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ON MAXIMUM NORM ESTIMATES FOR RITZ-VOLTERRA PROJECTION WITH APPLICATIONS TO SOME TIME DEPENDENT PROBLEMS*[1)]

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Abstract

The stability in L^∞-norm is considered for the Ritz-Volterra projection and some applications are presented in this paper. As a result, point-wise error estimates are established for the finite element approximation for the parabolic integro-differential equation, Sobolev equations, and a diffusion equation with non-local boundary value problem.

1. Introduction

We are concerned with the finite element method for parabolic integro-differential equation

$$\begin{aligned} &u_t(t) + V(t)u(t) = f(t), \qquad t \in (0,T), \\ &u(0) = v, \end{aligned} \tag{1.1}$$

where $V(t)$ is in general an integro-differential operator defined on a Hilbert space X and that u and f are X-valued functions defined on $J = (0,T)$ with a positive time T. A typical example of the Hilbert space X in the application will be the Sobolev space H_0^1 consisting of functions defined on an open bounded domain Ω with vanished boundary value and first order derivatives summable in L^2, while the operator $V(t)$ is the one defined by

$$V(t)u(t) \equiv A(t)u(t) + \int_0^t B(t,\tau)u(\tau)d\tau, \quad \text{in } \Omega \tag{1.2}$$

for any $u(t) \in H_0^1(\Omega)$, where $A(t)$ is a linear elliptic operator of second order and that $B(t,\tau)$ any linear operator of no more than second order. Although more examples of integro-differential operators will be considered in this paper, we shall illustrate our results for the operator $V(t)$ defined by (1.2), since the others can be modified to fit the strategy designed for (1.2).

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Numerical methods to the equation (1.1) have been studied by several authors recently. For finite difference schemes we refer to [23] and the references cited therein. The finite element method for this problem has also been studied; in [23] both smooth and non-smooth data cases were considered and optimal error estimates in L^2 were obtained, the semi-linear equation with non-smooth data and an operator B of zero order was treated in [12] along with a particular attention paid to the computation of the memory term by the quadrature rule. Recently, a different approach to the error analysis was proposed in [3] and [4]. Their idea can be summerized as introducing a so-called Ritz-Volterra projection to decompose the error. A systematic study of Ritz-Volterra projection and its applications to parabolic and hyperbolic integro-differential equations, Sobolev equation, and the equations of visco-elasticity can be found from [14].

For the sake of convenience of the analysis, we shall take Ω to be a plane convex polygonal domain. Let $\mathcal{T}_h$ be a quasi-uniform triangulation so that $\Omega_h = \cup_{K\in\mathcal{T}_h} K = \Omega$. Let S_h be the finite element subspace associated with $\mathcal{T}_h$. Without loss of generality, we shall assume that S_h is made up of piece-wise linear functions.

The object of this paper is to study the convergence behavior of the finite element approximation in the L^∞-norm. As a matter of fact, this problem had been considered by Lin and Zhang[15], where an optimal maximum norm has been obtained for piecewise linear elements for a very special case, that is, the operators A and B are divergence form which allows us to use the standard regularized Green function[16,22], and by Lin, Thomee, and Wahlbin in [14], where the following estimate for any small $\varepsilon > 0$

$$\|u(t) - u_h(t)\|_{0,\infty} \le C(u,\varepsilon)h^{r-\varepsilon}$$

was derived based on their estimate in L^p. Here r is the optimal order in the approximation and $C(u)$ a constant dependent upon the exact solution u only. It is clear that such an estimate is not optimal in compare with the results for the elliptic and parabolic equations[17,22,10,20,18]. We shall, therefore, study this problem from a different point of view in order to get a sharp estimate in the L^∞-norm. The main idea of our approach can be summarized as firstly introducing an auxiliary problem associated with the Ritz-Volterra operator V and then establishing our main results with the help of the solution of this auxiliary problem. The auxiliary problem to be introduced in next section is an analogy of the regularized Green's function in the study of the L^∞-stability for the elliptic equation of second order. Thus, the only contribution of the authors would be to apply the known technique appropriately to the current problem. However, such an extension is not trivial due to the memory term involved in the operator V.

