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ON THE LEAST SQUARES PROBLEM OF A MATRIX EQUATION*1)

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Abstract

Least squares solution of F=PG with respect to positive semidefinite symmetric P is considered,a new necessary and sufficient condition for solvablity is given,and the expression of solution is derived in the some special cases. Based on the expression, the least spuares solution of an inverse eigenvalue problem for positive semidefinite symmetric matrices is also given.

Key words: Least squares solution, Matrix equation, Inverse eigenvalue problem, Positive semidefinite symmetric matrix.

1. Introduction

The purpose of this paper is to study the least squares problem of the matrix equation F=PG with respect to P$\in S^n_{\geq}$,i.e.

(P_1) $\qquad \min_{P\in S^n_{\geq}} \|F-PG\|,\ where\ F,G\in R^{n\times m}\ and\ G\neq 0.$

Where $\|\cdot\|$ denotes the Frobenius norm, and $S^n_{\geq}=\{X\in S^n|X\geq 0\}, S^n=\{X\in R^{n\times n}|X=X^T\}$.Problem$(P_1)$ was first formulated by Allwright [1],A necessary and sufficient condition for the existence of the minimizer $\hat{P}$ in (P_1) was given in [2],where exact global solutions for (P_1) are denoted throughout by $\hat{P}$.The expressions of solution and the numerical solution for (P_1) had been studied in [3]. But the expression of solution is given only for two special cases, i.e. case a): $\hat{P}=FG^+$ if rank(G)=n and $G^TF\in S^m_{\geq}$; and case b):$\hat{P}=0$ iff rank(G)=n and $-FG^T-GF^T\in S^n_{\geq}$.

Problem (P_1) is often appeared in many fields such as structural analysis, system parameter identification ,automatic control, nonlinear programming and so on. A relevant work is [4].

When $S=S^n_{\geq}$,the following inverse eigenvalue problem

$$(P_2')\qquad \min_{P\in S}\|G\wedge -PG\|, where\ G\in R^{n\times m}\ and\ \wedge=diag(\lambda_1,\lambda_2,...,\lambda_m)$$

is a special case of (P_1).A necessary and sufficient condition for solvability and the expression of solution of (P_2') were given for $S=R^{n\times n}$ and $S=S^n$ in [5,6].The

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following special inverse eigenvalue problem

$$(P_2) \qquad \min_{P \in S^n_{\geq}} \|G \wedge - PG\|, where\ G \in R^{n\times m}\ and\ \wedge = diag(\lambda_1, \lambda_2, ..., \lambda_m) \geq 0$$

is solved by using dual cone theory[10].

Although the least squares solution of the following problem

$$(P_2'') \qquad \min_{X \in S^n_{\geq}} \|A^T XA - D\|,\ where\ A \in R^{n\times m}, D \in R^{m\times m}$$

was successfully solved by Dai and Lancaster[7], the approach adopted there is based on symmetry of (P_2''), and yet there is not such property in (P_1).So the approach adopted in [7] is not suitable to (P_1).

The aim of this paper is to give a new necessary and sufficient condition for solvability of (P_1) and then derive a expression of solution in the some special cases. Based on the expression we have also solved (P_2).This paper extends the results in [10].

The notation used in the sequel can be summarized as follows. For $A, B \in R^{n\times m}$, A^+ and $A * B$ respectively denote the Moore-Penrose pseudoinverse of A and the Hadamard product of A and B.$OR^{n\times n}$ denotes the set of all orthogonal matrices in $R^{n\times n}$.The notation $A \geq 0 (> 0)$ means that A is positive semidefinite (definite). For $\Sigma = diag(\sigma_1, \sigma_2, ..., \sigma_r) > 0, \Phi_\Sigma$ denotes the matrix $(\varphi_{ij})_{r\ \times r}$,where $\varphi_{ij} = (\sigma_i^2 + \sigma_j^2)^{-1}, 1 \leq i, j \leq r$. In addition, a unit matrix is denoted by I,and the set $\{X \in S^n | X > 0\}$ is denoted by $S^n_>$.

This paper is organized as follows. A new necessary and sufficient condition for solvability of (P_1) is given in section 2. Based on the condition, in section 3 the expression of solution of (P_1) is given in some special cases. Problem (P_2) is solved in section 4.

