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SUPERCONVERGENCES OF THE ADINI'S ELEMENT FOR SECOND ORDER EQUATION*1)

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Abstract

In this paper, the asymptotic error expansions of Adini's element for the second order imhomogeneous Neumann problem are given and the superconvergence estimations are obtained. Moreover, a numerical example to support our theoretical analysis is reported.

Key words: Adini's element, Superconvergence estimation, Asymptotic expansion.

1. Introduction

It is well known that the Adini's element is commonly used for approximating the solution of high order partial differential equations (such as plate problem). What results can we get if we solve second order imhomogeneous Neumann elliptic problem using Adini's element ? We shall find by this paper that Adini's element is a natural and simple superconvergence element and it has many advantages— it not only provides directly values of the finite element solution and its derivatives at the vertices of rectangular elements, but also has fewer degree of freedoms and higher approximate accuracy than that of the standard bicubic element. Moreover we have further obtained the natural superconvergences of derivatives at the vertices.

We consider the following inhomogeneous Neumann problem:

$$\begin{cases} -\triangle u + u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a rectangle (or a paralleogram).

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The weak form of equation (1) is given by :

$$\begin{cases} \text{find } u \in & H^1(\Omega) \text{ suth that} \\ a(u,v) = & (f,v), \qquad \forall v \in H^1(\Omega), \end{cases} \tag{2}$$

where

$$a(u,v) = \int_\Omega (\nabla u \nabla v + uv) dxdy.$$

Let $T^h = \{e\}$ be a rectangular partition of Ω, h denote the diameter of the largest element in T^h and $V^h \subset H^1(\Omega)$ be the Adini's finite element space satisfying boundary condition associated with T^h. Then the FE–approximation of the problem (2) is as follows:

$$\begin{cases} \text{find } R_h u \in & V^h \text{ such that} \\ a(R_h u,v) = & (f,v), \qquad \forall v \in V^h. \end{cases} \tag{3}$$

Let $i^h : C(\Omega) \longrightarrow V^h(\Omega)$ be a standard interpolation operator. For a fixed rectangle $e = (x_e - h_e, x_e + h_e) \times (y_e - k_e, y_e + k_e)$ with the center $p = (x_e, y_e)$ and two widths $2h_e$ and $2k_e$, we define the following quadratic error functions as that in [5]–[7]:

$$E(x) = \frac{1}{2}\left((x - x_e)^2 - h_e^2\right), \quad F(y) = \frac{1}{2}\left((y - y_e)^2 - k_e^2\right).$$

Lemma 1. *For any $x_0 \in e \in T^h$, we have*

$$\begin{aligned} f(x_0)\int_e u dxdy &= \int_e uf dxdy + \mathcal{O}(h)||f||_{1,2}||u||_{0,2} \\ &= \int_e uf dxdy + \mathcal{O}(h^2)||f||_{2,2}||u||_{1,2}. \end{aligned}$$

Lemma 2. *For any $v \in V^h$, we have the following expansion formulas if the partition T^h of Ω is a uniform mesh*

$$\sum_e \int_e (u - i_h u)_x v_x = \mathcal{O}(h^4)||u||_5||v||_1, \tag{4}$$

$$\sum_e \int_e (u - i_h u)_y v_y = \mathcal{O}(h^4)||u||_5||v||_1, \tag{5}$$

where v_x denotes the partial derivative of v for x.

Proof. We have by Taylar expansion formula:

$$\begin{aligned} v_x(x,y) &= v_x(x_e,y_e)+v_{x^2}(x_e,y_e)E_x+v_{xy}(x_e,y_e)F_y+\tfrac{1}{2}v_{x^3}(x_e,y_e)(x-x_e)^2 \\ &\quad +v_{x^2y}(x_e,y_e)E_xF_y+\tfrac{1}{2}v_{xy^2}(x_e,y_e)(y-y_e)^2+\tfrac{1}{2}v_{x^3y}(x_e,y_e)(x-x_e)^2(y-y_e) \\ &\quad +\tfrac{1}{6}v_{xy^3}(x_e,y_e)(y-y_e)^3. \end{aligned}$$

We get by a detailed calculation

$$\int_e (u-i_hu)_x v_x(x_e,y_e)=\frac{1}{6}v_x(x_e,y_e)\int_e F^2u_{xy^4}=\mathcal{O}(h^4)||u||_5||v||_1,$$

$$\int_e (u-i_hu)_x F_y v_{xy}(x_e,y_e)=\frac{1}{90}v_{xy}(x_e,y_e)\int_e (F^3)_y u_{xy^4}=\mathcal{O}(h^4)||u||_5||v||_1,$$

$$\int_e (u-i_hu)_x (x-x_e)^2 v_{x^3}(x_e,y_e)=\mathcal{O}(h^4)||u||_5||v||_1,$$

$$\int_e (u-i_hu)_x E_xF_y v_{x^2y}(x_e,y_e)=\mathcal{O}(h^4)||u||_5||v||_1,$$

$$\int_e (u-i_hu)_x (y-y_e)^2 v_{xy^2}(x_e,y_e)=\mathcal{O}(h^4)||u||_5||v||_1,$$

$$\int_e (u-i_hu)_x (x-x_e)^2 F_y v_{x^3y}(x_e,y_e)=\mathcal{O}(h^4)||u||_5||v||_1,$$

$$\int_e (u-i_hu)_x (y-y_e)^3 v_{xy^3}(x_e,y_e)=\mathcal{O}(h^4)||u||_5||v||_1,$$

$$\begin{aligned} \int_e (u-i_hu)_x v_{x^2}(x_e,y_e)E_x &= -\frac{h_e^4}{45}v_{x^2}(x_e,y_e)\int_e u_{x^4}-\frac{h_e^2k_e^2}{9}v_{x^2}(x_e,y_e)\int_e u_{x^2y^2} \\ &\quad +\mathcal{O}(h^4)||u||_5||v||_1. \end{aligned}$$

