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad C_{21}\pi^T = (0, B_2),$$

where A_{22} *is an* $\bar{s} \times \bar{s}$ *matrix,* B_2 *is an* $r \times \bar{s}$ *matrix,* $1 \leq \bar{s} < s$. *The method (1.3) is said to be irreducible if it is not reducible.*

2. Existence

Lemma 2.1.(cf.[6]) *If the method (1.3) is irreducible and* (k, p, q)*- weakly algebraically stable, then when* $p \geq 0$, D *must be positive definite.*

Let $\bar{D}$ is the set of positive definite diagonal matrices. For $D \in \bar{D}$, we define an inner product on R^{ms} by

$$< X, Y >_D = X^T (D \otimes I_m) Y$$

and the corresponding norm on R^{ms} by

$$\|Y\|_D = < Y, Y >_D = \sum_{i=1}^{s} d_i \|Y_i\|^2,$$

where $X, Y \in R^{ms}$. Let

$$\psi_D(C_{11}) = \inf_{\xi \neq 0} \frac{< C_{11}\xi, \xi >_D}{< \xi, \xi >_D}, \quad \psi_0(C_{11}) = \sup_{D \in \bar{D}} \psi_D(C_{11}), \quad \xi \in R^s.$$

Lemma 2.2.(cf.[5]) *If* C_{11}^{-1} *exists,* $\psi_D(C_{11}) \geq 0$, *then*

$$\psi_D(C_{11}) \|C_{11}\|_D^{-2} \leq \psi_D(C_{11}^{-1}) \leq \psi_D(C_{11}) \|C_{11}^{-1}\|_D^2,$$

where

$$\|C_{11}\|_D = \sup_{\xi \neq 0} \frac{\|C_{11}\xi\|_D}{\|\xi\|_D}, \quad \|C_{11}^{-1}\|_D = \sup_{\xi \neq 0} \frac{\|C_{11}^{-1}\xi\|_D}{\|\xi\|_D}.$$

Lemma 2.3. *If the method (1.3) is irreducible and weakly algebraically stable (about the matrices* $G, D, \phi(\widetilde{D})$*),* C_{11}^{-1} *exists, then*

$$\psi_D(C_{11}^{-1}) \geq 0, \quad \psi_0(C_{11}^{-1}) \geq 0.$$

Proof. It follows from Lemma 2.1 that D is positive definite and

$$< \xi, \xi >_D = \sum_{i=1}^{s} d_i \xi_i^2 > 0$$

for all $\xi = (\xi_1, \xi_2, \cdots, \xi_s)^T \in R^s$, $\xi \neq 0$. Further, it follows from

$$< C_{11}\xi, \xi >_D = \xi^T D C_{11} \xi = \frac{1}{2} \xi^T M_{22} \xi + \frac{1}{2} (C_{21}\xi)^T G (C_{21}\xi) \geq 0$$

that $\psi_D(C_{11}) \geq 0$. Thus it follows from Lemma 2.2 that the conclusions hold.

Theorem 2.1. *If* C_{11} *is invertible with*

$$\psi_0(C_{11}^{-1}) + h\beta > 0, \tag{2.1}$$

then the implicit equations of the method (1.3) applied to the problem (1.1)-(1.2) are soluble.

Proof. It easily follows from (2.1) that there exists $D \in \bar{D}$ such that
$$\psi_D(C_{11}^{-1}) + h\beta > 0.$$
Using this D we have that
$$Y^T(DC_{11}^{-1}\otimes I_m)Y \geq \psi_D(C_{11}^{-1})\|Y\|_D^2,$$
$$Y^T(DC_{11}^{-1}\otimes I_m)z \leq \|Y\|_D \|(C_{11}^{-1}\otimes I_m)z\|_D,$$
$$Y^T(D\otimes I_m)F(Y) \leq \alpha\sum_{i=1}^{s} d_i - \beta\|Y\|_D^2 .$$
where $z = \widetilde{C}_{12}y^{(n-1)}$. Let
$$\Delta(Y) = (C_{11}^{-1}\otimes I_m)(Y - z - h(C_{11}\otimes I_m)F(Y)). \tag{2.2}$$
It follows from the above four formulae that
$$< Y, \Delta(Y) >_D \geq (\psi_D(C_{11}^{-1}) + h\beta)\|Y\|_D^2 - \|Y\|_D \|(C_{11}^{-1}\otimes I_m)z\|_D - h\alpha\sum_{i=1}^{s} d_i.$$
It follows from (2.1) that for R sufficiently large we have that
$$< Y, \Delta(Y) >_D > 0$$
for all Y satisfying $\|Y\|_D \geq R$. It follows from a classical result, given by Ortega & Rheinboldt[9], that there exists a $Y \in R^{ms}$ such that $\|Y\|_D < R$ and $\Delta(Y) = 0$, i.e.
$$Y - z - h(C_{11}\otimes I_m)F(Y)) = 0.$$
It follows from Lemma 2.1, 2.3 and Theorem 2.1 that

