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A LEGENDRE PSEUDOSPECTRAL METHOD FOR SOLVING NONLINEAR KLEIN-GORDON EQUATION*

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Abstract

A Legendre pseudospectral scheme is proposed for solving initial-boundary value problem of nonlinear Klein-Gordon equation. The numerical solution keeps the discrete conservation. Its stability and convergence are investigated. Numerical results are also presented, which show the high accuracy. The technique in the theoretical analysis provides a framework for Legendre pseudospectral approximation of nonlinear multi-dimensional problems.

1. Introduction

As we know, the Klein-Gordon equation is an important mathematical model in quantum mechanics. It is of the form

$$\begin{cases} \dfrac{\partial^2 U}{\partial t^2}(x,t) - \triangle U(x,t) + bU(x,t) + g(U(x,t)) = f(x,t), & x \in \Omega,\ 0 < t \le T, \\ U(x,t) = 0, & x \in \partial\Omega,\ 0 \le t \le T, \\ \dfrac{\partial U}{\partial t}(x,0) = U_1(x), & x \in \Omega, \\ U(x,0) = U_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega = (-1,1)^n$, $x = (x_1, x_2, \cdots, x_n)$, $\triangle = \sum\limits_{j=1}^{n} \dfrac{\partial^2}{\partial x_j^2}$, $g(z) = |z|^\alpha z$, $p = \alpha + 2$ and b is a real number. Assume that $U_0(x) = U_1(x) = 0$ on $\partial\Omega$ and

$$\begin{cases} \alpha \ge 0, & \text{for } n \le 2, \\ 0 \le \alpha \le \dfrac{2}{n-2}, & \text{for } n \ge 3. \end{cases} \tag{1.2}$$

As in [1], it can be shown that if $U_0 \in H_0^1(\Omega) \cap L^p(\Omega)$, $U_1 \in L^2(\Omega)$ and $f \in L^2(0,T;L^2(\Omega))$, then (1.1) has unique solution $U \in C(0,T;H_0^1(\Omega) \cap L^p(\Omega))$. If U_0, U_1 and f are smoother, then U is smoother also. On the other hand, some finite difference schemes were proposed with strict proof of generalized stability and convergence. Their numerical solutions keep the discrete conservations. One of special cases ($\alpha = 2$) was considered also in [4]. But for all these finite difference approximations, the convergence rate is of order 2 in the space. To overcome it, some Fourier spectral and Fourier

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pseudospectral schemes were studied for periodic problems (see [5,6]). Their numerical solutions possess the convergence rate of "infinite order". Recently, Legendre spectral scheme was also studied for the initial-boundary value problem(see[7]). Its numerical results also show that it is more accurate than the corresponding finite difference scheme. However, because of the nonlinear term $g(U)$, it is very difficult to implement the spectral method strictly, unless α is a small integer. In this paper, we discuss the pseudospectral method for solving (1.1). In the next section, we construct a Legendre pseudospectral scheme which simulates the energy conservation law reasonably. In particular, it can be easily implemented for all α. We present the numerical results in section 3, which show the advantages of such approximation. Then we list some lemmas and prove the generalized stability and convergence in the last three sections. The technique in the theoretical analysis provides a framework for Legendre pseudospectral approximation of nonlinear multi-dimensional problems arising in quantum mechanics and other fields.

2. The Scheme

Let $L^q(\Omega) = \{v|v$ is Lebesgue measureable on Ω and $\|v\|_{L^q} < \infty\}$, where

$$\|v\|_{L^q(\Omega)} = \begin{cases} \Big(\int_\Omega |v|^q dx\Big)^{\frac{1}{q}}, & \text{if } 1 \le q < \infty, \\ ess \cdot \sup\limits_{x\in\Omega} |v(x)|, & \text{if } q = \infty. \end{cases}$$

For $q = 2$, we denote the inner product and the norm of $L^2(\Omega)$ by $(\cdot,\cdot)$ and $\|\cdot\|$ respectively. Let Z be the set of all non-negative integers, and $\gamma_l \in Z$. Set $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n)$, $|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ and $D^\gamma = \dfrac{\partial^{|\gamma|}}{\partial {x_1}^{\gamma_1} \partial {x_2}^{\gamma_2} \cdots \partial {x_n}^{\gamma_n}}$. For any non-negative integer m, define $H^m(\Omega) = \{v|D^\gamma v \in L^2(\Omega),\ 0 \le |\gamma| \le m\}$, with the semi-norm $|\cdot|_m$ and the norm $\|\cdot\|_m$ as follows

$$|v|_m = \Big(\sum_{|\gamma|=m} \|D^\gamma v\|^2\Big)^{1/2}, \quad ||v||_m = (||v||_{m-1}^2 + |v|_m^2)^{1/2}.$$

For non–negative real number s, we define $H^s(\Omega)$ by the interpolation between the spaces $H^{[s]}(\Omega)$ and $H^{[s+1]}(\Omega)$. Its norm and semi-norm are denoted by $||\cdot||_s$ and $|\cdot|_s$ respectively.

Let $j_l \in Z$, $j = (j_1, j_2, \cdots, j_n)$ and $|j| = \max\limits_{1\le l\le n} |j_l|$. Set $L_j(x) = \prod\limits_{l=1}^{n} L_{j_l}(x_l)$, $L_{j_l}(x_l)$ being the Legendre polynomial of degree j_l with respect to x_l. For Legendre spectral approximation in spatial directions, we define that for any positive integer N,

$$S_N = span\{L_j(x) \mid |j| \le N\}, \quad V_N = S_N \cap H_0^1(\Omega).$$

Let $P_N : L^2(\Omega) \longmapsto V_N$ be the L^2-orthogonal projection operator, i.e., for any $v \in L^2(\Omega)$, we have $(P_N v - v, \varphi) = 0$, $\forall \varphi \in V_N$.

In this paper, we consider the n-dimensional interpolation. Let $k_l \in Z$, $k = (k_1, k_2, \cdots, k_n)$, $|k| = \max\limits_{1\le l\le n} |k_l|$. Set $x^{(k)} = (x_1^{(k_1)}, x_2^{(k_2)}, \cdots, x_n^{(k_n)})$ and $\omega^{(k)} = \omega_1^{(k_1)}$

$\omega_2^{(k_2)} \cdots \omega_n^{(k_n)}$, $x_l^{(k_l)}$ and $\omega_l^{(k_l)}$ being the nodes and weights of the Gauss-Lobatto quadrature formula on $\overline{I}_l = [-1,1]$, i.e., $x_l^{(0)} = -1$, $x_l^{(N)} = 1$, $x_l^{(k_l)}$ are the zeroes of $L'_N(x_l)$, $k_l = 1, \cdots, N-1$, and

$$\omega_l^{(k_l)} = \frac{2}{N(N+1)} \cdot \frac{1}{[L'_N(x_l^{(k_l)})]^2}, \quad k_l = 0, \cdots, N.$$

Let $\Omega_N = \{x^{(k)} | x^{(k)} \in \overline{\Omega}\}$. Then

$$\int_\Omega v(x)dx = \sum_{x^{(k)} \in \Omega_N} v(x^{(k)})\omega^{(k)}, \quad \forall v \in S_{2N-1}.$$

Let $P_c : C(\overline{\Omega}) \longmapsto S_N$ be the interpolation operator, i.e., for any $v \in C(\overline{\Omega})$, $P_c v \in S_N$ satisfies $P_c v(x^{(k)}) = v(x^{(k)})$, $\forall x^{(k)} \in \Omega_N$. We introduce the discrete L^q-norm and the discrete L^2-inner product associated with the above collocation points as

$$||v||_{L^q,N} = \begin{cases} \Big(\sum\limits_{x^{(k)} \in \Omega_N} |v(x^{(k)})|^q \omega^{(k)} \Big)^{\frac{1}{q}}, & \text{if } 1 \le q < \infty, \\ \sup\limits_{x^{(k)} \in \Omega_N} |v(x^{(k)})|, & \text{if } q = \infty \end{cases}$$

and $(v,w)_N = \sum\limits_{x^{(k)} \in \Omega_N} v(x^{(k)})w(x^{(k)})\omega^{(k)}$. It is not difficult to verifty that (see [8])

$$\begin{aligned} &P_c v = v, \quad \forall v \in S_N, \\ &(v,w)_N = (P_c v, P_c w)_N, \quad \forall v, w \in C(\overline{\Omega}). \end{aligned}$$

Let τ be the mesh size in variable $t, S_\tau = \Big\{ t = k\tau | k = 1, 2, \cdots, \Big[\frac{T}{\tau}\Big] \Big\}$ and $\overline{S}_\tau = S_\tau \cup \{0\}$. For simplicity, we denote $v(x,t)$ by $v(t)$ or v sometimes. Define

$$\begin{aligned} \hat{v}(t) &= \frac{1}{2}(v(t+\tau) + v(t-\tau)), \\ v_{\hat{t}}(t) &= \frac{1}{2\tau}(v(t+\tau) - v(t-\tau)), \\ v_t(t) &= \frac{1}{\tau}(v(t+\tau) - v(t)), \\ v_{\bar{t}}(t) &= v_t(t-\tau), \\ v_{t\bar{t}}(t) &= \frac{1}{\tau}(v_t(t) - v_{\bar{t}}(t)). \end{aligned}$$

It can be verified that

$$2(v_{\hat{t}}(t), \hat{v}(t))_N = (||v(t)||_N^2)_{\hat{t}}, \tag{2.1}$$
$$2(v_{\hat{t}}(t), v_{t\bar{t}}(t))_N = (||v_{\bar{t}}(t)||_N^2)_t. \tag{2.2}$$

It is well known that the solution of (1.1) possesses the conservation

$$E(U,t) = E(U,0) + 2\int_0^t \Big(\frac{\partial U}{\partial t'}(t'), f(t') \Big) dt' \tag{2.3}$$

where $E(U,t) = \left\|\frac{\partial U}{\partial t}(t)\right\|^2 + |U(t)|_1^2 + b||U(t)||^2 + \frac{2}{p}||U(t)||_{L^p}^p$. Clearly, a reasonable discretization of (1.1) should simulate this property. The key point is to approximate the nonlinear term $g(U(x,t))$ suitably. To do this, let (see [2, 3])

$$G(v(x,t)) = \int_0^1 g(\sigma v(x,t+\tau) + (1-\sigma)v(x,t-\tau))d\sigma. \tag{2.4}$$

