Journal of Computational Mathematics, Vol.15, No.2, 1997, 127–137.

CONVERGENCE OF VORTEX WITH BOUNDARY ELEMENT METHODS*

P.W. Zhang
(*Department of Mathematics, Peking University*)

Abstract

In this work, the vortex methods for Euler equations with initial boundary value problem is considered, Poisson equations are solved using boundary element methods which can be seen to require less operations to compute the velocity field from the vorticity by Chorin[6]. We prove that the rate of convergence of the boundary element schemes can be independent of the vortex blob parameters.

1. Introduction

The paper written by Chorin[6] in 1973 was the basis of the vortx methods. He divided numerical program into three steps: the first step is to solve the Euler equation with the vortex method, where the velocity flied is computed from the vorticity field with the boundary element method; the second step is to produce the vorticity on the boundary; the third step is to simulate diffusion with random method. It is very difficult that to build the fully mathematic theorey of vortex methods. None can get the convergence of Chorin's algorithm now. In 1978, Chorin, Hughes, McCracken, Marsden[7] regarded the methods as

$$\omega(n \triangle t) = (H(\triangle t)\Theta E(\triangle t))^n \omega_0, \tag{1.1}$$

where Θ is "operator created vorticity", $E(\cdot)$ is Euler's operator, $H(\cdot)$ is Stoke's operator.

People have more studied (1.1) in order to build the mathematical theory of vortex methods. For the simple model, it can be divided as convergence of viscous splitting; convergence of vortex method for Euler equation; and convergence of random vortex method.

The problem in viscous splitting is to consider convergence of the approximate solution, where in every time step, Euler's operator and Stoke's operator both exact, and "operator created vorticity" is considered as a projection operator. Beale and Majda[3] got a fully result for the initial prblems. L. Ying and P. Zhang[21] have studied the initial boundary problems and got a series result. About the random vortex methods, the main result is to see Goodman[8] and D. Long[14]. The convergence of vortex methods for Euler equation is always the main direction. There are many results about the

* Received November 18, 1994.

convergence of Euler equation for initial problems, we can rerefence Hald[10], Beale and Majda[3,4], Anderson and Greengard[2], Raviart[16] and so. L. Ying[18] first considered the initial boundary problem with extrapolation method and got the convergence of semi-discretization. L. Ying and P. Zhang[20] got completely result for the vortex in finite element method. A similar result was got by P. Zhang[22]. Chorin[6] used the boundary element method to disretize Laplace equation. Since the boundary element method is simple and fast for numerical computation instead of the Green function method, especially for the exterior domain which it is not fit for the finite element method. J. Yang[17] have studied the vortex with boundary element methods. convergence results were given for semi-discretization, but constants in the error bounds depended on the vortex blob parameters.

One purpose of this paper is to prove that the rate of convergence of the boundary element scheme is independent of vortex blob parameters.

2. Boundary Element Method

Boundary element method can be divided into two cases: one is only to consider the error produced by approximation function when the boundary is exact; and the other is both to consider the errors produced by boundary and approximation function. For simplity we only consider the first case.

Let Γ be a smooth curve, $x = x(s)$, $s \in [0, L]$, s is parameter of curve, and $\dfrac{dx}{ds}$ is not zero in any point.

$L = L(\Gamma)$ is the length of curve Γ, if Γ is smooth curves in C^k, then $x(s) \in (C^k)^2$.

We choose NE points A_e $(1 \le e \le NE)$, such that

$$A_e = x(s_e) \qquad 1 \le e \le NE$$

We define $s_0 = 0$, $s_{NE} = L$ and $A_0 = A_{NE}$, $\Gamma = \cup_{e=1}^{NE} \Gamma_e$ for closed curve Γ. Γ_e may be expressed as in the local frame

$$\begin{cases} u = \xi h_e \qquad 0 \le \xi \le 1 \\ v = f_e(\xi) = v_e \circ \overline{A_{e-1} x}(s) \end{cases}$$

where $h_e = |x(s_e) - x(s_{e-1})|$, and denote $h = \max_{1 \le e \le NE}(s_e - s_{e-1})$, and s is function of ξ, their relation is

$$u_e \circ \overline{A_{e-1} x}(s) = \xi h_e,$$

since $x(s)$ is continuous differential, s is uniqne according to ξ if h is small enough.