Our main result regards to the maximum norm error estimate for the Ritz-Volterra projection V_h defined by

$$V(t; V_h u(t), \phi) = V(t; u(t), \phi), \quad \phi \in S_h \tag{1.3}$$

for each $t \in J$, where $V(t;\cdot,\cdot)$ is the bilinear form associated with the Ritz-Volterra operator $V(t)$ defined by

$$V(t; u(t), v(t)) = A(t; u(t), v(t)) + \int_0^t B(t,\tau; u(\tau), v(t))d\tau \tag{1.4}$$

for $u(t), v(t) \in H_0^1$ with $t \in J$. Applications to finite element approximations for the parabolic integro-differential equation, Sobolev equation, and a diffusion equation with non-local boundary condition are presented in this paper.

This paper is organized as follows. In section 2, we shall introduce and study an auxiliary problem associated with the operator V. The solution of this problem can be regarded as a certain regularized Green's function associated with the Ritz-Volterra operator. In section 3, we shall establish an estimate in the L^∞-norm for the Ritz-Volterra projection onto the finite element subspace S_h, while the applications to the parabolic integro-differential equation, Sobolev equation, and a diffusion equation with non-local boundary condition will be given in section 4.

A preliminary of this paper can be summarized as follows. Denote by $W^{m,p}$ the Sobolev space on the domain Ω defined by

$$W^{m,p} = \{v; \quad D^j v \in L^p \text{ with } |j| \le m\}$$
$$\|v\|_{m,p} = \Big(\sum_{|j|=0}^{m} \|D^j v\|_{0,p}^p \Big)^{\frac{1}{p}}$$

for non-negative integers m and $p \in [1, \infty]$, where D^j is the differential operator of order $|j|$ with a multi-index j and that $\|\cdot\|_{0,p}$ the L^p norm of the corresponding function. In the case of $p = 2$ we shall use the notation H^m with norm $\|\cdot\|_m$ rather than $W^{m,p}$. With an abuse of notation, $\|\cdot\|_\infty$ will be used to indicate the L^∞-norm.

Along with the operator $V(t)$ of (1.2), we define a new operator $V^*(t)$ by

$$V^*(t)u(t) \equiv A(t)u(t) + \int_t^T B^*(\tau, t)u(\tau)d\tau, \quad u(t) \in H_0^1 \tag{1.5}$$

for each $t \in J$, where B^* is the adjoint of the linear operator B in H_0^1. By changing the order of integration it is not hard to check that

$$\int_0^T V(t; u(t), v(t))dt = \int_0^T V^*(t; v(t), u(t))dt, \tag{1.6}$$

where $V(t; \cdot, \cdot)$ and $V^*(t; \cdot, \cdot)$ are the bilinear forms associated with the operators $V(t)$ and $V^*(t)$, respectively. Thus, the operator $V^*(t)$ can be considered as the adjoint of $V(t)$.

To analyze the solution associated with the operator $V^*(t)$, the Gronwall's lemma in the following version will be used. Let ψ and ϕ are two non-negative functions defined on $[0, T]$ and

$$\psi(t) \le \phi(t) + C \int_t^T \psi(\tau)d\tau, \quad t \in J.$$

Then,

$$\psi(t) \le C\Big\{\phi(t) + \int_t^T \phi(\tau)d\tau\Big\}.$$

We shall refer the last relation as 'back-ward' Gronwall's inequality. Here and throughout this paper we shall use C to denote genetic non-negative constant independent of the mesh size h and any functions involved. But it may depend upon the time interval $[0, T]$.