Clearly, $G(v(x,t))$ is an approximate to $g(v(x,t))$. Futhermore, since

$$g(z) = \frac{1}{p}\frac{d}{dz}|z|^p, \tag{2.5}$$

we have $2v_{\hat{t}}(x,t)G(v(x,t)) = \frac{1}{\tau p}(|v(x,t+\tau)|^p - |v(x,t-\tau)|^p)$, and so

$$(G(v(t)), v_{\hat{t}}(t))_N = \frac{1}{p}(||v(t)||_{L^p,N}^p)_{\hat{t}}. \tag{2.6}$$

Now, let u be the approximation to U. We approximate the nonlinear term $g(U)$ by $P_cG(u)$ instead of $P_cg(u)$. Then the Legendre pseudospectral scheme for (1.1) is to find $u(t) \in V_N$ for all $t \in \overline{S}_\tau$ such that

$$\begin{cases} (u_{t\bar{t}}(t) + b\hat{u}(t) + G(u(t)), v)_N + (\bigtriangledown\hat{u}(t), \bigtriangledown v)_N = (\hat{f}(t), v)_N, \quad \forall v \in V_N, t \in S_\tau, \\ u_t(0) = u_1, \\ u(0) = u_0, \end{cases} \tag{2.7}$$

where $u_0 = P_cU_0$ and $u_1 = P_cU_1 + \frac{\tau}{2}P_c(\triangle U_0 - bU_0 - g(U_0) + f(0))$. We next check the conservation. By taking $v = 2u_{\hat{t}}$ in the first equation of (2.7), we have from (2.1), (2.2) and (2.6) that

$$(||u_{\bar{t}}(t)||_N^2)_t + (||\bigtriangledown u(t)||_N^2)_{\hat{t}} + b(||u(t)||_N^2)_{\hat{t}} + \frac{2}{p}(||u(t)||_{L^p,N}^p)_{\hat{t}} = 2(\hat{f}(t), u_{\hat{t}}(t))_N.$$

A summation of the above equality for $t \in S_\tau$ yields that

$$E^*(u,t) = E^*(u,\tau) + 2\tau\sum_{t' \le t-\tau}(u_{\hat{t}}(t'), \hat{f}(t'))_N \tag{2.8}$$

where

$$\begin{aligned} E^*(u,t) =& ||u_{\bar{t}}(t)||_N^2 + \frac{1}{2}(||\bigtriangledown u(t)||_N^2 + ||\bigtriangledown u(t-\tau)||_N^2) + \frac{b}{2}(||u(t)||_N^2 + ||u(t-\tau)||_N^2) \\ &+ \frac{1}{p}(||u(t)||_{L^p,N}^p + ||u(t-\tau)||_{L^p,N}^p). \end{aligned}$$

Obviously (2.8) is a reasonable analogy of (2.3). Thus scheme (2.7) can give better numerical results.

3. Numerical Results

This section is devoted to numerical experiments. We shall use (2.7) to solve (1.1). For comparison, we also consider a Legendre spectral scheme (see[7]) and a finite difference scheme (see[3, 4]). Let u^s be the Legendre spectral approximation to U. We approximate the nonlinear term $g(U)$ by $P_N G(u^s)$ instead of $P_N g(u^s)$. The Legendre spectral scheme for (1.1) is

$$\begin{cases} (u^s_{t\bar{t}}(t) + b\hat{u}^s(t) + G(u^s(t)), v) + (\bigtriangledown \hat{u}^s(t), \bigtriangledown v) = (\hat{f}(t), v), \quad \forall v \in V_N, d \quad t \in S_\tau, \\ u^s_t(0) = P_N U_1 + \dfrac{\tau}{2} P_N(\triangle U_0 - bU_0 - g(U_0) + f(0)), \\ u^s(0) = P_N U_0. \end{cases} \tag{3.1}$$

We now consider the finite difference scheme. Let $h = \dfrac{1}{N}$ and $\Omega_h = \{x | x = (j_1 h, j_2 h, \cdots, j_n h), -N+1 \le j_l \le N-1, 1 \le l \le n\}$. Define $e_j = (\underbrace{0, \cdots, 0}_{j-1}, 1, 0, \cdots, 0)$, and

$$\triangle_h v(x,t) = \frac{1}{h^2} \sum_{j=1}^{n} (v(x + he_j, t) - 2v(x,t) + v(x - he_j, t)).$$

The finite difference scheme is

$$\begin{cases} u^h_{t\bar{t}}(x,t) - \triangle_h \hat{u}^h(x,t) + b\hat{u}^h(x,t) + G(u^h(x,t)) = \hat{f}(t), & x \in \Omega_h,\ t \in S_\tau, \\ u^h(x,t) = 0, & x \in \partial\Omega_h, t \in \overline{S}_\tau, \\ u^h_t(x,0) = U_1(x) + \dfrac{\tau}{2}(\triangle_h U_0(x) - bU_0(x) - g(U_0(x)) + f(x,0)), & x \in \Omega_h, \\ u^h(x,0) = U_0(x), & x \in \Omega_h. \end{cases} \tag{3.2}$$

For describing the error, let

$$\tilde{E}(u,t) = \frac{||U(t) - u(t)||_N}{||U(t)||_N}, \quad \tilde{E}^s(u^s,t) = \frac{||U(t) - u^s(t)||}{||U(t)||}$$

and

$$\tilde{E}^h(u^h,t) = \frac{\left(\sum_{x \in \Omega_h} |U(x,t) - u^h(x,t)|^2\right)^{1/2}}{\left(\sum_{x \in \Omega_h} |U(x,t)|^2\right)^{1/2}}.$$

For simplicity, we take $n = b = T = 1$ and $\alpha = 2$ in all calculations. The test function is as follows $U(x,t) = A(x^2 - 1)\cos(B(x+t))e^{\omega t}$.

In Table 1, the calculation is carried out with $A = 0.5$, $B = \omega = 1.0$, $N = 8$ and $\tau = 0.005$. The numerical results show that scheme (2.7) gives much better results than (3.2). Scheme(2.7) and (3.1) provide the numerical solutions with very high accracy even if N is small. We also know from Table 1 that scheme(2.7) and (3.1) have the same accuracy. Whereas for scheme(3.1), we have to calculate the coefficients of Legendre expansion by numerical integration, which is quite difficult job. In particular, it takes much time for the nonlinear terms. Table 2 shows the numerical results of scheme(2.7) and (3.2) with $A = B = 1.0$ and $\omega = 2.0$. We find that if N increases and τ decreases proportionally, then the errors become smaller quickly. Table 2 shows the convergences of scheme(2.7) and (3.2). But scheme(2.7) gives much better numerical results and possesses higher convergence rate than (3.2).

Table 1. The errors $\tilde{E}(u,t), \tilde{E}^s(u^s,t)$ and $\tilde{E}^h(u^h,t)$.

	Scheme(2.7)	Scheme(3.1)	Scheme(3.2)
t=0.2	9.21281E-7	9.50890E-7	1.23154E-3
t=0.4	5.91656E-7	6.52785E-7	4.34661E-3
t=0.6	1.52947E-6	1.39562E-6	8.66063E-3
t=0.8	3.82931E-6	3.64652E-6	1.36452E-2
t=1.0	6.61214E-6	6.31079E-6	1.88353E-2

Table 2. The errors $\tilde{E}(u,1.0)$ and $\tilde{E}^h(u^h,1.0)$.

	Scheme(2.7)			Scheme(3.2)		
N	$\tau=0.005$	$\tau=0.001$	$\tau=0.0005$	$\tau=0.005$	$\tau=0.001$	$\tau=0.0005$
4	1.52081E-3	1.54167E-3	1.54159E-3	8.79735E-3	8.78729E-3	8.78698E-3
8	2.95840E-5	1.16568E-6	2.97163E-7	2.65475E-3	2.63820E-3	2.63768E-3
16	2.94326E-5	1.16412E-6	2.93173E-7	6.76090E-4	6.58751E-4	6.58222E-4
32	2.92907E-5	1.16264E-6	2.90184E-7	1.83250E-4	1.64651E-4	1.64116E-4
64	2.91293E-5	1.16087E-6	2.87146E-7	6.33188E-5	4.16680E-5	4.11237E-5
128	2.88781E-5	1.15844E-6	2.83207E-7	3.66429E-5	1.09870E-5	1.04145E-5

4. Some Lemmas

In order to derive the error estimations, we need some notations and lemmas. Let B be a Banach space. Define $C(0,T;B)=\{v|v:[0,T]\longmapsto B$ is strongly continuous $\}$, equipped with the norm $||v||_{C(0,T;B)}=\max\limits_{0\le t\le T}||v(t)||_B$. Furthermore

$$C^m(0,T;B)=\Big\{v\Big|\frac{\partial^k v}{\partial t^k}\in C(0,T;B),0\le k\le m\Big\}$$

and

$$||v||_{C^m(0,T;B)}=\max_{0\le k\le m}\Big\|\frac{\partial^k v}{\partial t^k}\Big\|_{C(0,T;B)}.$$

Let $I=(-1,1)$ and $L^2(I;B)=\{v|v:I\longmapsto B$ is strongly measurable and $||v||_{L^2(I;B)}<\infty\}$, equipped with the norm

$$||v||_{L^2(I;B)}=\Big(\int_I||v(z)||_B^2dz\Big)^{\frac{1}{2}}.$$

Furthermore, for any non-negative integer m, we have

$$H^m(I;B)=\Big\{v\Big|\frac{\partial^k v}{\partial z^k}\in L^2(I;B),\ 0\le k\le m\Big\}$$

and

$$||v||_{H^m(I;B)}=\Big(\sum_{k=0}^m\Big\|\frac{\partial^k v}{\partial z^k}\Big\|^2_{L^2(I;B)}\Big)^{\frac{1}{2}}.$$

For non-negative real number s, we define $H^s(I;B)$ by the interpolation between the spaces $H^{[s]}(I;B)$ and $H^{[s+1]}(I;B)$.

Let c be a positive constant independent of τ, N and any function. But its value could be different in different cases. We shall list some lemmas which are the modifications of results in [3,5,7].