Denote

$$s = \phi_e(\xi), \; \xi \in [0, 1],$$

ϕ_e is one to one in $[0, 1] \longmapsto [s_{e-1}, s_e]$, while equation of Γ_e in local coordinates (u_e, v_e) is

$$x = \Phi_e(\xi), \qquad \Phi_e(\xi) = x(\phi_e(\xi)) = x(s).$$

If we use $P_m(\xi)$ to express the polynomial function spaces that degree is less than m in $[0, 1]$. Then we can define function spaces P_m^e

$$P_m^e = \{p : p = \widetilde{p} \circ \Phi_e^{-1}, \widetilde{p} \in P_m\},$$

then the boundary element spaces $V_h(\Gamma)$ may denote by

$$V_h(\Gamma) = \{v_h : v_h|_{\Gamma_e} \in P_m^e\}.$$

For two-dimensional Poisson equation

$$\begin{cases} -\triangle\psi = \omega, & \text{in}\Omega, \\ \psi|_{\partial\Omega} = 0, \end{cases}$$

the solution $\psi(x)$ of this equation can be expressed as

$$\psi(x) = -\frac{1}{2\pi}\int_\Gamma \frac{\partial\psi}{\partial n}(y)\ln|x-y|ds_y - \frac{1}{2\pi}\int_\Omega \omega(y)\ln|x-y|dy$$

let

$$v(x) = -\frac{1}{2\pi}\int_\Gamma \frac{\partial\psi}{\partial n}(y)\ln|x-y|ds_y = -\frac{1}{2\pi}\int_\Gamma \sigma(y)\ln|x-y|ds_y$$

where $\sigma(y) = \dfrac{\partial\psi}{\partial n}(y)$, $y \in \Gamma$, and $v(x)$ satisfies the equation

$$\begin{cases} \triangle v = 0, & \text{in} \quad \Omega\cup\Omega', \\ v|_{\partial\Omega} = \dfrac{1}{2\pi}\displaystyle\int_\Omega \omega(y)\ln|x-y|dy \end{cases}$$

where Ω' is the outside domain of Ω.

The boundary element method for Poisson equation is to find $\sigma_h \in V_h(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma)$, such that

$$\begin{cases} b(\sigma_h, \sigma_h') = \dfrac{1}{2\pi}\displaystyle\int_\Gamma\int_\Omega \omega(x)\sigma_h'\ln|x-y|dxds_y \\ b(\sigma_h, \sigma_h') = -\dfrac{1}{2\pi}\displaystyle\int_\Gamma\int_\Gamma \sigma_h(x)\sigma_h'\ln|x-y|ds_xds_y. \end{cases}$$

The boundary element solution of Poisson equation is

$$\psi_h(x) = -\frac{1}{2\pi}\int_\Gamma \sigma_h(y)\ln|x-y|ds_y - \frac{1}{2\pi}\int_\Omega \omega(y)\ln|x-y|dy.$$

Then we have

Theorem 2.1. *Let ψ be solution of two-dimensional Poisson equation, and satisfies*

$$\begin{cases} -\triangle\psi = \omega, & \text{in } \Omega \\ \psi|_{\partial\Omega} = 0, \end{cases}$$

and $\omega \in H^{k-1}(\Omega)$, then $\psi(x)$ can be expressed as

$$\psi(x) = -\frac{1}{2\pi}\int_\Gamma \frac{\partial\psi}{\partial n}(y)\ln|x-y|ds_y - \frac{1}{2\pi}\int_\Omega \omega(y)\ln|x-y|dy$$

and ψ_h is approximation solution

$$\psi_h(x) = -\frac{1}{2\pi}\int_\Gamma \sigma_h(y)\ln|x-y|ds_y - \frac{1}{2\pi}\int_\Omega \omega(y)\ln|x-y|dy.$$

where $\sigma_h \in V_h(\Gamma)$, then

$$\|\psi-\psi_h\|_{s+1,\Omega} \le Ch^{m+\frac{3}{2}-s}\|\omega\|_{m+\frac{1}{2},\Omega}, \quad \forall -1 \le s \le \frac{3}{2}. \tag{2.1}$$