2. The Solvability Conditions for (P_1)

To Study the solvability of (P_1), we decompose the given matrix G by the singular value decomposition(SVD):

$$G = U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^T = U_1 \Sigma V_1^T \tag{2.1}$$

where $U = (U_1, U_2) \in OR^{n\times n}, U_1 \in R^{n\times r}, V = (V_1, V_2) \in OR^{m\times m}, V_1 \in R^{m\times r}, \Sigma = diag(\sigma_1, \sigma_2, ..., \sigma_r) > 0, r = rank(G)$.

Theorem 2.1. *Suppose that rank(G)<n, and a SVD of the matrix G is (2.1). Then (P_1) has a solution if and only if rank($\hat{P}_{11}$)= rank($\hat{P}_{11}|\hat{P}_{12}$),where $\hat{P}_{11}$ is a unique minimizer of $\|U_1^T FV_1 - P_{11}\Sigma\|$ with respect to $P_{11} \in S^r_{\geq}$,and $\hat{P}_{12} = (U_2^T FV_1\Sigma^{-1})^T$.*

If (P_1) has a solution, then the expression of solution is

$$\hat{P} = U \begin{pmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{12}^T & \hat{P}_{12}^T \hat{P}_{11}^+ \hat{P}_{12} + B \end{pmatrix} U^T \tag{2.2}$$

where $B \in S^{n-r}_{\geq}$ is arbitrary.

To prove Theorem 2.1, it will be convenient to give the following three lemmas.

Lemma 2.1.[1] *The minimizer in (P_1) exists and is unique when rank(G)=n.*

Lemma 2.2. *Suppose that F,G* $\in R^{n\times m}$ *and there is a minimizer* $\hat{P}$ *in* (P_1). *Then*

$$\|F-\hat{P}G\| = \min_{P\in S^n_{\geq}} \|F-PG\| = \inf_{P\in S^n_{>}} \|F-PG\|.$$

Proof. It is obvious that $\|F-\hat{P}G\| = \min_{P\in S^n_{\geq}} \|F-PG\| \leq \inf_{P\in S^n_{>}} \|F-PG\|$.On the other hand,for any $\epsilon>0$, let $\delta =(2\|G\|)^{-1}\epsilon$ (note $G\neq 0$), then $\hat{P}+\delta I>0$ (note $\hat{P}\geq 0$) and $\|F-(\hat{P}+\delta I)G\| \leq \|F-\hat{P}G\|+\|\delta G\| < \|F+\hat{P}G\|+\epsilon$.Hence $\|F-\hat{P}G\| = \inf_{P\in S^n_{>}} \|F-PG\|$.

Lemma 2.3.[8] *Suppose that a real symmetric matrix is partitioned as*

$$\begin{pmatrix} E & F \\ F^T & G \end{pmatrix}$$

where E and G are square. Then this matrix is positive semidefinite if and only if

$$E\geq 0, G-F^TE^+F\geq 0 \text{ and } rank(E)=rank(E|F).$$

Proof of Theorem 2.1. For P$\in S^n_{\geq}$,partition P as

$$P=U\begin{pmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{pmatrix}U^T \tag{2.3}$$

where $P_{11}\in S^r_{\geq}$.It follows from (2.1) and (2.3) that

$$\|F-PG\|^2 = \|\begin{pmatrix} U_1^TFV_1-P_{11}\Sigma & U_1^TFV_2 \\ U_2^TFV_1-P_{12}^T\Sigma & U_2^TFV_2 \end{pmatrix}\|^2 = \|U_1^TFV_1-P_{11}\Sigma\|^2$$
$$+\|U_2^TFV_1-P_{12}^T\Sigma\|^2+\|U_1^TFV_2\|^2+\|U_2^TFV_2\|^2 \tag{2.4}$$