Lemma 1. *If $0 \le r \le 1$ and $s > \frac{n}{2} + \frac{r}{2}$, then there exists a positive constant c depending on s such that for any function $v \in H^s(\Omega)$, $||v - P_c v||_r \le cN^{r-s}||v||_s$.*

Proof. We know from section 4 and section 5 of [9] that

(i) There exists a positive conctant c such that for any $v \in H^1(I)$,

$$||P_c v||_{H^1(I)} \le c||v||_{H^1(I)}. \tag{4.1}$$

(ii) If $0 \le r \le 1$ and $s > \frac{1}{2} + \frac{r}{2}$, then there exists a positive constant c depending on s such that for any function $v \in H^s(I)$,

$$||v - P_c v||_{H^r(I)} \le cN^{r-s}||v||_{H^s(I)}. \tag{4.2}$$

(iii) If $0 \le r \le 1$ and $s > 1 + \frac{r}{2}$, then there exists a positive constant c depending on s such that for any function $v \in H^s(I^2)$,

$$||v - P_c v||_{H^r(I^2)} \le cN^{r-s}||v||_{H^s(I^2)}. \tag{4.3}$$

We shall apply the above results and the induction to prove this lemma. Firstly, (4.2) and (4.3) that show the conclusion is true for $n = 1$ and $n = 2$. Now, we assume that the result is true for $n - 1$, i.e., for any real numbers s and r, $0 \le r \le 1$ and $s > \frac{n-1}{2} + \frac{r}{2}$, there exists a positive constant c depending on s such that for any function $v \in H^s(I^{n-1})$,

$$||v - P_c v||_{H^r(I^{n-1})} \le cN^{r-s}||v||_{H^s(I^{n-1})}. \tag{4.4}$$

Let $P_c^{x_j}$ be the one-dimensional interploation operator with respect to the variable x_j and $P_c = P_c^{x_j} \cdot P_c^{\hat{x}_j} = P_c^{\hat{x}_j} \cdot P_c^{x_j}$, where $P_c^{\hat{x}_j} = P_c^{x_1} \cdot P_c^{x_2} \cdot \dots \cdot P_c^{x_{j-1}} \cdot P_c^{x_{j+1}} \cdot \dots \cdot P_c^{x_n}$. We first deal with the case with $r = 0$. Let ϑ be the identity operator. Then

$$\begin{aligned} ||v - P_c v||_{L^2(I^n)} \le & ||v - P_c^{x_j} v||_{L^2(I;L^2(I^{n-1}))} + ||v - P_c^{\hat{x}_j} v||_{L^2(I;L^2(I^{n-1}))} \\ & + ||(\vartheta - P_c^{x_j}) \cdot (\vartheta - P_c^{\hat{x}_j}) v||_{L^2(I;L^2(I^{n-1}))}. \end{aligned}$$

Let $s_1 = \frac{1}{n} s$ and $s_2 = \frac{n-1}{n} s$. Since $s > \frac{n}{2}$, we have $s_1 > \frac{1}{2}$ and $s_2 > \frac{n-1}{2}$. By (4.2) and (4.4),

$$\begin{aligned} ||v - P_c v||_{L^2(I^n)} \le & \, c(N^{-s}||v||_{H^s(I;L^2(I^{n-1}))} + N^{-s}||v||_{L^2(I;H^s(I^{n-1}))} \\ & + N^{-s_1}||(\vartheta - P_c^{\hat{x}_j}) v||_{H^{s_1}(I;L^2(I^{n-1}))}) \\ \le & \, c(N^{-s}||v||_{H^s(I;L^2(I^{n-1}))} + N^{-s}||v||_{L^2(I;H^s(I^{n-1}))} \\ & + N^{-s}||v||_{H^{s_1}(I;H^{s_2}(I^{n-1}))}). \end{aligned}$$

Since $H^s(I^n) \hookrightarrow H^s(I;L^2(I^{n-1}))$, $H^s(I^n) \hookrightarrow L^2(I;H^s(I^{n-1}))$, and $H^s(I^n) \hookrightarrow H^{s_1}(I;H^{s_2}(I^{n-1}))$, we obtain $||v-P_c v||_{L^2(I^n)} \le cN^{-s}||v||_{H^s(I^n)}$. We next consider the case with $r=1$. Using (4.1), (4.2), (4.4) and embedding theory, we have that for $1 \le j \le n$,

$$
\begin{aligned}
\Big\|\frac{\partial}{\partial x_j}(v-P_c v)\Big\|_{L^2(I^n)} \le\, & \Big\|\frac{\partial}{\partial x_j}(v-P_c^{x_j}v)\Big\|_{L^2(I;L^2(I^{n-1}))} \\
& + \Big\|\frac{\partial}{\partial x_j}(P_c^{x_j}\cdot(\vartheta-P_c^{\hat{x}_j})v)||_{L^2(I;L^2(I^{n-1}))} \\
\le\, & cN^{1-s}||v||_{H^s(I;L^2(I^{n-1}))} + c\Big\|\frac{\partial}{\partial x_j}((\vartheta-P_c^{\hat{x}_j})v)\Big\|_{L^2(I;L^2(I^{n-1}))} \\
\le\, & cN^{1-s}||v||_{H^s(I;L^2(I^{n-1}))} + c||(\vartheta-P_c^{\hat{x}_j})\frac{\partial v}{\partial x_j}||_{L^2(I;L^2(I^{n-1}))} \\
\le\, & cN^{1-s}||v||_{H^s(I;L^2(I^{n-1}))} + cN^{1-s}||v||_{H^1(I;H^{s-1}(I^{n-1}))} \\
\le\, & cN^{1-s}||v||_{H^s(I^n)}.
\end{aligned}
$$

Finally, by an argument for the interpolation between the spaces $L^2(I^n)$ and $H^1(I^n)$, we can obtain the desired result.

Lemma 2. *If $v \in S_N$, then $||v|| \le ||v||_N \le c_N||v||$, $c_N = \Big(2+\frac{1}{N}\Big)^{\frac{n}{2}}$.*

Proof. Let

$$\varphi_{j_l}(x_l) = \Big(\frac{2}{2j_l+1}\Big)^{-\frac{1}{2}} L_{j_l}(x_l), \quad \varphi_j(x) = \prod_{l=1}^{n} \varphi_{j_l}(x_l).$$

Then

$$v(x) = \sum_{|j|\le N} a_j\varphi_j(x), \quad ||v||^2 = \sum_{|j|\le N} a_j^2.$$

We define the discrete inner product in $\mathcal{P}_N(\overline{I}_l)$ as

$$(v,w)_N^{(l)} = \sum_{k=0}^{N} v(x_l^{(k_l)})w(x_l^{(k_l)})\omega_l^{(k_l)}, \quad \forall v,w \in \mathcal{P}_N(\overline{I}_l).$$

By the orthogonality of Legendre polynomials,

$$(\varphi_{j_l},\varphi_{j_l'})_N^{(l)} = \begin{cases} 0, & \text{if } j_l \ne j_l', \\ 1, & \text{if } j_l = j_l' < N, \\ 2+\dfrac{1}{N}, & \text{if } j_l = j_l' = N. \end{cases}$$

We have

$$(\varphi_j,\varphi_{j'})_N = \prod_{l=1}^{n}(\varphi_{j_l},\varphi_{j_l'})_N^{(l)}$$

and so

$$||v||_N^2 = \sum_{|j|<N} a_j^2 + \sum_{|j|=N} a_j^2 \prod_{l=1}^{n}(\varphi_{j_l},\varphi_{j_l'})_N^{(l)}.$$

Since

$$1 \le \prod_{l=1}^{n} (\varphi_{j_l}, \varphi_{j'_l})_N^{(l)} \le \Big(2 + \frac{1}{N}\Big)^n,$$

we obtain the desired result.

Lemma 3. *For all* $v \in S_N$,

$$||v||_{L^\infty} \le a_N^n ||v||, \quad a_N = \Big(\frac{1}{2}(N+1)(N+2)\Big)^{1/2}$$

and $||v||_{L^q,N} \le c_N^{\frac{2}{q}} a_N^{\frac{n(q-2)}{q}} ||v||, \quad q \ge 2.$

Proof. The first conclusion comes from Lemma 2 of [7]. By Lemma 2, for $q \ge 2$,

$$||v||_{L^q,N}^q \le ||v||_{L^\infty,N}^{q-2} ||v||_N^2 \le c_N^2 ||v||_{L^\infty}^{q-2} ||v||^2 \le c_N^2 a_N^{n(q-2)} ||v||^q.$$

Lemma 4. (*Lemma* 3 *of* [7]). *For all* $v \in S_N$,

$$|v|_1 \le q n^{\frac{1}{2}} N^2 ||v||, \quad q = 1 + \frac{1}{2N} \le \frac{3}{2}.$$

Lemma 5. *For all* $v \in V_N$, $||v||_{L^q,N}^q \le c_q^n ||v||_{L^q}^q$, *where* c_q *is a positive constant dependent of* q.

Proof. Let $\tilde{N}$ be a positive integer and $\mathcal{P}_{\tilde{N}}(I)$ be the set of all polynomials of degree $\le \tilde{N}$ on I. Nevai proved the following result (Theorem 9.25 of [10], also see Section 2 of [11]).

Let μ be a Jacobi weight, $1 \le q < \infty$. If $c^* > 1$ is a fixed number and f are an arbitrary, not necessarily integrable Jacobi weight, then for any $v \in \mathcal{P}_{c^*\tilde{N}}(I)$,

$$\sum_{i=1}^{\tilde{N}} |v(\xi_i)|^q f(\xi_i) \rho_i(\mu) \le c_q \int_{-1}^{1} |v(y)|^q f(y) \mu(y) dy, \tag{4.5}$$

where ξ_i and ρ_i are the nodes and the weights of Gauss quadrature with respect to the weight μ on $I = (-1, 1)$.