The following a priori estimates for the operators $V(t)$ and $V^*(t)$ are valid due to the Gronwall's lemma. Let $f(t) \in L^2$ for each $t \in J$ and u and w satisfy

$$V(t)u(t) = f(t), \text{ for all } t \in J \tag{1.7}$$

and

$$V^*(t)w(t) = f(t), \text{ for all } t \in J \tag{1.8}$$

with homogeneous Dirichlet boundary condition, respectively. Then, $u, w \in H_0^1 \cap H^2$ for each $t \in J$ and there exists a constant C such that

$$\|u(t)\|_2 \leq C\Big(\|f(t)\|_0 + \int_0^t \|f(\tau)\|_0 d\tau\Big), \tag{1.9}$$

$$\|w(t)\|_2 \leq C\Big(\|f(t)\|_0 + \int_t^T \|f(\tau)\|_0 d\tau\Big). \tag{1.10}$$

It can also be seen easily that the following estimate holds

$$\|\nabla w(t)\|_0 \leq C\Big(\|f(t)\|_{-1} + \int_t^T \|f(\tau)\|_{-1} d\tau\Big) \tag{1.11}$$

with $\|\cdot\|_{-1}$ being the norm in the space H^{-1}.

2. An Auxiliary Problem

In studying the convergence in the L^∞-norm for the finite element method associated with elliptic equations, one needs to introduce and study the approximation for the Green's function in $W^{1,\infty}$[22,10,18,21,9,27] or alternatively, to employ a weighted norm in the analysis[16,17]. The Green's function is usually defined to be the solution of a conjugate problem of the problem under consideration. Here in our problem (1.1) we have the operator V as an analogue of the elliptic operator of the second order, so that we need to introduce an auxiliary problem whose solution plays the role as the Green's function in the elliptic case.

For this purpose we now define an operator V^* on $H_0^1 \times J$, understood to be the adjoint of V, so that for any $u(t) \in H_0^1$

$$V^*u(t) \equiv A(t)u(t) + \int_t^T B^*(\tau, t)u(\tau)d\tau, \tag{2.1}$$

where B^* is the adjoint of the operator B. Formally, let $g(t) = g(z, t, z_0, t_0)$ satisfy

$$V^*g(t) = \delta^{z_0}(z)\ \delta^{t_0}(t), \tag{2.2}$$

where $z = (x, y)$ is the space variable and that $\delta^{z_0}(z)$ and $\delta^{t_0}(t)$ the dirac δ-function associated with the points z_0 and t_0, respectively. Thus, for any sufficiently smoothing function $w(z, t)$ one may have

$$w(z_0, t_0) = \langle \delta^{z_0}(z)\ \delta^{t_0}(t), w(z, t)\rangle = \int_0^T V^*(t; g(t), w(t))dt$$

$$= \int_0^T V(t; w(t), g(t))dt, \tag{2.3}$$

where we have used the relation (1.6) to derive (2.3). It is clear that the function $g(t)$ acts as the Green's function in a certain space. However, to make everything be more precise from the mathematical point of view, we define a function $G(t) \equiv G(z, t; z_0)$ to be the solution of the equation

$$\begin{aligned} &A(t)G(t) + \int_t^T B^*(\tau, t)G(\tau)d\tau = \delta_h^{z_0}(z)\phi(t), \text{ in } \Omega, \\ &G(t) = 0, \qquad \text{on } \partial\Omega, \end{aligned} \tag{2.4}$$

where $\phi(t) \in C^\infty(0, T)$ and $\delta_h^{z_0}(z)$ is a smoothed δ-function associated with the point z_0 and the piecewise linear finite element subspace S_h. They are required to satisfy the following properties

1. $(\delta_h^{z_0}, \chi) = \chi(z_0)$, $\chi \in S_h$,
2. $|\delta_h^{z_0}(z)| \le Ch^{-2}$, sup $(\delta_h^{z_0}) \subset \{x;\ |x - z| \le Ch\}$,
3. $\|\phi\|_{L^1(0,T)} \le 1$.