It follows from Lemma 2.1 that there is unique $\hat{P}_{11}$ which minimizes $\|U_1^TFV_1-P_{11}\Sigma\|$ with respect to $P_{11}\in S^r_{\geq}$.If $\hat{P}_{12}=\Sigma^{-1}V_1^TF^TU_2$ and rank($\hat{P}_{11}$) $= rank(\hat{P}_{11}|\ \hat{P}_{12})$, then

$$P^*=U\begin{pmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{12}^T & P_{22} \end{pmatrix}U^T$$

is optimal for (P_1) for any $P_{22}\in S^{n-r}_{\geq}$ such that $P_{22}\geq \hat{P}_{12}^T\hat{P}_{11}^+\hat{P}_{12}$. In fact, it follows from Lemma 2.3 and (2.4) that $P^*\in S^n_{\geq}, \hat{P}_{11}$ and $\hat{P}_{12}$ minimize the right hand side of (2.4) with respect to $P_{11}\in S^r_{\geq}$ and $P_{12}\in R^{r\times(n-r)}$. Hence ($P_1$) certainly has a minimum when rank($\hat{P}_{11}$) $= rank(\hat{P}_{11}|\hat{P}_{12})$,and in which case $\hat{P}$ can be expressed by (2.2).

Conversely,it only remains to be shown that there is no minimum when $rank(\hat{P}_{11})\neq rank(\hat{P}_{11}|\hat{P}_{12})$.

Suppose that $rank(\hat{P}_{11})\neq rank(\hat{P}_{11}|\hat{P}_{12})$ and that there is a minimum, say at $A_0\in S^n_{\geq}$. Partition A_0 as

$$A_0=U\begin{pmatrix} B_0 & C_0 \\ C_0^T & D_0 \end{pmatrix}U^T, \tag{2.5}$$

where $B_0\in R^{r\times r}$, it follows from Lemma 2.3 that $B_0\in S^r_{\geq}$ and $rank(B_0)=rank(B_0|C_0)$.

If $C_0 = \hat{P}_{12}$, then $B_0 \neq \hat{P}_{11}$. Otherwise it follows that $rank(\hat{P}_{11}) = rank(B_0) = rank(B_0|C_0) = rank(\hat{P}_{11}|\hat{P}_{12})$, which contradicts the initial assumption that $rank(\hat{P}_{11}) \neq rank(\hat{P}_{11}|\hat{P}_{12})$.

Hence, whether $C_0 = \hat{P}_{12}$ or $C_0 \neq \hat{P}_{12}$, it follows from (2.1),(2.5) and Lemma 2.2 that

$$\begin{aligned}
\|F - A_0G\|^2 &= \|U_1^T FV_1 - B_0\Sigma\|^2 \\
&+ \|U_2^T FV_1 - C_0^T\Sigma\|^2 + \|U_1^T FV_2\|^2 + \|U_2^T FV_2\|^2 \\
&> \|U_1^T FV_1 - \hat{P}_{11}\Sigma\|^2 + \|U_1^T FV_2\|^2 + \|U_2^T FV_2\|^2 \\
&= \min_{P_{11}\in S^r_{\geq}} \|U_1^T FV_1 - P_{11}\Sigma\|^2 + \|U_1^T FV_2\|^2 + \|U_2^T FV_2\|^2 \\
&= \inf_{P_{11}\in S^r_{>}} \|U_1^T FV_1 - P_{11}\Sigma\|^2 + \|U_1^T FV_2\|^2 + \|U_2^T FV_2\|^2 \\
&= \inf_{P\in S^n_{>}} \|F - PG\|^2 \\
&= \min_{P\in S^n_{\geq}} \|F - PG\|^2
\end{aligned}$$

This contradicts the optimality of A_0 and therefore contradicts the existence of a minimum for (P_1) when $rank(\hat{P}_{11}) \neq rank(\hat{P}_{11}|\hat{P}_{12})$.Which completes the proof.

3. The Expression of Solution for (P_1)

As it is stated in [3], it is difficult to find expression of solution for (P_1).In this section, we get the expression of solution for (P_1) in the some special cases.