We shall use the above inequality and the induction to prove this lemma. Firstly, let $n = 1$. As we know, the Legendre polynomial $L_{k_1}(x_1)$ satisfies the differential equation

$$((1 - x_1^2) L'_{k_1}(x_1))' + k_1(k_1 + 1) L_{k_1}(x_1) = 0.$$

Therefore $\{L'_{k_1}(x_1)\}$ is an orthogonoal system with respect to the weight $1 - x_1^2$. This leads to that the interior nodes $x_1^{(k_1)} (0 \le k_1 \le N)$ of a Gauss-Lobatto quadrature with $N + 1$ nodes coincide with the nodes $\xi_{k_1} (1 \le k_1 \le N - 1)$ of a Gauss quadrature with $N - 1$ nodes, i.e., $x_1^{(k_1)} = \xi_{k_1}$, $1 \le k_1 \le N - 1$. Besides the weights are linked by the following equality $\omega_1^{(k_1)} = (1 - \xi_{k_1}^2)^{-1} \rho_{k_1}$, $1 \le k_1 \le N - 1$, where $\omega_1^{(k_1)} (0 \le k_1 \le N)$ are the Gauss-Lobatto weights and $\rho_{k_1} (1 \le k_1 \le N - 1)$ are the Gauss weights. Let $f(x_1) = (1 - x_1^2)^{-1}$ and $\mu(x_1) = 1 - x_1^2$, we have

$$||v||_{L^q,N}^q = \sum_{k_1=0}^{N} |v(x_1^{(k_1)})|^q \omega_1^{(k_1)} = \sum_{k_1=1}^{N-1} |v(x_1^{(k_1)})|^q \omega_1^{(k_1)} = \sum_{k_1=1}^{N-1} |v(\xi_{k_1})|^q (1 - \xi_{k_1}^2)^{-1} \rho_{k_1}(\mu).$$

Thus by (4.5),

$$||v||^q_{L^q,N} \le c_q \int_{-1}^{1} |v(x_1)|^q (1-x_1^2)^{-1}(1-x_1^2)dx_1 = c_q \int_{-1}^{1} |v(x_1)|^q dx_1 = c_q ||v||^q_{L^q}. \quad (4.6)$$

Next, assume that the result is true for $n-1$. Then we have from (4.6) that

$$\begin{aligned}
||v||^q_{L^q,N} &= \sum_{x^{(k)}\in\Omega_N} |v(x^{(k)})|^q \omega^{(k)} \\
&\le \sum_{k_n=0}^{N} c_q^{n-1} \int\cdots\int_{I^{n-1}} |v(x_1,x_2,\cdots,x_{n-1},x_n^{(k_n)})|^q dx_1 dx_2\cdots dx_{n-1}\omega_n^{(k_n)} \\
&= c_q^{n-1}\int\cdots\int_{I^{n-1}} \sum_{k_n=0}^{N} |v(x_1,x_2,\cdots,x_{n-1},x_n^{(k_n)})|^q \omega_n^{(k_n)} dx_1dx_2\cdots dx_{n-1} \\
&\le c_q^n \int\cdots\int_{I^n} |v(x_1,x_2,\cdots,x_{n-1},x_n)|^q dx_1dx_2\cdots dx_{n-1}dx_n = c_q^n ||v||^q_{L^q}.
\end{aligned}$$

Lemma 6. *For all* $v \in H_0^1(\Omega)$, $||v||^2 \le \frac{4}{n\pi^2}|v|_1^2$. *If* $v\in V_N$, *then*

$$||v||_N^2 \le \frac{4e_N}{n\pi^2}||\bigtriangledown v||_N^2, \quad e_N = 2+\frac{1}{N}.$$

Proof. The first conclusion is Lemma 9 of [7]. Let $I_l=(-1,1)$. By Lemma 2 and the first conclusion,

$$\sum_{k_l=0}^{N} |v(x_1,\cdots,x_l^{(k_l)},\cdots,x_n)|^2 \omega_l{}^{(k_l)} \le \frac{4}{\pi^2}\Big(2+\frac{1}{N}\Big)\sum_{k_l=0}^{N}\Big|\frac{\partial v}{\partial x_l}(x_1,\cdots,x_l^{(k_l)},\cdots,x_n)\Big|^2 \omega_l{}^{(k_l)}$$

Hence

$$||v||_N^2 \le \frac{4}{\pi^2}\Big(2+\frac{1}{N}\Big)\Big\|\frac{\partial v}{\partial x_l}\Big\|_N^2$$

which leads to the second conclusion.

Lemma 7. *For all* $v\in C^4(0,T;C(\overline{\Omega}))$,

$$\begin{aligned}
&||\hat{v}(t)-v(t)||_N \le c\tau^2||v||_{C^2(0,T;L^\infty(\Omega))}, \\
&\Big\|v_{t\bar{t}}(t)-\frac{\partial^2 v}{\partial t^2}(t)\Big\|_N \le c\tau^2||v||_{C^4(0,T;L^\infty(\Omega))}, \\
&\Big\|v_t(t)-\frac{\partial v}{\partial t}(t)-\frac{\tau}{2}\frac{\partial^2 v}{\partial t^2}(t)\Big\|_N \le c\tau^2||v||_{C^3(0,T;L^\infty(\Omega))}.
\end{aligned}$$

Proof. By the mean value theorem, we have

$$\begin{aligned}
|\hat{v}(t)-v(t)| &= \Big|\frac{1}{2}(v(t+\tau)-v(t))-\frac{1}{2}(v(t)-v(t-\tau))\Big| \\
&= \Big|\frac{\tau}{2}\frac{\partial v}{\partial t}(t_0)-\frac{\tau}{2}\frac{\partial v}{\partial t}(t_1)\Big| = \frac{\tau^2}{2}\Big|\frac{\partial^2 v}{\partial t^2}(t_2)\Big|
\end{aligned}$$

where $t \le t_0 \le t+\tau$, $t-\tau \le t_1 \le t$ and $t-\tau \le t_2 \le t+\tau$. Hence $||\hat{v}(t)-v(t)||_N \le \tau^2 ||v||_{C^2(0,T;L^\infty(\Omega))}$. We can prove the other conclusions similarly.

Lemma 8. *For all* $v \in C^1(0,T;C(\overline{\Omega}))$,

$$||G(P_c v(t)) - \hat{g}(v(t))||_N \le \begin{cases} c\tau ||v||^{\alpha+1}_{C^1(0,T;L^\infty(\Omega))}, & \text{for } 0 \le \alpha < 1, \\ c\tau^2 ||v||^{\alpha+1}_{C^1(0,T;L^\infty(\Omega))}, & \text{for } \alpha \ge 1. \end{cases}$$

Proof. By Taylor's expansion,

$$g(\sigma v(x,t+\tau)+(1-\sigma)v(x,t-\tau)) = g(v(x,t-\tau)) + \sigma(v(x,t+\tau)-v(x,t-\tau))$$
$$\cdot \frac{dg}{dz}(\theta(\sigma)v(x,t+\tau)+(1-\theta(\sigma))v(x,t-\tau))$$

where $0 \le \theta(\sigma) \le \sigma$. Thus the first mean value theorem leads to

$$\begin{aligned} G(v(x,t)) = g(v(x,t-\tau)) &+ \frac{1}{2}(v(x,t+\tau)-v(x,t-\tau)) \\ &\cdot \frac{dg}{dz}(\theta_1 v(x,t+\tau)+(1-\theta_1)v(x,t-\tau)), \quad 0 \le \theta_1 \le 1. \end{aligned}$$

Similarly,

$$\begin{aligned} G(v(x,t)) = g(v(x,t+\tau)) &- \frac{1}{2}(v(x,t+\tau)-v(x,t-\tau)) \\ &\cdot \frac{dg}{dz}(\theta_2 v(x,t+\tau)+(1-\theta_2)v(x,t-\tau)), \quad 0 \le \theta_2 \le 1. \end{aligned}$$

Moreover, we have

$$\frac{dg}{dz}(z) = (\alpha+1)|z|^\alpha$$

and

$$|v(x,t+\tau)-v(x,t-\tau)| \le 2\tau \Big|\frac{\partial v}{\partial t}(x,t_0)\Big| \quad t-\tau \le t_0 \le t+\tau. \tag{4.7}$$

Also we know that $G(P_c v(x,t)) = G(v(x,t))$ for all $x \in \Omega_N$. Therefore

$$||G(P_c v(t)) - \hat{g}(v(t))||_N \le c\tau ||v||^\alpha_{C(0,T;L^\infty(\Omega))} \Big\|\frac{\partial v}{\partial t}\Big\|_{C(0,T;L^\infty(\Omega))} \le c\tau ||v||^{\alpha+1}_{C^1(0,T;L^\infty(\Omega))}.$$

If $\alpha \ge 1$, then by the expression of remainder term of trapezoidal quadrature, we have

$$|G(v(t)) - \hat{g}(v(t))| = \frac{1}{12}\Big|\frac{d^2 g}{dz^2}(\theta_3 v(x,t+\tau)+(1-\theta_3)v(t-\tau))\Big| \cdot |v(x,t+\tau)-v(x,t-\tau)|^2$$

where $0 \le \theta_3 \le 1$. Moreover,

$$\frac{d^2 g}{dz^2}(z) = \alpha(\alpha+1)|z|^{\alpha-2}z$$

which together with (4.7), yields the desired conclusion for $\alpha \ge 1$.

Lemma 9. *For all* $v, w \in C(0,T;V_N)$, $G(v(x,t)+w(x,t)) = G(v(x,t)) + R(x,t)$, *with*

$$||R(t)||_N^2 \le c(||v||^{2\alpha}_{C(0,T;H^1(\Omega))} + ||w||^{2\alpha}_{C(0,T;H^1(\Omega))})(||w(t+\tau)||_1^2 + ||w(t-\tau)||_1^2).$$

Proof. Let

$$\begin{aligned} V(\sigma) &= \sigma v(x,t+\tau) + (1-\sigma)v(x,t-\tau),\\ W(\sigma) &= \sigma w(x,t+\tau) + (1-\sigma)w(x,t-\tau). \end{aligned}$$

Then by Taylor's expansion and that (see [12])

$$(a_1+a_2)^\alpha \le c(a_1^\alpha + a_2^\alpha), \quad \forall a_1,\ a_2 \ge 0,$$

we have

$$\begin{aligned} |R(x,t)| \le &\int_0^1 |g(V(\sigma)+W(\sigma)) - g(V(\sigma))|d\sigma\\ &= (\alpha+1)\int_0^1 |V(\sigma)+\theta(\sigma)W(\sigma)|^\alpha |W(\sigma)|d\sigma\\ &\le c(|v(x,t+\tau)|^\alpha + |v(x,t-\tau)|^\alpha + |w(x,t+\tau)|^\alpha + |w(x,t-\tau)|^\alpha)\\ &\quad \cdot (|w(x,t+\tau)| + |w(x,t-\tau)|) \end{aligned}$$

where $0 \le \theta(\sigma) \le 1$. Taking $\beta = \max\left(\frac{3}{2}, \frac{n}{2}, \frac{1}{2\alpha}\right)$, we have from Hölder inequality that

$$\begin{aligned} ||R(x,t)||_N^2 \le c(&||v(x,t+\tau)||^{2\alpha}_{L^{2\alpha\beta},N} + ||v(x,t-\tau)||^{2\alpha}_{L^{2\alpha\beta},N}\\ &+ ||w(x,t+\tau)||^{2\alpha}_{L^{2\alpha\beta},N} + ||w(x,t-\tau)||^{2\alpha}_{L^{2\alpha\beta},N})\\ &\cdot (||w(x,t+\tau)||^2_{L^{\frac{2\beta}{\beta-1}},N} + ||w(x,t-\tau)||^2_{L^{\frac{2\beta}{\beta-1}},N}) \end{aligned}$$

Since $H^1(\Omega) \hookrightarrow L^{2\alpha\beta}(\Omega)$ and $H^1(\Omega) \hookrightarrow L^{\frac{2\beta}{\beta-1}}(\Omega)$, we complete the proof by Lemma 5.