The solution of the auxiliary problem (2.4) plays the same role as the regularized Green's function used in the L^∞ error analysis for the finite element method of the elliptic problem of second order[22,10,21,9,27], though the function on the right hand side of (2.4) is not precisely a regularized δ-function. In fact, since our analysis will be made only for the semi-discretization algorithm the function $\delta_h^{z_0}\phi(t)$ can be regarded as a regularized δ-function in the space variable. Thus, the solution of the problem (2.4) shall be referred to as the regularized Green's function for the Ritz-Volterra operator V.

Let $G_h(t)$ be the finite element approximation of the regularized Green's function for $t \in [0, T]$; i.e.

$$A(t; G(t) - G_h(t), \chi) + \int_t^T B^*(\tau, t; G(\tau) - G_h(\tau), \chi)d\tau = 0, \ \chi \in S_h. \tag{2.5}$$

It is not hard to see that (cf. [3] [4] [14])

$$\|G - G_h\|_0 + h\|\nabla(G - G_h)\|_0 \le Ch^2\Big\{\|G\|_2 + \int_t^T \|G(\tau)\|_2 d\tau\Big\}. \tag{2.6}$$

Thus, the a priori estimate (1.10) implies that

$$\|G - G_h\|_0 + h\|\nabla(G - G_h)\|_0 \le Ch(1 + |\phi(t)|), \tag{2.7}$$

since

$$\|G\|_2 + \int_t^T \|G(\tau)\|_2 d\tau \le Ch^{-1}(1 + |\phi(t)|).$$

We are now ready to establish the result for the regularized Green's function.

Theorem 2.1. *Assume the triangulation $\mathcal{T}_h$ to be regular* [6]*. Then, there exists a constant C , independent of h, z, and $\phi(t)$, such that*

$$\|G(t) - G_h(t)\|_{1,1} \le Ch \log\frac{1}{h}(1 + |\phi(t)|). \tag{2.8}$$

Proof. Let $\sigma_{z_0}(z) = (|z - z_0|^2 + K^2h^2)^{1/2}$ be the weight function used in [10] (see also [16] [17] [22]) for the standard Galerkin approximation. We omit the well known properties regarding the weight function for sufficiently large K. Since no confusion is possible, we shall take $K = 1$ in our analysis. Thus, by Schwarz inequality,

$$\begin{aligned}\|G - G_h\|_{1,1} &\le \|G - G_h\|_{0,1} + \|\nabla(G - G_h)\|_{0,1}\\ &\le C\|G - G_h\|_0 + C\Big(\log\frac{1}{h}\Big)^{1/2}\|\nabla(G - G_h)\|_{\sigma^2},\end{aligned}\tag{2.9}$$

where, as usual, $\|\cdot\|_{\sigma^\alpha,Q}$ is a weighted norm defined by

$$\|\varphi\|^2_{\sigma^\alpha,Q} = \int_Q \sigma^\alpha\varphi^2 dQ \quad \text{for } \alpha \text{ real}$$

and $\|\cdot\|_{\sigma^\alpha}$ the weighted norm for $Q = \Omega$. It follows from (2.7) and (2.9) that it suffices to prove the following

$$\|\nabla(G - G_h)\|_{\sigma^2} \le Ch\Big(\log\frac{1}{h}\Big)^{1/2}(1 + |\phi(t)|).\tag{2.10}$$

By the ellipticity of the operator $A(t)$, we may assume, without loss of generality, that

$$\|\nabla w\|_0^2 \le A(t; w, w), \quad w \in H_0^1(\Omega)$$

is valid. Thus, a simple calculation shows that

$$\|\nabla(G - G_h)\|^2_{\sigma^2} \le C\|G - G_h\|_0^2 + A(t; G - G_h, \psi),\tag{2.11}$$

where $\psi = \sigma^2(G - G_h)$. Let ψ_I be the piece-wise linear interpolation of ψ on S_h. Then,

$$\begin{aligned}A(t; G - G_h, \psi) =& A(t; G - G_h, \psi - \psi_I) - \int_t^T B^*(\tau, t; G(\tau) - G_h(\tau), \psi - \psi_I)d\tau\\ &+ \int_t^T B^*(\tau, t; G(\tau) - G_h(\tau), \psi)d\tau.\end{aligned}\tag{2.12}$$