Proof. Suppose S_h is a projection operator from $L^2(\Gamma)$ to $V_h(\Gamma)$. Since we have $\psi|_{\partial\Omega}=0$, then we have

$$\|\sigma-\sigma_h\|_{-\frac{1}{2},\Gamma} \le C\inf_{\widetilde{\sigma}\in V_h}\|\sigma-\widetilde{\sigma}\|_{-\frac{1}{2},\Gamma} \le C\|\sigma-S_h\sigma\|_{-\frac{1}{2},\Gamma} \le Ch^{m+\frac{3}{2}}\|\sigma\|_{m+1,\Gamma}.$$

According to the theory of finite element, we know $\sigma_h \in H^1(\Gamma)$, by inverse inequality, we have

$$\|\sigma-\sigma_h\|_{1,\Gamma} \le Ch^m\|\sigma\|_{m+1,\Gamma}.$$

In virtue of theorey of elliptic differential equations

$$\|\psi-\psi_h\|_{s+1,\Omega} \le C\|\sigma-\sigma_h\|_{s-\frac{1}{2},\Gamma} \le Ch^{m+\frac{3}{2}-s}\|\sigma\|_{m+1,\Gamma} \le Ch^{m+\frac{3}{2}-s}\|\omega\|_{m+\frac{1}{2},\Omega}.$$

3. Vortex Method Scheme for Semi-discretization

Let $\Omega \subset R^2$ be a convex and bounded domain, whose boundary $\partial\Omega$ is sufficiently smooth. Denote by $x=(x_1,x_2)$ the points in R^2. We consider the following initial boundary value problems

$$\frac{\partial u}{\partial t} + (u\cdot\bigtriangledown)u + \frac{1}{\rho}\bigtriangledown\pi = f, \tag{3.1}$$

$$\bigtriangledown\cdot u = 0, \tag{3.2}$$

$$u\cdot n\,|_{x\in\partial\Omega} = 0, \tag{3.3}$$

$$u\,|_{t=0} = u_0, \tag{3.4}$$

where $u=(u_1,u_2)$ stands for velocity, π stands for pressure, $f=(f_1,f_2)$ is the external force, the density ρ is a positive constant, n is the unit outward normal vector along $\partial\Omega$.

Let $\omega = -\bigtriangledown\wedge u$, $\omega_0 = -\bigtriangledown\wedge u_0$ and ψ be the stream function corresponding to u, then (3.1)—(3.4) is equivalent to

$$\frac{\partial\omega}{\partial t} + u\cdot\bigtriangledown\omega = -\bigtriangledown\wedge f \equiv F, \tag{3.5}$$

$$-\triangle\psi = \omega, u = \bigtriangledown\wedge\psi, \tag{3.6}$$

$$\psi\,|_{x\in\partial\Omega} = 0, \tag{3.7}$$

$$\omega\mid_{t=0}=\omega_0. \tag{3.8}$$

We extend functions u_0 and f, still denoted by u_0 and f, such that they are sufficiently smooth on R^2 and $R^2\times[0,T]$ respectively and the supports of them are compact. Let d be any positive constant, we define

$$\Omega_d=\{x,\ \mathrm{dist}\,(x,\overline{\Omega})<d\}.$$

The "blob function" is defined as follows, $\zeta(x)$ is a cutoff function, such that $\zeta\equiv 0$ for $|x|>1$ and

$$\zeta_\varepsilon(x)=\frac{1}{\varepsilon^2}\zeta\Big(\frac{x}{\varepsilon}\Big).$$

and the kth-moment condition

$$\begin{aligned}&\int_{R^2}\zeta(x)dx=1,\\ &\int_{R^2}x^\alpha\zeta(x)dx=0,\quad \forall\alpha\in N^2\quad \text{with } 1\le|\alpha|\le k-1.\end{aligned} \tag{3.9}$$