We now consider a special case, i.e.,

$$\begin{cases} 1 \le \alpha \le 2, & \text{for } n = 1,\\ 1 \le \alpha < 2, & \text{for } n = 2,\\ \alpha = 1, & \text{for } n = 3. \end{cases} \tag{4.8}$$

In this case, we can improve the result of the previous lemma.

Lemma 10. *If* α *satisfies* (4.8), *then for all* $v, w \in C(0,T;V_N)$, *we have*

$$G(v(x,t)+w(x,t)) = G(v(x,t)) + G(w(x,t)) + R(x,t)$$

with

$$||R(t)||_N^2 \le d(v)(||w(t+\tau)||_1^2 + ||w(t-\tau)||_1^2 + ||w(t+\tau)||^p_{L^p,N} + ||w(t-\tau)||^p_{L^p,N})$$

where $d(v)$ is a positive constant depending on α and $||v||_{C(0,T;H^1(\Omega))}$.

Proof. Let $V(\sigma)$ and $W(\sigma)$ be the same as in the proof of lemma 8. Then by Taylor's expansion

$$|V(\sigma)+W(\sigma)|^\alpha = |V(\sigma)|^\alpha + \alpha|V(\sigma)+\theta_1 W(\sigma)|^{\alpha-2}(V(\sigma)+\theta_1 W(\sigma))W(\sigma),$$
$$|V(\sigma)+W(\sigma)|^\alpha = |W(\sigma)|^\alpha + \alpha|W(\sigma)+\theta_2 V(\sigma)|^{\alpha-2}(W(\sigma)+\theta_2 V(\sigma))V(\sigma)$$

where $0 \le \theta_1,\, \theta_2 \le 1$. Hence $g(V(\sigma)+W(\sigma)) = g(V(\sigma)) + g(W(\sigma)) + R(\sigma)$, where

$$\begin{aligned}|R(\sigma)| \le & c(|v(x,t+\tau)|^\alpha + |v(x,t-\tau)|^\alpha)(|w(x,t+\tau)| + |w(x,t-\tau)|)\\ & + c(|w(x,t+\tau)|^\alpha + |w(x,t-\tau)|^\alpha)(|v(x,t+\tau)| + |v(x,t-\tau)|).\end{aligned}$$

By taking $\beta = \max\left(\frac{3}{2}, \frac{n}{2}, \frac{1}{2\alpha}\right)$, we have from Hölder inequality and Lemma 5 that

$$\begin{aligned}||\, |v(t+\tau)|^\alpha w(t+\tau)||_N^2 &\le ||v(t+\tau)||^{2\alpha}_{L^{2\alpha\beta},N} ||w(t+\tau)||^2_{L^{\frac{2\beta}{\beta-1}},N}\\ &\le ||v(t+\tau)||_1^{2\alpha} ||w(t+\tau)||_1^2.\end{aligned}$$

We can estimate the term $||\, |v(t+\tau)|^\alpha w(t-\tau)||_N$ similarly, etc. Next we consider the norm $||\, |w(t+\tau)|^\alpha v(t+\tau)||_N$. If $n=1$ or $n=2$, then $H^1(\Omega) \hookrightarrow L^{\frac{2(\alpha+2)}{2-\alpha}}(\Omega)$. Thus Hölder inequality and Lemma 5 lead to

$$\begin{aligned}||\, |w(t+\tau)|^\alpha v(t+\tau)||_N^2 &\le ||v(t+\tau)||^2_{L^{\frac{2(\alpha+2)}{2-\alpha}},N} ||w(t+\tau)||^{2\alpha}_{L^{\alpha+2},N}\\ &\le ||v(t+\tau)||_1^2 ||w(t+\tau)||^{2\alpha}_{L^p,N}.\end{aligned}$$

Note that (see [12]) for $q, q' \ge 1$ satisfying $\frac{1}{q} + \frac{1}{q'} = 1$,

$$a_1 a_2 \le \frac{a_1^q}{q} + \frac{a_2^{q'}}{q'}, \quad \forall a_1, a_2 \ge 0. \tag{4.9}$$

Hence we obtain from Lemma 5 that

$$\begin{aligned}||w(t+\tau)||^{2\alpha}_{L^p,N} &\le \frac{2-\alpha}{\alpha}||w(t+\tau)||^2_{L^p,N} + \frac{2(\alpha-1)}{\alpha}||w(t+\tau)||^p_{L^p,N}\\ &\le c(||w(t+\tau)||_1^2 + ||w(t+\tau)||^p_{L^p,N})\end{aligned}$$

which leads to the conclusion for $n=1,2$ and $1 \le \alpha < 2$. If $n=1$ and $\alpha=2$, we can prove the conclusion directly. We can also obtain the same result for $n=3$ and $\alpha=1$.

Lemma 11. (*Lemma* 4.16 *of* [13]). *Assume that*
(i) *$Q(t)$ and $\rho(t)$ are non-negative functions defined on S_τ, and $\rho(t)$ is non-decreasing in t;*
(ii) *M is a non-negative constant;*
(iii) *$Q(0) \le \rho(0)$ and for $t \in S_\tau$,*

$$Q(t) \le \rho(t) + M \sum_{t' \le t-\tau} Q(t').$$

Then for all $t \in S_\tau$,

$$Q(t) \le \rho(t) e^{Mt}.$$

5. The Analysis of Generalized Stability

Firstly we derive a priori estimation for the approximate solution of (2.7). Assume $\tau N^2 = r < \infty$. By the conservation (2.8), we need only to bound the initial values $E^*(u,\tau)$ and $2\tau \sum\limits_{t' \in S_\tau, t' \le t-\tau} (\hat{f}(t'), u_{\hat{t}}(t'))_N$. By Lemma 2 and Lemma 4, we have

$$||u(\tau)||_N^2 \le 2c_N^2 ||u_0||^2 + 2c_N^2 \tau^2 ||u_1||^2, \quad || \bigtriangledown u(\tau)||_N^2 \le 2c_N^2 |u_0|_1^2 + \frac{9}{2} c_N^2 n r^2 ||u_1||^2. \tag{5.1}$$

It is not difficult to show that

$$\Big| 2\tau \sum_{\substack{t' \in S_\tau \\ t' \le t-\tau}} (\hat{f}(t'), u_{\hat{t}}(t'))_N \Big| \le \tau ||u_{\bar{t}}(t)||_N^2 + 2\tau \sum_{\substack{t' \in S_\tau \\ t' \le t-\tau}} (||u_{\bar{t}}(t')||_N^2 + ||\hat{f}(t')||_N^2) \tag{5.2}$$

and

$$\frac{1}{p} ||u(\tau)||_{L^p,N}^p \le \frac{2^{p-1}}{p} (||u_0||_{L^p,N}^p + \tau^p ||u_1||_{L^p,N}^p). \tag{5.3}$$

Then we have from (2.8) that

$$\begin{aligned}(1-\tau)||u_{\bar{t}}(t)||_N^2 &+ \frac{1}{2}(|| \bigtriangledown u(t)||_N^2 + || \bigtriangledown u(t-\tau)||_N^2) + \frac{b}{2}(||u(t)||_N^2 + ||u(t-\tau)||_N^2) \\ &+ \frac{1}{p}(||u(t)||_{L^p,N}^p + ||u(t-\tau)||_{L^p,N}^p) \\ &\le c(||u_0||_1^2 + ||u_0||_{L^p,N}^p + ||u_1||^2 + \tau^p ||u_1||_{L^p,N}^p) \\ &+ 2\tau \sum_{\substack{t' \in S_\tau \\ t' \le t-\tau}} (||u_{\bar{t}}(t')||_N^2 + ||\hat{f}(t')||_N^2).\end{aligned} \tag{5.4}$$

On the other hand,

$$(u(t))^2 = \Big(u_0 + \tau \sum_{\substack{t' \in S_\tau \\ t' \le t}} u_{\bar{t}}(t'))^2 \le 2u_0^2 + 2t\tau \sum_{\substack{t' \in S_\tau \\ t' \le t}} u_{\bar{t}}^2(t')$$

which implies

$$||u(t)||_N^2 \le 2||u_0||_N^2 + 2t\tau \sum_{\substack{t' \in S_\tau \\ t' \le t}} ||u_{\bar{t}}(t')||_N^2.$$

Hence

$$\Big| \frac{b}{2}(||u(t)||_N^2 + ||u(t-\tau)||_N^2) \Big| \le |b| t\tau ||u_{\bar{t}}(t)||_N^2 + 2|b|(||u_0||_N^2 + t\tau \sum_{\substack{t' \in S_\tau \\ t' \le t-\tau}} ||u_{\bar{t}}(t')||_N^2). \tag{5.5}$$

Let $b_0 > 0$ and

$$\varphi(b) = \begin{cases} 0, & \text{for } b > -\dfrac{n\pi^2}{8e_N}, \\ |b| - \dfrac{n\pi^2}{4e_N}\Big(\dfrac{1}{2} - b_0\Big), & \text{for } b \le -\dfrac{n\pi^2}{8e_N}, \end{cases}$$

$$\psi(b) = \begin{cases} \dfrac{1}{2}, & \text{for } b \ge 0, \\ \dfrac{1}{2} - \dfrac{4e_N|b|}{n\pi^2}, & \text{for } -\dfrac{n\pi^2}{8e_N} < b < 0, \\ b_0, & \text{otherwise}, \end{cases}$$

$$\chi(b) = \begin{cases} \dfrac{b}{2}, & \text{for } b \ge 0, \\ 0, & \text{otherwise}. \end{cases}$$

By Lemma 2, Lemma 6 and the above functions, we have from (5.4) that

$$\begin{aligned} &(1 - \tau - \tau t\varphi(b))||u_{\bar{t}}(t)||_N^2 + \psi(b)(||\bigtriangledown u(t)||_N^2 + ||\bigtriangledown u(t-\tau)||_N^2) \\ &+ \chi(b)(||u(t)||_N^2 + ||u(t-\tau)||_N^2) + \frac{1}{p}(||u(t)||_{L^p,N}^p + ||u(t-\tau)||_{L^p,N}^p) \\ \le& \rho(u_0, u_1, f) + 2\tau(1 + t\varphi(b)) \sum_{\substack{t' \in S_\tau \\ t' \le t-\tau}} ||u_{\bar{t}}(t')||_N^2 \end{aligned} \tag{5.6}$$

where

$$\rho(u_0, u_1, f) = c(||u_0||_1^2 + ||u_0||_{L^p,N}^p + ||u_1||^2 + \tau^p||u_1||_{L^p,N}^p) + c\tau \sum_{\substack{t' \in S_\tau \\ t' \le t}} ||f(t')||_N^2.$$

Let τ be sufficiently small and define

$$E^{**}(u,t) = ||u_{\bar{t}}(t)||_N^2 + ||\bigtriangledown u(t)||_N^2 + ||u(t)||_{L^p,N}^p.$$

By applying Lemma 11 to (5.6), we get

$$E^{**}(u,t) \le c\rho(u_0, u_1, f)e^{ct}. \tag{5.7}$$

Remark 1. Indeed we have from (1.2) and Lemma 5 that $||u_0||_{L^p,N}^p \le c||u_0||_1^2$. Also by Lemma 3 and $\tau = O\Big(\dfrac{1}{N^2}\Big)$, $\tau^p||u_1||_{L^p,N}^p \le c\tau^p N^{n(p-2)}||u_1||^p \le c||u_1||^p$. Hence $\rho(u_0, u_1, f)$ only depends on $||u_0||_1, ||u_1||$ and $\sum\limits_{t' \in S_\tau, t' \le t} ||f(t')||_N^2$. On the other hand, if $b \ge 0$, then we do not use (5.1). Also $\tau^p||u_1||_{L^p,N}^p \le c||u_1||^p$ when $\tau = O(N^{\frac{2n-np}{p}})$.