Thus, by applying the Schwarz inequality we get

$$\begin{aligned}A(t, G - G_h, \psi) \le& \frac{1}{4}\|\nabla(G - G_h)\|^2_{\sigma^2} + Ch^2\|\nabla^2\psi\|^2_{\sigma^{-2}} + C\|G - G_h\|_0^2\\ &+ \int_t^T B^*(\tau, t; G(\tau) - G_h(\tau), \psi)d\tau + C\Big(\int_t^T \|\nabla(G - G_h)\|_{\sigma^2}d\tau\Big)^2\end{aligned}\tag{2.13}$$

and

$$\begin{aligned}\int_t^T B^*(\tau, t; G(\tau) - G_h(\tau), \psi)d\tau \le& \frac{1}{4}\|\nabla(G - G_h)\|^2_{\sigma^2} + C\|G - G_h\|_0^2\\ &+ C\Big(\int_t^T (\|G - G_h\|_0 + \|\nabla(G - G_h)\|_{\sigma^2})d\tau\Big)^2,\end{aligned}\tag{2.14}$$

where

$$\|\nabla^2\psi\|_{\sigma^{-2}}^2 = \sum_{K\in\mathcal{T}_h} \|\nabla^2\psi\|_{\sigma^{-2},K}^2$$

and $\nabla^2\psi$ denotes a general second order derivative operator. As in the case for elliptic equations of second order, it is not hard to see that

$$h^2\|\nabla^2\psi\|_{\sigma^{-2}}^2 \le C\Big(\|G-G_h\|_0^2 + h^2\|\nabla(G-G_h)\|_0^2 + h^2\|\nabla^2 G\|_{\sigma^2}^2\Big). \tag{2.15}$$

Set

$$g(t) = \|G-G_h\|_0 + h\|\nabla(G-G_h)\|_0 + h\|\nabla^2 G\|_{\sigma^2}$$

and

$$\rho(t) = \|\nabla(G-G_h)\|_{\sigma^2}.$$

It follows from (2.13), (2.14), and (2.15) that

$$A(t;G-G_h,\psi) \le \frac{1}{2}\rho^2(t) + C\Big(\int_t^T (\rho(s)+g(s))ds\Big)^2. \tag{2.16}$$

This last inequality combined with (2.11) yields

$$\rho(t) \le Cg(t) + C\int_t^T (\rho(s)+g(s))ds. \tag{2.17}$$

Thus, by applying the back-ward Gronwall's lemma we get

$$\rho(t) \le C\Big(g(t) + \int_t^T g(s)ds\Big).$$

Now (2.8) follows from the last inequality combined with the relation (2.18) below.

To complete the proof of Theorem 2.1, we present here an estimate for the function $g(t)$ defined above.

Lemma 2.1. *Let the function $g(t)$ be defined as before. Then, there exists a constant C such that*

$$g(t) \le Ch\sqrt{\log\frac{1}{h}}(1+|\phi(t)|). \tag{2.18}$$

Proof. It follows from (2.7) that

$$g(t) \le Ch(1+|\phi(t)|) + h\|\nabla^2 G\|_{\sigma^2}. \tag{2.19}$$

Thus, one only needs to estimate $h^2\|\nabla^2 G\|_{\sigma^2}^2$ in order to conclude (2.18). The estimate of this term can be given along the same line as in the case for the elliptic problem of second order. For the sake of completness, an outline is presented as follows.