With those notations, the semi-discretization scheme for (3.5)—(3.8) is

$$\omega^\varepsilon(x,t)=\sum_{j\in J_1}\alpha_j^\varepsilon(t)\zeta_\varepsilon(x-X_j^\varepsilon(t)), \tag{3.10}$$

$$\frac{d\alpha_j^\varepsilon}{dt}=h^2F(X_j^\varepsilon(t),t),\alpha_j^\varepsilon(0)=\alpha_j, \tag{3.11}$$

$$\frac{dX_j^\varepsilon}{dt}=g^\varepsilon(X_j^\varepsilon(t),t),X_j^\varepsilon(0)=X_j, \tag{3.12}$$

$$-\triangle\psi^\varepsilon=\omega^\varepsilon,\psi^\varepsilon\mid_{x\in\partial\Omega}=0, \tag{3.13}$$

$$u^\varepsilon=\bigtriangledown\wedge\psi^\varepsilon, \tag{3.14}$$

$$g^\varepsilon(x,t)=\sum_{i=1}^{M}a_iu^\varepsilon(x^{(i)},t), \tag{3.15}$$

where $j=(j_1,j_2)$, $X_j=(j_1h,j_2h)$, $\alpha_j=h^2\omega_0(X_j)$ and $J_1=\{j,\ X_j\in\Omega_d\}$; if $x\in\overline{\Omega}$ then $x^{(i)}=x$ otherwise

$$x^{(i)}=(i+1)Y-ix,$$

where Y is the nearest point on $\partial\Omega$ to x; the terms a_i are the solutions of the system

$$\sum_{i=1}^{M}(-i)^ja_i=1\quad j=0,\cdots,M.$$

Equations (3.15) makes sense only if $x^{(i)}$ belongs to $\overline{\Omega}$, but it is proved in [10] that this fact is true provided d is small enough. In this scheme the function g^ε plays the role of velocity which is equal to u^ε in the domain and interpolated to the exterior part of Ω. This is a natural way to deal with blobs near the boundary. Using g^ε and a "slightly larger" domain Ω_d in computation, all blobs move according to a uniform formula (3.12).

The notation $W^{m,p}(\Omega)$ for conventional Sobolev spaces and $\|\cdot\|_{m,p}$ for the norms of them are applied throughout this paper. Let $X_j(t)$ be characteristic curves which satisfy

$$\frac{dX_j(t)}{dt} = u(X_j(t),t), \quad X_j(0) = X_j.$$

As a rule, we admit the value of u as an extension if $X_j(t)\overline{\in}\overline{\Omega}$. Then set

$$J_2 = \{j; X_j \in \Omega_{C_0\varepsilon} \cap \Omega_d\},$$
$$\|e(t)\|_p = \Big(h^2 \sum_{j\in J_2} |X_j(t) - X_j^\varepsilon(t)|^p\Big)^{1/p}, \ 1 \le p < \infty,$$

where C_0 is a positive constant to be determined. The following theorem is proved in [10]:

Theorem 1. *If we have $m \ge 1$, $k \ge 2$, such that $\zeta \in W^{m+1,\infty}(R^2)$ and kth-moment condition* (3.9) *is satisfied, if there is a constant $\widetilde{C}$, such that*

$$\widetilde{C}^{-1}\varepsilon^a \le h \le \widetilde{C}\varepsilon^{1+\frac{k-1}{m}} \tag{3.16}$$

where $a \ge 1 + \dfrac{k-1}{m}$, and if the constant in expression (3.15)*, $M \ge k$, then for any $p \in [1,\infty)$, there are positive constants d_0, C_0, C_1 and C_2 such that if $d \le d_0$, then the solution of problem* (3.10)–(3.15) *satisfy*

$$|\bigtriangledown u^\varepsilon(x,t)| \le C_1, \quad x \in \overline{\Omega}, \tag{3.17}$$
$$\|u - u^\varepsilon\|_{0,p,\Omega} + \|e(t)\|_p \le C_2\varepsilon^k, \tag{3.18}$$

for $t \in [0,T]$.

For our later use, we need the following Corollary which is proved in [7]:

Corollary. Under the assumption of Theorem 1, Let $C_3 > 0$ be given, then there is a constant C_4, such that

$$\|\omega^\varepsilon(\cdot,t)\|_{k-1,p,\Omega_{C_3\varepsilon}} \le C_4, \qquad t \in [0,T]. \tag{3.19}$$

4. Further Discretization with Boundary Element Method

In pratical computation, we can not exactly find the solution of the Laplace equations

$$\begin{cases} -\triangle\psi^\varepsilon = \omega^\varepsilon, \\ \psi^\varepsilon|_{\partial\Omega} = 0. \end{cases}$$