Theorem 2.2. *If the method (1.3) is irreducible and weakly algebraically stable (about the matrices $G, D, \phi(\widetilde{D})$), C_{11} is invertible, then the implicit equations of the method (1.3) applied to the problem (1.1)-(1.2) are soluble for any stepsize $h > 0$ and any $z = \widetilde{C}_{12}y^{(n-1)} \in R^{ms}$.*

Now we consider the more general case where C_{11} may be singular. Let
$$z = (z_1^T, z_2^T, \cdots, z_s^T)^T = \widetilde{C}_{12}y^{(n-1)} \in R^{ms},$$
$$X = Y - z, \quad \bar{F}(X) = hF(X + z).$$
Using the notation, the method (1.4) may be expressed in the form
$$\begin{cases} Y &= \widetilde{C}_{11}\bar{F}(X) + z, \\ y^{(n)} &= \widetilde{C}_{21}\bar{F}(X) + \widetilde{C}_{22}y^{(n-1)}, \\ y_n &= \widetilde{\sigma}y^{(n)}. \end{cases} \tag{2.3}$$
Lemma 2.4. *Suppose there exists $D \in \bar{D}$ and $p \geq 0$ such that*
$$DC_{11} + C_{11}^T D - pC_{11}^T DC_{11} \geq 0.$$
Then $C_{11} + \tau I_s$ is invertible for all $\tau > 0$ and
$$< ((C_{11} + \tau I_s)^{-1}\otimes I_m)X, X >_D \geq \frac{p}{2(1 + p\tau)}\|X\|_D^2 .$$

Theorem 2.3. *If the method (1.3) is irreducible and $(k, p, 0)$-weakly algebraically stable with $k > 0$, $p \geq 0$ (about the matrices $G, D, \phi(\widetilde{D})$), then the implicit equations of the method (1.3) applied to the problem (1.1)-(1.2) are soluble for any stepsize $h > 0$ and any $z = \widetilde{C}_{12}y^{(n-1)} \in R^{ms}$.*

Proof. It follows from Lemma 2.1 that D is positive definite and
$$DC_{11} + C_{11}^T D - pC_{11}^T DC_{11} \geq 0.$$
For given $\tau > 0$, let
$$\Psi_\tau(X) = ((C_{11} + \tau I_s)^{-1}\otimes I_m)X - \bar{F}(X).$$

It follows from (1.2) and Lemma 2.4

$$< \Psi_\tau(X), X >_D \geq (\frac{p}{2(1+p\tau)} + h\beta)\|X\|_D^2 - h\alpha\sum_{i=1}^{s} d_i$$

This implies $< \Psi_\tau(X), X >_D \geq 0$ for all X satisfying

$$\|X\|_D \geq ((\frac{p}{2(1+p\tau)} + h\beta)^{-1} h\alpha \sum_{i=1}^{s} d_i)^{\frac{1}{2}}.$$

It follows from a classical result, given by Ortega & Rheinboldt[9], that there exists an $X(\tau)\in R^{ms}$ with

$$\Psi_\tau(X(\tau)) = 0, \quad \|X(\tau)\|_D < ((\frac{p}{2(1+p\tau)} + h\beta)^{-1} h\alpha \sum_{i=1}^{s} d_i)^{\frac{1}{2}}.$$

Now consider a positive sequence $\{\tau_k\}$ with limit zero. Since the corresponding sequence $\{X(\tau_k)\}$ is bounded, there is a sub-sequence $\{X(t_k)\}$ which converges to some $X^* \in R^{ms}$. Since $\{t_k\}$ has limit zero and

$$X(t_k) - ((C_{11} + t_k I_s)\otimes I_m)\bar{F}(X(t_k)) = 0, \quad k = 1, 2, 3, \cdots,$$

it follows from the continuity of F that $Y^* = X^* + z$ is a solution of the implicit equations of the method (1.3).