Let $z=(x,y)$ and $z_0=(x_0,y_0)$. Then, there exists a constant C such that

$$\|\nabla^2 G\|_{\sigma^2}^2 \le C(\|\nabla^2\xi(t)\|_0 + \|\nabla^2\eta(t)\|_0 + \|\nabla G\|_0 + h\|\nabla^2 G\|_0)^2, \tag{2.20}$$

where

$$\xi(t) = (x-x_0)G(t)$$

and

$$\eta(t) = (y - y_0)G(t).$$

Since $G(t)$ is the solution of (2.4), there exist two functions $w_1(t)$ and $w_2(t)$ so that

$$A(t)\xi(t) + \int_t^T B^*(\tau, t)\xi(\tau)d\tau = (x - x_0)\delta_h^{z_0}(z)\phi(t) + w_1(t) \tag{2.21}$$

and

$$A(t)\eta(t) + \int_t^T B^*(\tau, t)\eta(\tau)d\tau = (y - y_0)\delta_h^{z_0}(z)\phi(t) + w_2(t), \tag{2.22}$$

By the structure of $\xi(t)$ and $\eta(t)$ we know that $w_1(t)$ and $w_2(t)$ are made of the first order derivatives of $G(t)$ and its integral in time from t to T. Thus, they can be estimated as follows

$$\|w_i(t)\|_0 \leq C(\|\nabla G(t)\|_0 + \int_t^T \|\nabla G(\tau)\|_0 d\tau) \tag{2.23}$$

for $i = 1, 2$. Applying the a priori estimate (1.10) yields

$$\|\nabla^2 \xi(t)\|_0 \leq C\Big(\|r(t)\|_0 + \int_t^T \|r(\tau)\|_0 d\tau\Big), \tag{2.24}$$

where

$$r(t) = (x - x_0)\delta_h^{z_0}(z)\phi(t) + w_1(t).$$

Clearly,

$$\begin{aligned} \|r(t)\|_0 &\leq \|(x - x_0)\delta_h^{z_0}(z)\phi(t)\|_0 + \|w_1(t)\|_0 \\ &\leq C\Big(1 + |\phi(t)| + \|\nabla G(t)\|_0 + \int_t^T \|\nabla G(\tau)\|_0 d\tau\Big). \end{aligned} \tag{2.25}$$

It follows from (2.24), (2.25), and (2.23) that

$$\|\nabla^2 \xi(t)\|_0 \leq C\Big(1 + |\phi(t)| + \|\nabla G(t)\|_0 + \int_t^T \|\nabla G(\tau)\|_0 d\tau\Big). \tag{2.26}$$

Similarly, the following is valid for $\eta(t)$

$$\|\nabla^2 \eta(t)\|_0 \leq C\Big(1 + |\phi(t)| + \|\nabla G(t)\|_0 + \int_t^T \|\nabla G(\tau)\|_0 d\tau\Big). \tag{2.27}$$

Combining (2.20) with (2.26) and (2.27) gives

$$\|\nabla^2 G(t)\|_{\sigma^2}^2 \leq C\Big((1 + |\phi(t)|) + \|\nabla G(t)\|_0 + \int_t^T \|\nabla G(\tau)\|_0 d\tau\Big)^2, \tag{2.28}$$

where we have used the fact that $h\|\nabla^2 G\|_0 \leq C$ for a positive constant C. It remains to estimate $\|\nabla G(t)\|_0^2 + \int_t^T \|\nabla G(\tau)\|_0^2 d\tau$ in the last inequality. This can be done by applying the a priori estimate (1.11) to the problem (2.4). Thus,

$$\|\nabla G(t)\|_0 + \int_t^T \|\nabla G(\tau)\|_0 d\tau \leq C(1 + \phi(t))\sqrt{\log \frac{1}{h}}. \tag{2.29}$$

It follows from (2.29) and (2.28) that

$$\|\nabla^2 G(t)\|^2_{\sigma^2} \le C(1+|\phi(t)|)^2 \log\frac{1}{h}, \tag{2.30}$$

which, together with (2.19), demonstrates (2.18).

3. Estimate of the Ritz-Volterra Projection

Let's apply the result in § 2 to establish an estimate for the Ritz-Volterra projection V_h in the maximum norm. We shall consider in the rest of this paper the finite element method for a parabolic integro-differential equation, Sobolev equation, and a diffusion equation with non-local boundary condition. The maximum norm error estimates for those problems converge to the estimate for Ritz-Volterra type projections.