Now we consider the generalized stability of (2.7). Suppose that u_0, u_1 and $P_c f$ have the errors $\tilde{u_0}, \tilde{u_1}$ and $\tilde{f}$ respectively which induce the error of u denoted by $\tilde{u}$. Then they satisfy the following error equation

$$\begin{cases} (\tilde{u}_{t\bar{t}}(t) + b\hat{\tilde{u}}(t) + \tilde{G}(u(t)), v)_N + (\bigtriangledown\hat{\tilde{u}}(t), \bigtriangledown v)_N = (\hat{\tilde{f}}(t), v)_N, \quad \forall v \in V_N, \\ \tilde{u}_t(0) = \tilde{u}_1, \\ \tilde{u}(0) = \tilde{u}_0 \end{cases} \tag{5.8}$$

where $\tilde{G}(x,t) = G(u(x,t)+\tilde{u}(x,t)) - G(u(x,t))$. By taking $v = 2\tilde{u}_{\hat{t}}$ in the first formula of (5.8), we have from (2.1) and (2.2) that

$$(||\tilde{u}_{\bar{t}}(t)||_N^2)_t + (||\bigtriangledown\tilde{u}(t)||_N^2)_{\hat{t}} + b(||\tilde{u}(t)||_N^2)_{\hat{t}} + 2(\tilde{G}(t),\ \tilde{u}_{\hat{t}}(t))_N = 2(\hat{\tilde{f}}(t), \tilde{u}_{\hat{t}}(t))_N. \quad (5.9)$$

Let $d(u)$ and $d(\tilde{u})$ be two positive constants depending only on $||u||_{C(0,T;H^1(\Omega))}$ and $||\tilde{u}||_{C(0,T;H^1(\Omega))}$ respectively. Then we get from Lemma 2 and Lemma 9 that

$$\begin{aligned}|2(\tilde{G}(t), \tilde{u}_{\hat{t}}(t))_N| &\le ||\tilde{u}_t(t)||_N^2 + ||\tilde{u}_{\bar{t}}(t)||_N^2 + (d(u)+d(\tilde{u}))(||\tilde{u}(t+\tau)||_1^2 + ||\tilde{u}(t-\tau)||_1^2)\\ &\le ||\tilde{u}_t(t)||_N^2 + ||\tilde{u}_{\bar{t}}(t)||_N^2 + (d(u)+d(\tilde{u}))(||\bigtriangledown\tilde{u}(t+\tau)||_N^2 + ||\bigtriangledown\tilde{u}(t-\tau)||_N^2).\end{aligned}$$

By an argument similar to the derivation of (5.6), we obtain

$$\begin{aligned}&(1-2\tau-2\tau t\varphi(b))||\tilde{u}_{\bar{t}}(t)||_N^2 + (\psi(b)-\tau d(u)-\tau d(\tilde{u}))(||\bigtriangledown\tilde{u}(t)||_N^2 + ||\bigtriangledown u(t-\tau)||_N^2)\\ &+\chi(b)(||\tilde{u}(t)||_N^2 + ||\tilde{u}(t-\tau)||_N^2)\\ \le&\tilde{\rho}_1(\tilde{u}_0,\tilde{u}_1,\tilde{f}) + \tau(2+2t\varphi(b)+d(u)+d(\tilde{u}))\sum_{\substack{t'\in S_\tau\\ t'\le t-\tau}}(||\tilde{u}_{\bar{t}}(t')||_N^2 + ||\bigtriangledown\tilde{u}(t)||_N^2)\end{aligned} \quad (5.10)$$

where

$$\tilde{\rho}_1(\tilde{u}_0,\tilde{u}_1,\tilde{f}) = (c+\tau d(u_0)+\tau d(\tilde{u_0}))(||\tilde{u}_0||_1^2 + ||\tilde{u}_1||^2) + c\tau\sum_{\substack{t'\in S_\tau\\ t'\le t}}||\tilde{f}(t')||_N^2.$$

On the other hand, by a priori estimation (5.7), we have $||u||_{C(0,T:H^1(\Omega))} \le c\rho(u_0,u_1,f)e^{cT}$. Similarly $||u+\tilde{u}||_{C(0,T:H^1(\Omega))} \le c\rho(u_0+\tilde{u_0}, u_1+\tilde{u_1}, f+\tilde{f})e^{cT}$. Thus if $\tilde{\rho}_1 \le M_0$ for certain $M_0 > 0$, then we conclude that $\rho(u_0+\tilde{u_0}, u_1+\tilde{u_1}, f+\tilde{f})$, and furthermore $d(\tilde{u})$ are bounded above by a positive constant depending only on $\rho(u_0,u_1,f)$ and M_0. Consequently if τ is sufficiently small, then (5.10) implies that

$$||\tilde{u}_{\bar{t}}(t)||_N^2 + ||\bigtriangledown\tilde{u}(t)||_N^2 \le M_1\rho_1(\tilde{u}_0,\tilde{u}_1,\tilde{f})e^{M_2t}.$$

Theorem 1. *Let* (1.2) *hold,* $\tau N^2 < r$ *for* $b<0$ *and* $\tau = O(N^{\frac{2n-np}{p}})$ *for* $b\ge 0$. *If* $\tilde{\rho}_1(\tilde{u}_0,\tilde{u}_1,\tilde{f}) \le M_0$, *then for suitably large* N *and all* $t\in S_\tau$, $||\tilde{u}_{\bar{t}}(t)||_N^2 + ||\bigtriangledown\tilde{u}(t)||_N^2 \le M_1\tilde{\rho}_1(\tilde{u}_0,\tilde{u}_1,\tilde{f})e^{M_2t}$, M_1 *and* M_2 *being positive constants depending only on* $||u_0||_1$, $||u_1||$, $||f||_{C(0,T;L^2(\Omega))}$ *and* M_0.

Remark 2. Theorem 1 shows that scheme (2.7) is not stable in the sense of Lax (see [14]). But if the errors of data are bounded, then the error of numerical solution is still controlled by the errors of data. Indeed, it means that (2.7) is of generalized stability with the index $s\le 0$ (see[15]).

Next we consider the case with (4.8). We have from Lemma 10 that $\tilde{G}(x,t) = G(\tilde{u}(x,t)) + \tilde{R}(x,t)$, with

$$\begin{aligned}||\tilde{R}(t)||_N^2 &\le d(u)(||\tilde{u}(t+\tau)||_1^2 + ||\tilde{u}(t-\tau)||_1^2 + ||\tilde{u}(t+\tau)||_{L^p,N}^p + ||\tilde{u}(t-\tau)||_{L^p,N}^p)\\ &\le d(u)(||\bigtriangledown\tilde{u}(t+\tau)||_N^2 + ||\bigtriangledown\tilde{u}(t-\tau)||_N^2 + ||\tilde{u}(t+\tau)||_{L^p,N}^p + ||\tilde{u}(t-\tau)||_{L^p,N}^p).\end{aligned}$$

By taking the inner product with $2\tilde{u}_{\hat{t}}(t)$ in the first equation of (5.8). We get

$$\begin{aligned}&(\|\tilde{u}_{\bar{t}}(t)\|_N^2)_t + (\| \bigtriangledown \tilde{u}(t)\|_N^2)_{\hat{t}} + b(\|\tilde{u}(t)\|_N^2)_{\hat{t}} + \frac{2}{p}(\|\tilde{u}(t)\|_{L^p,N}^p)_{\hat{t}} + 2(\tilde{R}(t), \tilde{u}_{\hat{t}}(t))_N \\ =&2(\hat{\tilde{f}}(t), \tilde{u}_{\hat{t}}(t))_N. \end{aligned} \tag{5.11}$$

Besides, (5.3) implies

$$\frac{1}{p}\|\tilde{u}(\tau)\|_{L^p,N}^p \leq \frac{2^{p-1}}{p}(\|\tilde{u}_0\|_{L^p,N}^p + \tau^p\|\tilde{u}_1\|_{L^p,N}^p).$$

By an argument similar to the derivation of (5.6), we obtain that

$$\begin{aligned}&(1-2\tau t\varphi(b))\|\tilde{u}_{\bar{t}}(t)\|_N^2 + (\psi(b) - \tau d(u))\| \bigtriangledown \tilde{u}(t)\|_N^2 \\ &+ \chi(b)\|\tilde{u}(t)\|_N^2 + \Big(\frac{1}{p} - \tau d(u)\Big)\|\tilde{u}(t)\|_{L^p,N}^p \\ \leq&\tilde{\rho}_2(\tilde{u}_0, \tilde{u}_1, \tilde{f}) + \tau(c + d(u)) \sum_{\substack{t' \in S_\tau \\ t' \leq t-\tau}} (\|\tilde{u}_{\bar{t}}(t')\|_N^2 + \| \bigtriangledown \tilde{u}(t')\|_N^2 + \|\tilde{u}(t')\|_{L^p,N}^p)\end{aligned} \tag{5.12}$$

where

$$\begin{aligned}\tilde{\rho}_2(\tilde{u}_0, \tilde{u}_1, \tilde{f}) =&(c + \tau d(u_0))(\|\tilde{u}_0\|_1^2 + \|\tilde{u}_1\|^2) + \Big(\frac{1}{p} + \frac{2^{p-1}}{p} + \tau d(u_0)\Big)\|\tilde{u}_0\|_{L^p,N}^p \\ &+ \frac{2^{p-1}}{p}\tau^p\|\tilde{u}_1\|_{L^p,N}^p + 2\tau \sum_{\substack{t' \in S_\tau \\ t' \leq t}} \|\tilde{f}(t')\|_N^2.\end{aligned}$$

If N is suitably large, then we can verify the boundedness of $d(u)$ as before. Thus by applying Lemma 11 to (5.12), we get

$$\|\tilde{u}_{\bar{t}}(t)\|_N^2 + \| \bigtriangledown \tilde{u}(t)\|_N^2 + \|\tilde{u}(t)\|_{L^p,N}^p \leq M_3\rho_2(\tilde{u}_0, \tilde{u}_1, \tilde{f})e^{M_4 t}.$$

Theorem 2. *Let $\tau N^2 < r$ and (4.8) hold, Then for suitably large N and all $t \in S_\tau$,*

$$\|\tilde{u}_{\bar{t}}(t)\|_N^2 + \| \bigtriangledown \tilde{u}(t)\|_N^2 + \|\tilde{u}(t)\|_{L^p,N}^p \leq M_3\tilde{\rho}_2(\tilde{u}_0, \tilde{u}_1, \tilde{f})e^{M_4 t}$$

where M_3 and M_4 are positive constants depending only on b, α and $\|u\|_{C(0,T;H^1(\Omega))}$.