Here we will find the approximation solution of Lapace equation with the boundary element method. Now the scheme for the resolution of Euler equation

$$\omega^\delta(x,t) = \sum_{j\in J_1} \alpha_j^\delta(t)\zeta_\varepsilon(x - X_j^\delta(t)), \tag{4.1}$$

$$\frac{d\alpha_j^\delta}{dt} = h^2 F(X_j^\delta(t), t), \alpha_j^\delta(0) = \alpha_j, \tag{4.2}$$

$$\frac{dX_j^\delta}{dt} = g^\delta(X_j^\delta(t), t), X_j^\delta(0) = X_j, \tag{4.3}$$

$$\psi^\delta(x) = -\frac{1}{2\pi}\int_\Gamma \sigma_\delta(y)\ln|x-y|ds_y - \frac{1}{2\pi}\int_\Omega \omega^\delta(y)\ln|x-y|dy \tag{4.4}$$

where δ is the scale of boundary division.

$$g^\delta(x,t) = \sum_{i=1}^m a_i u^\delta(x^{(i)}, t), \qquad u^\delta = \bigtriangledown \wedge \psi^\delta. \tag{4.5}$$

And $\sigma_\delta \in V_\delta(\Gamma)$ satisfy

$$\begin{cases} b(\sigma_\delta, \sigma'_\delta) = \dfrac{1}{2\pi}\displaystyle\int_\Gamma\int_\Omega \omega^\delta(y)\sigma'_\delta(x)\ln|x-y|ds_x dy \\ b(\sigma_\delta, \sigma'_\delta) = -\dfrac{1}{2\pi}\displaystyle\int_\Gamma\int_\Gamma \sigma_\delta(x)\sigma'_\delta(y)\ln|x-y|ds_x ds_y, \end{cases}$$

for any $\sigma'_\delta \in V_\delta(\Gamma)$, where $V_\delta(\Gamma)$ is the boundary element space.

We need to estimate $u^\varepsilon - u^\delta$ and $X_j^\varepsilon - X_j^\delta$. Let

$$\|e(t)\|_p = \Big(h^2 \sum_{j\in J_1} |X_j^\varepsilon(t) - X_j^\delta(t)|^p\Big)^{\frac{1}{p}},$$
$$\|e(t)\|_\infty = \max_{j\in J_1} |X_j^\varepsilon(t) - X_j^\delta(t)|.$$

and we define a function ψ_1 which solves

$$\begin{cases} -\Delta\psi_1 = \omega^\delta, \quad x\in\Omega, \\ \psi_1 \mid_{x\in\partial\Omega} = 0. \end{cases} \tag{4.6}$$

Then function ψ^δ determined by (4.2) is the boundary element approximation of ψ_1. Let

$$u_1 = \bigtriangledown \wedge \psi_1 = G\omega^\delta. \tag{4.7}$$

We suppose that there is a constant C_5, such that

$$|\bigtriangledown u_1| \le C_5. \tag{4.8}$$

Lemma 1. *On the assumption of Theorem* 1*, we have*

$$\|u^\varepsilon(\cdot,t) - u_1(\cdot,t)\|_{l,p,\Omega} \le \frac{C}{\varepsilon^l}\Big\{\Big(1 + \frac{\|e(t)\|_\infty}{\varepsilon}\Big)^{\frac{2}{q}}\|e(t)\|_p + \int_0^t \|e(s)\|_p ds\Big\}, \tag{4.9}$$

$$\|u_1(\cdot,t) - u^\delta(\cdot,t)\|_{s,2,\Omega}$$
$$\le C\delta^{m+\frac{3}{2}-s} + C\frac{\delta^{m+\frac{3}{2}-s}}{\varepsilon^{m+\frac{1}{2}}}\Big\{\Big(1 + \frac{\|e(t)\|_\infty}{\varepsilon}\Big)^{\frac{2}{q}}\|e(t)\|_p + \int_0^t \|e(s)\|_p ds\Big\}, \tag{4.10}$$

where $\dfrac{1}{p}+\dfrac{1}{q}=1,\ p>2,\ 0\le s\le \dfrac{3}{2}$.