Lemma 3.1. *Suppose that real numbers $\sigma_1, \sigma_2, ..., \sigma_r$ are all positive,$\Phi_\Sigma = (\varphi_{ij})_{r\times r}, \varphi_{ij} = (\sigma_i^2 + \sigma_j^2)^{-1}, 1 \leq i, j \leq r$.Then $\Phi_\Sigma \geq 0$.*

Proof. Note that $\varphi_{ij} = (\sigma_i^2 + \sigma_j^2)^{-1} = \int_0^\infty e^{-(\sigma_i^2+\sigma_j^2)t}dt$, then for any $X = (x_1, x_2, ..., x_r)^T \in R^r$,we see that

$$X^T\Phi_\Sigma X = \int_0^\infty (x_1, ..., x_r)(e^{-\sigma_1^2 t}, ..., e^{-\sigma_r^2 t})^T(e^{-\sigma_1^2 t}, ..., e^{-\sigma_r^2 t})(x_1, ..., x_r)^T dt$$

$$= \int_0^\infty (\sum_{i=1}^r x_i e^{-\sigma_i^2 t})^2 dt \geq 0,$$

so $\Phi_\Sigma \geq 0$.The Lemma is proved.

Lemma 3.2.[9] *If $A, B \in S^n_{\geq}$, then $A * B \in S^n_{\geq}$. If,in addition,$B\in S^n_{>}$ and A has no diagonal entry equal to 0,then $A * B \in S^n_{>}$.*

Theorem 3.1. *Suppose that $G \in R^{n\times m}$,rank(G) = n and the SVD of the matrix G is*

$$G = U(E, O)V^T = UEV_1^T \tag{3.1}$$

where $U \in OR^{n\times n}, V = (V_1, V_2) \in R^{m\times m}, E = diag(\sigma_1, \sigma_2, ..., \sigma_n) > 0$. Then

$$\hat{P} = U(\Phi_E * (U^T(FG^T + GF^T)U))U^T, \tag{3.2}$$

if $\Phi_E * (U^T(FG^T + GF^T)U) \geq 0$. Especially, if $FG^T + GF^T \geq 0$, then $\hat{P}$ can be expressed by(3.2).

Proof. For any P $\in S^n_{\geq}$, it follows from (3.1) that

$$\|F-PG\|^2 = \|U^TFV_1 - U^TPUE\|^2 + \|U^TFV_2\|^2 \tag{3.3}$$

On the other hand, it follows from [6,Lemma 2.1] that $\|U^TFV_1 - SE\|$ is minimized with respect to $S \in S^n$ by taking

$$\begin{aligned}\hat{S} &= \Phi_E * (U^TFV_1E + EV_1^TF^TU)\\ &= \Phi_E * (U^T(FG^T + GF^T)U)\end{aligned} \tag{3.4}$$

Hence,when $\hat{S} \geq 0$,the minimizer $\hat{P}$ of $\|U^TFV_1 - U^TPUE\|$ with respect to $P \in S^n_{\geq}$ is $U\hat{S}U^T$,i.e.$\hat{P} = U\hat{S}U^T$. That is to say,$\hat{P}$ can be expressed by (3.2) when $\hat{S} \geq 0$. Especially, if $FG^T + GF^T \geq 0$,it follows from Lemma 3.1,Lemma 3.2 and (3.4) that $\hat{S} \geq 0$. Thus we have proved Theorem 3.1.

Note: When rank (G) = n and $G^TF \geq 0$, we can prove that

$$U(\Phi_E * (U^T(FG^T + GF^T)U))U^T = FG^+ \geq 0,$$

thereby $\Phi_E * (U^T(FG^T + GF^T)U) \geq 0$. On the other hand,let $G = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $F = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$,then rank(G)=2 and $FG^T + GF^T = \begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix} \geq 0$, thereby $\Phi_E * (U^T(FG^T + GF^T)U) \geq 0$; but $G^TF = \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \notin S^2_{\geq}$. Hence,Theorem 3.1 is a generalization of the Theorem 2.4 in [3].