Remark 3. If the conditions of Theorem 2 are fulfilled, then scheme (2.7) is of generalized stability with the index $s = -\infty$ (see [15]). It means that there is no restriction on the errors of data and so (2.7) is stabler.

6. The Convergence

Setting $w = P_c U$, we get from (1.1) that

$$\begin{cases} (w_{t\bar{t}}(t) + b\hat{w}(t) + G(w(t)), v)_N + (\bigtriangledown\hat{w}(t), \bigtriangledown v)_N \\ = (\hat{f}(t) + \sum_{i=1}^{3} f_i(t), v)_N, & \forall v \in V_N,\ t \in S_\tau, \\ w_t(0) = P_c U_1 + \frac{\tau}{2}P_c(\triangle U_0 - bU_0 - g(U_0) + f(0)) + f_4, \\ w(0) = P_c U_0 \end{cases} \tag{6.1}$$

where

$$\begin{cases} f_1(t) = w_{t\bar{t}}(t) - \dfrac{\partial^2 \hat{w}}{\partial t^2}(t), \\ f_2(t) = P_c[G(w(t)) - \hat{g}(U(t))], \\ f_3(t) = P_c \triangle \hat{U}(t) - \triangle P_c \hat{U}(t), \\ f_4 = w_t(0) - \dfrac{\partial w}{\partial t}(0) - \dfrac{\tau}{2}\dfrac{\partial^2 w}{\partial t^2}(0). \end{cases}$$

Setting $\tilde{U} = u - w$, we get from (2.7) and (6.1) that

$$\begin{cases} (\tilde{U}_{t\bar{t}}(t) + b\hat{\tilde{U}}(t) + G(w(t) + \tilde{U}(t)) - G(w(t)), v)_N + (\bigtriangledown \hat{\tilde{U}}(t), \bigtriangledown v)_N \\ = -\Big(\displaystyle\sum_{i=1}^{3} f_i(t), v)_N, \quad \forall v \in V_N,\ t \in S_\tau, \\ \tilde{U}_t(0) = -f_4, \\ \tilde{U}(0) = 0. \end{cases} \tag{6.2}$$

By taking $v = 2\tilde{U}_{\hat{t}}$ in the first formula of (6.2), we have from (2.1) and (2.2) that

$$(||\tilde{U}_{\bar{t}}(t)||_N^2)_t + (|| \bigtriangledown \tilde{U}(t)||_N^2 + b||\tilde{U}(t)||_N^2)_{\hat{t}} + 2(G(w(t) + \tilde{U}(t)) - G(w(t)), \tilde{U}_{\hat{t}}(t))_N = -2\sum_{i=1}^{3}(f_i(t), \tilde{U}_{\hat{t}}(t))_N.$$

Evidently we can get the results similar to Theorem 1 and Theorem 2. But $||\tilde{u}_{\bar{t}}(t)||_N$, $|| \bigtriangledown \tilde{u}(t)||_N$ and $||\tilde{u}(t)||_{L^p,N}^p$ are replaced by $||\tilde{U}_{\bar{t}}(t)||_N$, $|| \bigtriangledown \tilde{U}(t)||_N$ and $||\tilde{U}(t)||_{L^p,N}^p$ respectively, while $\tilde{\rho}_1(\tilde{u}_0, \tilde{u}_1, \tilde{f})$ and $\tilde{\rho}_2(\tilde{u}_0, \tilde{u}_1, \tilde{f})$ become

$$\rho_1^*(t) = (c + \tau d(w(0)))||\tilde{U}_t(0)||^2 + c\tau \sum_{i=1}^{3} \sum_{\substack{t' \in S_\tau \\ t' \le t-\tau}} ||f_i(t')||_N^2, \tag{6.3}$$

and

$$\rho_2^*(t) = (c + \tau d(w(0)))||\tilde{U}_t(0)||^2 + \frac{2^{p-1}}{p}\tau^p||\tilde{U}_t(0)||_{L^p,N}^p + c\tau \sum_{i=1}^{3} \sum_{\substack{t' \in S_\tau \\ t' \le t-\tau}} ||f_i(t')||_N^2. \tag{6.4}$$

For the convergence, we have to estimate $\rho_1^*(t)$ and $\rho_2^*(t)$. We have from Lemma 7 that

$$||f_1(t)||_N^2 \le c\tau^4||U||_{C^4(0,T;L^\infty(\Omega))}^2.$$

By Lemma 8, we know that if $U \in C^1(0, T; C(\overline{\Omega}))$, then

$$||f_2(t)||_N^2 \le c\tau^{\beta(\alpha)}||U||_{C^1(0,T;L^\infty(\Omega))}^{2\alpha+2}$$

where $\beta(\alpha) = 4$ for $\alpha \ge 1$ and $\beta(\alpha) = 2$ for $0 \le \alpha \le 1$. On the other hand, the inverse inequality, Lemma 1 and Lemma 2 lead to that for $s > \frac{n}{2} + \frac{1}{2}$

$$||f_3(t)||_N^2 = ||P_c \triangle \hat{U}(t) - \triangle P_c\hat{U}(t)||_N^2 \le c||P_c \triangle \hat{U}(t) - \triangle P_c\hat{U}(t)||^2$$

$$\le c||P_c \triangle \hat{U}(t) - \triangle\hat{U}(t)||^2 + c|| \triangle \hat{U}(t) - \triangle P_c\hat{U}(t)||^2$$
$$\le cN^{-2s}|| \triangle \hat{U}(t)||_s + cN^4|| \bigtriangledown \hat{U}(t) - \bigtriangledown P_c\hat{U}(t)||^2$$
$$\le cN^{-2s}||\hat{U}(t)||^2_{s+2} + cN^{-2s}||\hat{U}(t)||^2_{s+3} \le cN^{-2s}||U||^2_{C(0,T;H^{s+3}(\Omega))}.$$

We obtain from Lemma 7 that

$$||\tilde{U}_t(0)||^2 \le ||\tilde{U}_t(0)||^2_N = ||f_4(t)||^2_N \le c\tau^4||U||^2_{C^3(0,T;L^\infty(\Omega))}.$$

So we can get the following result.

Theorem 3. *Let the conditions of Theorem* 1 *hold. We conclude that if* $s > \dfrac{n}{2} + \dfrac{1}{2}$ *and* $U \in C(0,T;H^1_0(\Omega) \cap H^{s+3}(\Omega)) \cap C^4(0,T;C(\Omega))$, *then for all* $t \in S_\tau$,

$$||\tilde{U}_{\bar{t}}(t)||^2_N + || \bigtriangledown \tilde{U}(t)||^2_N \le M_1^*(\tau^{\beta(\alpha)} + N^{-2s})$$

where M_1^* *is a positive constant depending only on the norms* $||U||_{C(0,T;H^{s+3}(\Omega))}$ *and* $||U||_{C^4(0,T;L^\infty(\Omega))}$.

We now consider the special case with (4.8). In this case, by Lemma 3 and Lemma 7, $\tau^p||\tilde{U}_t(0)||^p_{L^p,N} \le c\tau^pN^{n(p-2)}||\tilde{U}_t(0)||^p \le c\tau^{3p}N^{n(p-2)}||U||^p_{C^3(0,T;L^\infty(\Omega))}$.

Theorem 4. *Let the conditions of Theorem* 2 *hold. We conclude that if* $s > \dfrac{n}{2} + \dfrac{1}{2}$ *and* $U \in C(0,T;H^1_0(\Omega) \cap H^{s+3}(\Omega)) \cap C^4(0,T;C(\Omega))$, *then for all* $t \in S_\tau$,

$$||\tilde{U}_{\bar{t}}(t)||^2_N + || \bigtriangledown \tilde{U}(t)||^2_N + ||\tilde{U}(t)||^p_{L^p,N} \le M_2^*(\tau^4 + \tau^{3p}N^{n\alpha} + N^{-2s})$$

where M_2^* *is a positive constant depending only on the norms* $||U||_{C(0,T;H^{s+3}(\Omega))}$ *and* $||U||_{C^4(0,T;L^\infty(\Omega))}$.