Proof. We decomposition $u^\varepsilon - u_1$ as [5],

$$u^\varepsilon - u_1 = G\Big(\sum_{j\in J_1}\alpha_j^\varepsilon\zeta_\varepsilon(\cdot - X_j^\varepsilon(t)) - \sum_{j\in J_1}\alpha_j^\delta\zeta_\varepsilon(\cdot - X_j^\delta(t))\Big) = v_1 + v_2 \tag{4.11}$$

where

$$\begin{aligned} v_1 &= G\Big(\sum_{j\in J_1}\alpha_j^\delta(\zeta_\varepsilon(\cdot - X_j^\varepsilon(t)) - \zeta_\varepsilon(\cdot - X_j^\delta(t))\Big),\\ v_2 &= G\sum_{j\in J_1}(\alpha_j^\varepsilon - \alpha_j^\delta(t))\zeta_\varepsilon(\cdot - X_j^\varepsilon(t)). \end{aligned}$$

Repeating the argument in the proof of Lemma 6 in [18], we get (4.9).

By (2.1)

$$\begin{aligned} \|u_1(\cdot,t) - u^\delta(\cdot,t)\|_{s,2,\Omega} &\le \|\psi_1(\cdot,t) - \psi^\delta(\cdot,t)\|_{s+1,2,\Omega}\\ &\le C\delta^{m+\frac{3}{2}-s}\Big\|G\sum_{j\in J_1}\alpha_j^\delta\zeta_\varepsilon(\cdot - X_j^\delta(t))\Big\|_{m-\frac{1}{2},2,\Omega}\\ &\le C\delta^{m+\frac{3}{2}-s}\Big\|G\sum_{j\in J_1}\alpha_j^\delta\zeta_\varepsilon(\cdot - X_j^\delta(t))\Big\|_{m-\frac{1}{2},p,\Omega}\\ &\le C\delta^{m+\frac{3}{2}-s} + C\frac{\delta^{m+\frac{3}{2}-s}}{\varepsilon^{m+\frac{1}{2}}}\Big\{\Big(1+\frac{\|e(t)\|_\infty}{\varepsilon}\Big)\|e(t)\|_p + \int_0^t\|e(s)\|_p ds\Big\}, \end{aligned}$$

for $p>2$. This complete the proof of (4.10).

Lemma 2. *Under the assumption of Theorem 1 with $k\ge 2$, then we have*

$$\begin{aligned} \|e(t)\|_p \le{} & C_6(\varepsilon^k + \delta^m)\\ & + C_6\Big(1+\frac{\delta^m}{\varepsilon^{m+\frac{1}{2}}}\Big)\Big\{\Big(1+\frac{\|e(t)\|_\infty}{\varepsilon}\Big)^{\frac{2}{q}}\|e(t)\|_p + \int_0^t\|e(s)\|_p ds\Big\} \end{aligned} \tag{4.12}$$

where $p\in(1,\infty)$.

Proof. We consider the case of $p>2$, we define

$$g_1(x,t) = \sum_{i=1}^M a_i u_1(x^{(i)},t).$$

and

$$g_1^\delta(x,t) = \sum_{i=1}^M a_i u_1(x_\delta^{(i)},t).$$

Then taking (3.12) and (4.5) into accout, we have

$$\frac{dX_j^\varepsilon(t)}{dt} - \frac{dX_j^\delta(t)}{dt} = I_1 + I_2 + I_3, \quad X_j^\varepsilon(0) - X_j^\delta(0) = 0. \tag{4.13}$$

where

$$I_1 = g^\varepsilon(X_j^\varepsilon(t),t) - g_1(X_j^\varepsilon(t),t)$$
$$I_2 = g_1(X_j^\varepsilon(t),t) - g_1^\delta(X_j^\delta(t),t),$$
$$I_3 = g_1^\delta(X_j^\delta(t),t) - g^\delta(X_j^\delta(t),t)$$

We define a set

$$J_0 = \{j; X_j \in \overline{\Omega}\},$$

then by Lemma 5.4, Chapter 2 of [5],

$$\Big(h^2 \sum_{j\in J_0} |(u^\varepsilon - u_1)(X_j(t),t)|^p\Big)^{\frac{1}{p}} \le C(\|u^\varepsilon - u_1\|_{0,p,\Omega} + h|u^\varepsilon - u_1|_{1,p,\Omega}). \tag{4.14}$$