Using

$$v_{x^2}(x_e,y_e)=v_{x^2}(x,y)-E_xv_{x^3}(x,y)-F_yv_{x^2y}(x,y)+E_xF_yv_{x^3y}(x,y),$$

we have further expansion if the mesh is uniform:

$$\sum_e \frac{h_e^4}{45} v_{x^2}(x_e,y_e)\int_e u_{x^4} = \sum_e \frac{h_e^4}{45}\int_e [F u_{x^4 y}(v_x - \overline{i_h} v_x)_{xy} - F_y u_{x^5}(v_x - \overline{i_h} v_x)_x] + \mathcal{O}(h^4)||u||_5||v||_1 = \mathcal{O}(h^4)||u||_5||v||_1,$$

$$\sum_e \frac{h_e^2 k_e^2}{9} v_{x^2}(x_e,y_e)\int_e u_{x^2y^2} = \mathcal{O}(h^4)||u||_5||v||_1,$$

where $\overline{i_h}$ denotes the bilinear interpolation operator and we have used the continuity of $\overline{i_h} v_x$ acrross the adjacent elements and the zero boundary condition for $\overline{i_h} v_x$. This completes the proof of (4) by above all identities. Similarly the (5) can be proved.

Lemma 3. *For any $v \in V^h$, we have*

$$\int_e (u - i_h u)v = \mathcal{O}(h^4)||u||_4||v||_0$$

2. Superconvergences and Numerical Example

Theorem. *Suppose that the partition T^h of Ω is a uniform mesh, u and $R_h u$ satisfy (2) and (3), respectively. Then we have, if $u \in H^5(\Omega)$,*

$$||R_h u - i_h u||_1 = \mathcal{O}(h^4), \tag{6}$$

$$|(u - R_h u)(x)| = \mathcal{O}(h^4|\ln h|^{\frac{1}{2}}),\ \forall x \in \Omega \tag{7}$$

and, if $u \in W^{5,\infty}$,

$$|\bigtriangledown (u - R_h u)(z)| = \mathcal{O}(h^4|\ln h|), \tag{8}$$

where z is the vertex of rectangular element $e \in T^h$.

Proof. From Lemma 2, Lemma 3 and the v–elliptic of the bilinear form $a(u,v)$, we get (6) immediately. From the interpolation error estimate and

$$||i_h u - R_h u||_{0,\infty} \le c|\ln h|^{\frac{1}{2}}||i_h u - R_h u||_1,$$

we deduce (7).

Let $\partial_z G_z^h \in V^h$, for any $z \in \Omega$, denote the derivative of discrete Green function. We see by Lemma 2 and Lemma 3 that

$$\bigtriangledown(i_h u - R_h u)(z) = a(i_h u - R_h u, \partial_z G_z^h) = a(i_h u - u, \partial_z G_z^h) = \mathcal{O}(h^4)||u||_{5,\infty}||\partial_z G_z^h||_{1,1},$$

which shows (8) combining the definition of interpolation operator with $||\partial_z G_z^h||_{1,1} = \mathcal{O}(|\ln h|)$ (see [13]).

In order to demonstrate our theoretical analysis, we give a numerical example in the following. We consider

$$\begin{aligned} f &= sin(x - \tfrac{1}{3}x^3)sin(y - \tfrac{1}{3}y^3) + sin(x - \tfrac{1}{3}x^3)(1 - x^2)^2 sin(y - \tfrac{1}{3}y^3) \\ &\quad + sin(y - \tfrac{1}{3}y^3)(1 - y^2)^2 sin(x - \tfrac{1}{3}x^3) + 2x cos(x - \tfrac{1}{3}x^3) sin(y - \tfrac{1}{3}y^3) \\ &\quad + 2y cos(y - \tfrac{1}{3}y^3) sin(x - \tfrac{1}{3}x^3) \end{aligned}$$

and $\Omega = [-1, 1] \times [-1, 1]$ in equation (1). Then the solution of the equation (1) is

$$u = sin(x - \frac{1}{3}x^3) sin(y - \frac{1}{3}y^3).$$

The numerical results, written in the following tables, coincide with the above theory.

Table 1. Finite element errors.

Mesh size	max $\lvert\nabla(u - R_h u)(z)\rvert$	error order	max$\lvert(u - R_h u)(z)\rvert$	error order
4×4	2.007911E-2		4.861653E-4	
8×8	2.411752E-3	$h^4\lvert\ln h\rvert$	4.565716E-5	$h^4\lvert\ln h\rvert^{\frac{1}{2}}$
16×16	2.724797E-4	$h^4\lvert\ln h\rvert$	3.367662E-6	$h^4\lvert\ln h\rvert^{\frac{1}{2}}$

Table 2. Finite element errors at the fixed point $z_0 = (-0.5, -0.5)$.

Mesh size	$\lvert\nabla(u - R_h u)(z_0)\rvert$	error order	$\lvert(u - R_h u)(z_0)\rvert$	error order
4×4	2.314583E-3		4.303604E-4	
8×8	1.067789E-4	h^4	3.483759E-5	h^4
16×16	3.266993E-6	h^4	2.533197E-6	h^4

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