If we analyze the generalized stability and the convergence with the negative norm, then we can get better results. The negative norm $||\cdot||_{-1}$ is defined as

$$||v||_{-1} = \sup_{\varphi \in H^1_0(\Omega)} \frac{|(v,\varphi)_N|}{|| \bigtriangledown \varphi||_N}.$$

We also note that (9.7.15 of [16]) for any $v \in H^1_0(\Omega)$ and $s \ge 2$, there exists $v^N \in V_N$ with the same boundary behavior as v, such that

$$||v - v^N||_r \le cN^{r-s}||v||_s, \quad 0 \le r \le 2. \tag{6.5}$$

We can use the above techniques to improve the results. In these cases, the right terms $f_1(t), f_2(t)$ and $f_3(t)$ in (6.1) are replaced by $F_1(t), F_2(t)$ and $F_3(t) + \tilde{F}_3(t)$ respectively, where $F_1(t) = f_1(t), F_2(t) = f_2(t)$, $F_3(t) = P_c \triangle \hat{U}(t) - \triangle\hat{U}^N(t)$ and $\tilde{F}_3(t) = \triangle\hat{U}^N(t) - \triangle P_c\hat{U}(t)$. And so

$$(F_1(t) + F_2(t) + F_3(t) + \tilde{F}_3(t), v)_N = (F_1(t) + F_2(t) + F_3(t), v)_N + (\tilde{F}_3(t), v)_N, \forall v \in V_N.$$

We also note that (Lemma 1.5 of [17])

$$2\tau \sum_{\substack{t' \in S_\tau \\ t' \le t-\tau}} (v(t'), u_{\hat{t}}(t'))_N = (v(t-\tau), u(t))_N + (v(t-2\tau), u(t-\tau))_N - (v(2\tau), u(\tau))_N$$

$$- (v(\tau), u(0))_N - 2\tau \sum_{\substack{t' \in S'_\tau \\ t' \le t-2\tau}} (v_{\hat{t}}(t'), u(t'))_N,$$

where $S'_\tau = S_\tau \backslash \{\tau\}$. Hence we have

$$2\tau \sum_{\substack{t' \in S_\tau \\ t' \le t-\tau}} (\tilde{F}_3(t'), u_{\hat{t}}(t'))_N = (\tilde{F}_3(t-\tau), u(t))_N + (\tilde{F}_3(t-2\tau), u(t-\tau))_N - (\tilde{F}_3(2\tau), u(\tau))_N$$
$$- (\tilde{F}_3(\tau), u(0))_N - 2\tau \sum_{\substack{t' \in S'_\tau \\ t' \le t-2\tau}} (\tilde{F}_{3\hat{t}}(t'), u(t'))_N.$$

Evidently we can get the results similar to Theorem 3 and Theorem 4, but $\rho_1^*(t)$ and $\rho_2^*(t)$ become

$$\begin{aligned} \tilde{\rho}_1^*(t) = {} & (c + \tau d(w(0)))||\tilde{U}_t(0)||^2 + c\tau \sum_{i=1}^{3} \sum_{\substack{t' \in S_\tau \\ t' \le t-\tau}} ||F_i(t')||_N^2 + ||\tilde{F}_3(t-\tau)||_{-1}^2 \\ & + ||\tilde{F}_3(t-2\tau)||_{-1}^2 + ||\tilde{F}_3(2\tau)||_{-1}^2 + ||\tilde{F}_3(\tau)||_{-1}^2 + \sum_{\substack{t' \in S'_\tau \\ t' \le t-2\tau}} ||\tilde{F}_{3\hat{t}}(t')||_{-1}^2, \end{aligned} \tag{6.6}$$

and

$$\begin{aligned} \tilde{\rho}_2^*(t) = {} & (c + \tau d(w(0)))||\tilde{U}_t(0)||^2 + \frac{2^{p-1}}{p}\tau^p ||\tilde{U}_t(0)||_{L^p,N}^p + c\tau \sum_{i=1}^{3} \sum_{\substack{t' \in S_\tau \\ t' \le t-\tau}} ||F_i(t')||_N^2 \\ & + ||\tilde{F}_3(t-\tau)||_{-1}^2 + ||\tilde{F}_3(t-2\tau)||_{-1}^2 + ||\tilde{F}_3(2\tau)||_{-1}^2 + ||\tilde{F}_3(\tau)||_{-1}^2 \\ & + \sum_{\substack{t' \in S'_\tau \\ t' \le t-2\tau}} ||\tilde{F}_{3\hat{t}}(t')||_{-1}^2. \end{aligned} \tag{6.7}$$

If $U \in C(0, T; H_0^1(\Omega) \cap H^{s+2}(\Omega))$, then Lemma 1 and the inequality (6.5) lead to that for $s > \max\left(\frac{n}{2}, 2\right)$,

$$\begin{aligned} ||F_3(t)||_N^2 &= ||P_c \triangle \hat{U}(t) - \triangle \hat{U}^N(t)||_N^2 \le c||P_c \triangle \hat{U}(t) - \triangle \hat{U}^N(t)||^2 \\ &\le c||P_c \triangle \hat{U}(t) - \triangle \hat{U}(t)||^2 + c|| \triangle \hat{U}(t) - \triangle \hat{U}^N(t)||^2 \\ &\le cN^{-2s}|| \triangle \hat{U}(t)||_s + cN^{-2s}||\hat{U}(t)||_{s+2}^2 \le cN^{-2s}||\hat{U}(t)||_{s+2}^2 \\ &\le cN^{-2s}||U||_{C(0,T;H^{s+2}(\Omega))}^2. \end{aligned}$$

If $U \in C(0, T; H_0^1(\Omega) \cap H^{s+1}(\Omega))$, then Lemma 1 and (6.5) lead to that for $s > \max\left(\frac{n}{2} + \frac{1}{2}, 2\right)$,

$$\begin{aligned} |(\tilde{F}_3(t'), v)_N| &= |(\bigtriangledown(\hat{U}^N(t') - P_c\hat{U}(t')), \bigtriangledown v)_N| \le c||\hat{U}^N(t') - P_c\hat{U}(t')||_1|| \bigtriangledown v||_N \\ &\le c(||\hat{U}^N(t') - \hat{U}(t')||_1 + ||\hat{U}(t') - P_c\hat{U}(t')||_1)|| \bigtriangledown v||_N \\ &\le c(N^{-s}||\hat{U}(t')||_{s+1} + N^{-s}||\hat{U}(t')||_{s+1})|| \bigtriangledown v||_N \end{aligned}$$

$$\le cN^{-s}||U||_{C(0,T;H^{s+1}(\Omega))}|| \bigtriangledown v||_N$$

where $t' = t-\tau, t-2\tau, 2\tau$ and τ. Hence, $||\tilde{F}_3(t')||_{-1} \le cN^{-s}||U||_{C(0,T;H^{s+1}(\Omega))}$.

If $U \in C^1(0,T;H_0^1(\Omega)\cap H^{s+1}(\Omega))$, then Lemma 1 and (6.5) lead to that for $s > \max\left(\frac{n}{2}+\frac{1}{2}, 2\right)$,

$$\begin{aligned}|(\tilde{F}_{3\hat{t}}(t'), v)_N| &= |(\bigtriangledown(\hat{U}^N_{\hat{t}}(t') - P_c\hat{U}_{\hat{t}}(t')), \bigtriangledown v)_N| \le c||\hat{U}_{\hat{t}}(t') - P_c\hat{U}_{\hat{t}}(t')||_1|| \bigtriangledown v||_N \\ &\le c(||\hat{U}^N_{\hat{t}}(t') - \hat{U}_{\hat{t}}(t')||_1 + ||\hat{U}_{\hat{t}}(t') - P_c\hat{U}_{\hat{t}}(t')||_1)|| \bigtriangledown v||_N \\ &\le c(N^{-s}||\hat{U}_{\hat{t}}(t')||_{s+1} + N^{-s}||\hat{U}_{\hat{t}}(t')||_{s+1})|| \bigtriangledown v||_N \\ &\le cN^{-s}||U||_{C^1(0,T;H^{s+1}(\Omega))}|| \bigtriangledown v||_N\end{aligned}$$

where $t' \in S'_\tau$ and $t' \le t-2\tau$. Finally, we get $||\tilde{F}_{3\hat{t}}(t')||_{-1} \le cN^{-s}||U||_{C^1(0,T;H^{s+1}(\Omega))}$. So we obtain the following results.

Theorem 5. *Let the conditions of Theorem 1 hold. We conclude that if* $s > \max\left(\frac{n}{2}+\frac{1}{2}, 2\right)$ *and* $U \in C(0,T;H_0^1(\Omega)\cap H^{s+2}(\Omega)) \cap C^1(0,T;H_0^1(\Omega)\cap H^{s+1}(\Omega)) \cap C^4(0,T;C(\bar{\Omega}))$, *then for all* $t \in S_\tau$,

$$||\tilde{U}_{\bar{t}}(t)||^2_N + || \bigtriangledown \tilde{U}(t)||^2_N \le M_1^*(\tau^{\beta(\alpha)} + N^{-2s})$$

where M_1^* *is a positive constant depending only on the norms* $||U||_{C(0,T;H^{s+2}(\Omega))}$, $||U||_{C^1(0,T;H^{s+1}(\Omega))}$ *and* $||U||_{C^4(0,T;L^\infty(\Omega))}$.

Theorem 6. *Let the conditions of Theorem 2 hold. We conclude that if* $s > \max\left(\frac{n}{2}+\frac{1}{2}, 2\right)$ *and* $U \in C(0,T;H_0^1(\Omega)\cap H^{s+2}(\Omega)) \cap C^1(0,T;H_0^1(\Omega)\cap H^{s+1}(\Omega)) \cap C^4(0,T;C(\bar{\Omega}))$, *then for all* $t \in S_\tau$,

$$||\tilde{U}_{\bar{t}}(t)||^2_N + || \bigtriangledown \tilde{U}(t)||^2_N + ||\tilde{U}(t)||^p_{L^p,N} \le M_2^*(\tau^4 + \tau^{3p}N^{n\alpha} + N^{-2s})$$

where M_2^* *is a positive constant depending only on the norms* $||U||_{C(0,T;H^{s+2}(\Omega))}$, $||U||_{C^1(0,T;H^{s+1}(\Omega))}$ *and* $||U||_{C^4(0,T;L^\infty(\Omega))}$.

Remark 4. The above estimations for the convergence rate are not optimal. This is caused by our comparison between $u(t)$ and $P_cU(t)$ in the proof, which generates the terms $P_c(\triangle U(t)) - \triangle(P_cU(t))$, and so decreases the convergence rate. However, if α is an integer, then we can compare $u(t)$ with $\tilde{P}^1_N U(t)$, the H^1-orthogonal projection of $U(t)$ onto V_N, instead. Indeed, let $P^1_N : H^1_0(\Omega) \longmapsto V_N$ be the orthogonal projection, i.e., for any $v \in H_0^1(\Omega)$, $(\bigtriangledown(P^1_N v - v), \bigtriangledown\varphi) = 0$, $\forall\varphi \in V_N$. Furthermore for any $v \in H_0^1(\Omega)$, we define $(\bigtriangledown\tilde{P}^1_N v, \bigtriangledown\varphi)_N = (\bigtriangledown v, \bigtriangledown\varphi)$, $\forall\varphi \in V_N$. Then for any $v \in H_0^1(\Omega)$, we have $(\bigtriangledown\tilde{P}^1_N v, \bigtriangledown\varphi)_N = (\bigtriangledown v, \bigtriangledown\varphi) = (\bigtriangledown P^1_N v, \bigtriangledown\varphi)$, $\forall\varphi \in V_N$. Moreover for $v \in H^s(\Omega)$ and $s \ge 1$,

$$||v - \tilde{P}^1_N v||_r \le cN^{r-s}||v||_s, \quad 0 \le r \le 1. \tag{6.8}$$

By such technique, we can weaken the conditions in Theorem 3 - Theorem 6 and then the optimal error estimations follow.

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