On the other hand, denote by $\Phi^{(i)}$ the mapping $X_j(t) \to (X_j(t))^{(i)}$, then

$$\begin{aligned}
&\Big(h^2 \sum_{j\in J_1\backslash J_0} |(u^\varepsilon - u_1)((X_j(t))^{(i)},t)|^p\Big)^{\frac{1}{p}}\\
&= \Big(h^2 \sum_{j\in J_1\backslash J_0} |(u^\varepsilon - u_1)(\Phi^{(i)}X_j(t),t)|^p\Big)^{\frac{1}{p}}\\
&\le C(\|(u^\varepsilon - u_1)\circ\Phi^{(i)}\|_{0,p,\Omega_{C_0\varepsilon}\backslash\overline{\Omega}} + h|(u^\varepsilon - u_1)\circ\Phi^{(i)}|_{0,p,\Omega_{C_0\varepsilon}\backslash\overline{\Omega}})\\
&\le C(\|u^\varepsilon - u_1\|_{0,p,\Omega} + h|u^\varepsilon - u_1|_{1,p,\Omega}).
\end{aligned} \tag{4.15}$$

Combining (4.14) with (4.15) and noting (4.9) we get

$$\Big(h^2 \sum_{j\in J_1} |(u^\varepsilon - u_1)(X_j(t),t)|^p\Big)\frac{1}{p}s \le C_6\Big\{\Big(1+\frac{\|e(t)\|_\infty}{\varepsilon}\Big)^{\frac{2}{q}}\|e(t)\|_p + \int_0^t \|e(s)\|_p ds\Big\}. \tag{4.16}$$

By (3.17) (3.18) and (4.8), we have

$$\Big(h^2 \sum_{j\in J_1} |I_1|^p)^{\frac{1}{p}} \le \varepsilon^k + C_6\Big\{\Big(1+\frac{\|e(t)\|_\infty}{\varepsilon})^{\frac{2}{q}}\|e(t)\|_p + \int_0^t \|e(s)\|_p ds\Big\}. \tag{4.17}$$

In virtue of (4.8), we have

$$|I_2| \le C|(X_j^\varepsilon(t))^{(i)} - (X_j^\delta(t))_\delta^{(i)}|.$$

where $(X_j^\delta(t))_\delta^{(i)} = (X_j^\delta(t))^{(i)}$, and so

$$|I_2| \le C|X_j^\varepsilon(t) - X_j^\delta(t)|. \tag{4.18}$$

By (4.10) and interpolation inequality, we obtain

$$|I_3| \le C\|u_1(\cdot,t) - u^\delta(\cdot,t)\|_{0,\infty,\Omega} \le C\|u_1(\cdot,t) - u^\delta(\cdot,t)\|_{\frac{3}{2},2,\Omega}$$

$$\leq C\delta^m + C\frac{\delta^m}{\varepsilon^{m+\frac{1}{2}}}\Big\{\Big(1+\frac{\|e(t)\|_\infty}{\varepsilon}\Big)^{\frac{2}{q}}\|e(t)\|_p + \int_0^t \|e(s)\|_p ds\Big\}. \tag{4.19}$$

Combining (4.17) (4.18) and (4.19), we obtain (4.12).

Theorem 2. *If the assumption of Theorem 1 holds with $k \geq 2$, $m \geq 1$ and d_0 is small, and if $\delta \leq C\varepsilon^{1+\frac{1}{2m}}$, then*

$$|\bigtriangledown u_1| < C_7 \tag{4.20}$$
$$\|e(t)\|_p + \|u^\varepsilon(\cdot,t) - u^\delta(\cdot,t)\|_{0,p,\Omega^\delta} \leq C(\varepsilon^k + \delta^m), \tag{4.21}$$

for any $p \in [1,\infty)$.

Proof. From Lemma 2, we can claim that

$$\|e(t)\|_p \leq C_8(\varepsilon^k + \delta^m), \tag{4.22}$$

for large p. The proof is standard.