3. Local Uniqueness

Humphries and Stuart[1] have given a special example of the problem (1.1)-(1.2) for which the backward Euler method has multiple solutions with h arbitrarily small. Therefore we can only establish a local uniqueness result.

Lemma 3.1. *If f satisfies the globally Lipschitz condition with Lipschitz constant L, then the implicit equations of (1.3) is uniquely soluble when $h < \frac{1}{La}$, where*

$$a = \max_i \sum_{j=1}^{i-1} |c_{ij}^{11}| + \max_i \sum_{j=i}^{s} |c_{ij}^{11}|.$$

Furthermore, set the iteration

$$Y_i^{N+1} = z_i + h\sum_{j=1}^{i-1} C_{ij}^{11} f(Y_j^{N+1}) + h\sum_{j=i}^{s} C_{ij}^{11} f(Y_j^{N}), \quad Y_i^0 = z_i, \tag{3.1}$$

where $z_i = \sum_{j=1}^{r} c_{ij}^{12} y_j^{(n-1)}$, $i = 1, 2, \cdots, s$. Then $Y_i = \lim_{N\to\infty} Y_i^N$ exists and defines the solution of the implicit equations of (1.3).

Lemma 3.1 is an extension of Theorem 14 about Runge-Kutta methods in Butcher[3], their proofs are completely similar.

Theorem 3.1. *Let $B\subset R^m$ be a compact set, $\varepsilon > 0$ a real constant. Suppose f is Lipschitz on $N(B,\varepsilon)$ and that $R = \sup_{y\in N(B,\varepsilon)} \|f(y)\|$ is finite. Then when*

$$h < min(\frac{\varepsilon}{aR}, \frac{1}{La}),$$

for $\forall z_i\in B$, the implicit equations of the method (1.3) have the unique solution Y_i such that

$$\|Y_i - z_i\| < \varepsilon, \quad i = 1, 2, \cdots, s,$$

and hence $Y_i\in B(z_i,\varepsilon)\subseteq N(B,\varepsilon)$ for all i and the iteration (3.1) converges to this solution, where $B(z_i,\varepsilon)$ is an open ball with the radius ε and the center z_i, $N(B,\varepsilon)$ is the ε-neighbourhood of B with

$$N(B,\varepsilon) = \{x : \inf_{y\in B} \|x - y\| < \varepsilon\}.$$

Proof. Consider the iteration (3.1). Denote the cartesian product of s closed balls $\bar{B}(z_i,\varepsilon)$ by $\prod_{i=1}^{s}\bar{B}(z_i,\varepsilon)$. Now suppose

$$(Y_1^N, Y_2^N, \cdots, Y_s^N) \in \prod_{i=1}^{s} \bar{B}(z_i,\varepsilon).$$

Then (3.1) and $h < \frac{\varepsilon}{aR}$ imply

$$(Y_1^{N+1}, Y_2^{N+1}, \cdots, Y_s^{N+1}) \in \prod_{i=1}^{s} B(z_i,\varepsilon).$$

Furthermore, since f is Lipschitz on $N(B,\varepsilon)$, the iteration (3.1) defines a continuous map from the convex compact set $\prod_{i=1}^{s}\bar{B}(z_i,\varepsilon)$ to itself. Thus by Brouwer's Fixed Point Theorem[10] there exists a fixed point of the iteration (3.1) within $\prod_{i=1}^{s}\bar{B}(z_i,\varepsilon)$. Furthermore, it follows from Lemma 3.1 that the fixed point is unique when $h < \frac{1}{La}$. Therefore, when

$$h < min(\frac{\varepsilon}{aR}, \frac{1}{La}),$$

this defines the required unique solution of the implicit equations of the method (1.3).

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