Theorem 3.2. *Suppose that $G \in R^{n\times m}$, rank(G)< n and the SVD of the matrix G is (2.1). Then(P_1) has a solution if $\hat{S} \geq 0$ and $rank(\hat{S}) = rank(\hat{S}|\hat{P}_{12})$,in which case the expression of solution is*

$$\hat{P} = U\begin{pmatrix} \hat{S} & \hat{P}_{12} \\ \hat{P}_{12}^T & \hat{P}_{12}^T\hat{S}^+\hat{P}_{12} + B \end{pmatrix}U^T \tag{3.5}$$

where $B \in S^{n-r}_{\geq}$ is arbitrary ,$\hat{S} = \Phi_\Sigma*(U_1^TFV_1\Sigma + \Sigma V_1^TF^TU_1)$ and $\hat{P}_{12} = (U_2^TFV_1\Sigma^{-1})^T$. Especially, Problem($P_1$) has a solution if $\hat{S} > 0$ or $FG^T + GF^T > 0$, and its solution $\hat{P}$ can be expressed by (3.5).

Proof. When $\hat{S} \geq 0$,it is easy to know from the proof of Theorem 3.1 that the minimizer $\hat{P}_{11}$ of $\|U_1^TFV_1 - P_{11}\Sigma\|$ with respect to $P_{11} \in S^r_{\geq}$ is $\hat{S}$ i.e. $\hat{P}_{11} = \hat{S}$.So,when $\hat{S} \geq 0$ and $rank(\hat{S}) = rank(\hat{S}|\hat{P}_{12})$,$\hat{P}_{11} \geq 0$ and $rank(\hat{P}_{11}) = rank(\hat{P}_{11}|\hat{P}_{12})$. Thereby ($P_1$) has a solution and the expression of its solution is (3.5) by Theorem 2.1. It only remains to be shown that $\hat{S} > 0$ when $FG^T + GF^T > 0$. In fact,it follows from (2.1) that

$$\begin{aligned}\hat{S} &= \Phi_\Sigma * (U_1^TFV_1\Sigma + \Sigma V_1^TF^TU_1)\\ &= \Phi_\Sigma * (U_1^T(FG^T + GF^T)U_1),\end{aligned}$$

where $U_1 \in R^{n\times r}$ and rank(U_1)=r. Obviously, $U_1^T(FG^T + GF^T)U_1 > 0$ when $FG^T + GF^T > 0$,thereby $\hat{S} > 0$ by Lemma 3.1 and Lemma 3.2. Thus we complete the proof.

5. The Expression of Solution for (P_2)

Theorem 4.1. *Let a SVD of the G be (2.1). Then (P_2) has a solution , and the* *expression of solution is*

$$\hat{P} = U \begin{pmatrix} 2\Phi_\Sigma * (\Sigma V_1^T \wedge V_1\Sigma) & 0 \\ 0 & B \end{pmatrix} U^T, \tag{4.1}$$

where $B \in S_{\geq}^{n-r}$ *is arbitrary.*

Proof. Note that (P_2) is a special case of (P_1) when F= G$\wedge$. So,from Theorem 3.1 and Theorem 3.2,it is all right to be shown $\hat{S} = 2\Phi_\Sigma * (\Sigma V_1^T \wedge V_1\Sigma) \geq 0$ and $\hat{P}_{12} = 0$.Here $\hat{S}$ and $\hat{P}_{12}$ are the same as Theorem 3.2. In fact,it follow from (2.1) that

$$\begin{aligned} \hat{S} &= \Phi_\Sigma * (U_1^T F V_1 \Sigma + \Sigma V_1^T F^T U_1) \\ &= \Phi_\Sigma * (U_1^T G \wedge V_1\Sigma + \Sigma V_1^T \wedge G^T U_1) \\ &= \Phi_\Sigma * (U_1^T U_1 \Sigma V_1^T \wedge V_1 \Sigma + \Sigma V_1^T \wedge V_1 \Sigma U_1^T U_1) \\ &= 2\Phi_\Sigma * (\Sigma V_1^T \wedge V_1 \Sigma) \end{aligned}$$

and $\Phi_\Sigma * (\Sigma V_1^T \wedge V_1\Sigma) \geq 0$ by $\Phi_\Sigma \geq 0$ and $\wedge \geq 0$.In addition,it follows from $U_2^T U_1 = 0$ that

$$\hat{P}_{12} = (U_2^T F V_1 \Sigma^{-1})^T = (U_2^T G \wedge V_1 \Sigma^{-1})^T = (U_2^T U_1 \Sigma V_1^T \wedge V_1 \Sigma^{-1})^T = 0.$$

The theorem 4.1 is proved.

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