From (3.17), we know

$$|\bigtriangledown u^\varepsilon(x,t)| < C_1$$

and we also know

$$\|u_1(\cdot,0) - u^\varepsilon(\cdot,0)\|_{1,\infty,\Omega} = 0$$

then

$$|\bigtriangledown u_1| < C$$

for $t = 0$. By continuity, (4.8) holds for $t \in [0,T^*]$, where T^* is a certain positive constant. Thus Lemma 2 is valid. Let $p > 2a$ and $l = 0,1,2$, then in conjunction with Lemma 1 implies

$$|u_1(\cdot,t) - u^\varepsilon(\cdot,t)|_{l,p,\Omega} \leq C\frac{\varepsilon^k + \delta^m}{\varepsilon^l}.$$

Then by interpolation theorem

$$\|u_1(\cdot,t) - u^\varepsilon(\cdot,t)\|_{1,\infty,\Omega} \leq C\frac{\varepsilon^k + \delta^m}{\varepsilon^{1+s}} \leq C\varepsilon^{\frac{3}{2}-1-s} \leq C\varepsilon^{\frac{1}{2}-s}.$$

Let ε be small enough, such that

$$\|u_1(\cdot,t) - u^\varepsilon(\cdot,t)\|_{1,\infty,\Omega} \leq 1.$$

Then we get $|\bigtriangledown u_1| < C$. By continuous extension we get (4.20) for $t \in [0,T]$.

Using Lemma 1, 2 again, we obtain the estimate (4.21) for sufficiently large p and small ε, then for small p and any ε are also true.

References

[1] R.A. Adams, Sobolev Spaces, Academic Press (1975).

[2] C. Anderson and C. Greengard, On vortex methods *SIAM J. Numer. Anal.,* 22(1985), 413–439.

[3] J.T. Beale and A. Majda, Rate of convergence for viscous splitting of the Navier-Stokes equations *Math. Comp.,* 37(1981), 243–259.

[4] J.T. Beale and A. Majda, Vortex methods I: Convergence in three dimensions, *Math. Comp.,* 39(1982), 1–27.

[5] J.T. Beale and A. Majda, Vortex methods II: Higher order accuracy in two and three dimensions, *Math. Comp.,* 39(1982), 29–52.

[6] A.J. Chorin, Numerical study of slightly viscous flow, *J. Fluid Mech.,* 57(1973), 785–796.

[7] A.J. Chorin, T.J.R. Hughes, M.F. McCracken and J.E. Mardsen, A Mathematical Introduction to Fluid Mechanics, Springer-Verlag, 1979.

[8] J. Goodman, Convergence of the random vortex method *Comm., Pure Appl. Math.,* 40(1987), 189–220.

[9] J. Goodman, T.Y Hou and J. Lowengrub, Convergence of the point vortex method for the 2-D euler equations, *Comm. Pure Appl. Math.,* 43(1990), 415–430.

[10] O. Hald, The convergence of vortex methods II, *SIAM J. Numer. Anal.,* 16(1979), 726–755.

[11] O. Hald, V.M. DelPrete, Convergence of vortex methods for Euler's equations, *Math. Comp.,* 32(1978), 781–809.

[12] T.Y. Hou and J. Lowengrub, Convergence of the point vortex method for the 3-D euler equations, *Comm. Pure Appl. Math.,* XLIII (1990), 965–981.

[13] A. Leonard, Vortex methods for flow simulation, *J. Comput. Phys.,* 37(1980), 289–335.

[14] D.G. Long, Convergence of the random vortex method in two dimensions, *J. Amer. Math. Soc.,* 1(1988), 779–804.

[15] R. Rannacher and R. Scott, Some optimal error estimates for piecewise linear finite element approximations, *Math. Comp.,* 38(1982), 437–445.

[16] P.A. Raviart, An analysis of paticle methods, Lecture Notes in Mathematics, Springer-verlag, 1127(1985), 243–324.

[17] J. Yang, Peking University Master's thesis (1992).

[18] L.G. Ying, Convergence of vortex methods for initial boundary value problems, *Advances in Math.,* 20(1991), 86–102.

[19] L.G. Ying, Convergence of vortex methods for three dimensional euler equations in bounded domain, Institute of Math. and Dept. of Math., Peking Univ., Research Report, 65(1992).

[20] L.G. Ying and P.W. Zhang, Fully discrete convergence estimates for vortex methods in dounded domains, (to appear in *SIAM J. Numer. Anal.*)

[21] L.G. Ying and P.W. Zhang, Vortex Methods, Science Press (to appear).

[22] P.W. Zhang, Convergence of Vortex Methods for Exterior Problems (to appear).

[23] P.W. Zhang, Convergence of Vortex in Linear Finite Element Methods, (to appear).

[24] J.L. Zhu, Boundary Element Analysis of Elliptic Boundary Problems, Sciences Press, 1991. (in Chinese).