For the sake of convenience in the analysis, we consider a projection operator K_h defined by seeking $K_h u(t) \in S_h$ such that

$$V(t; u(t) - K_h u(t), \chi) = C(t; w(t), \chi), \quad \chi \in S_h, \tag{3.1}$$

where $w(t) \in W^{1,\infty} \cap H^1_0$ for each $t \in J$ and $C(t;\cdot,\cdot)$ is a bilinear form on H^1_0 associated with a second order differential operator. It is clear that the case $w(t) = 0$ for every $t \in J$ corresponds to the Ritz-Volterra projection. Our object here is to present an estimate for the projection operator K_h, and then it follows the estimate for the Ritz-Volterra projection. Assume that the bilinear form $C(t;\cdot,\cdot)$ is bounded in H^1_0 and there exists a constant C such that

$$C(t; u, v) \le C\|u\|_\infty \ \|v\|_{2,1}$$

for $u \in L^\infty \cap H^1_0$ and $v \in W^{2,1} \cap H^1_0$.

Lemma 3.1. *Let $u(t) \in W^{1,\infty} \cap H^1_0$ for each $t \in J$. Then, there exists a constant C such that*

$$\|u - K_h u\|_\infty \le C \log\frac{1}{h}\Big(\inf_{\chi\in S_h} |||u-\chi|||_h + |||w|||_h\Big), \tag{3.2}$$

where for any $\phi(t) \in H^1_0$, the mesh dependent norm $|||\phi|||_h$ is defined as follows

$$|||\phi|||_h = \max_{t\in J}(\|\phi(t)\|_\infty + h\|\nabla\phi(t)\|_\infty). \tag{3.3}$$

Proof. Clearly,

$$K_h u - u = (K_h u - \chi) + (\chi - u). \tag{3.4}$$

Thus, it suffices to estimate $\rho = K_h u - \chi$ in order to conclude (3.2). Since $\rho \in S_h$ holds, it follows from (2.4) and the definition of the regularized Dirac δ-function that

$$\int_0^T \rho(z_0, t)\ \phi(t)dt = \int_0^T (K_h u - \chi, \delta_h^{z_0}\phi(t))dt = \int_0^T V^*(t; G_h, K_h u - \chi)dt \tag{3.5}$$

Applying the relation (1.6) to (3.5) gives

$$\int_0^T \rho(z_0,t)\phi(t)dt = \int_0^T V(t; K_h u - \chi, G_h)dt$$

$$= \int_0^T \Big(V(t; u-\chi, G_h) - C(t; w(t), G_h)\Big) dt, \tag{3.6}$$

where we have used the error equation (3.1) in deriving the last equality. To estimate (3.6), let's rewrite (3.6) in the following way

$$\int_0^T \rho(z_0, t)\ \phi(t) dt = I_1 + I_2 - I_3 - I_4, \tag{3.7}$$

where

$$\begin{aligned} I_1 &= \int_0^T V(t; u-\chi, G_h - G), \\ I_2 &= \int_0^T V(t; u-\chi, G), \\ I_3 &= \int_0^T C(t; w(t), G_h - G) dt, \end{aligned}$$

and

$$I_4 = \int_0^T C(t; w(t), G) dt.$$

By Theorem (2.1) and the fact that $\int_0^T |\phi(t)| dt \le 1$,

$$I_1 \le Ch \log \frac{1}{h} \max_{t\in J} \|\nabla(u-\chi)\|_\infty \int_0^T (1+|\phi|) dt \le Ch \log \frac{1}{h} \max_{t\in J} \|\nabla(u-\chi)\|_\infty. \tag{3.8}$$

As far as I_2 was concerned, note that $G(t)$ is the solution of (2.4) and $\|\delta_h^{z_0}\|_{0,1} \le C$ for some constant C. Thus,

$$I_2 = \int_0^T (\delta_h^{z_0} \phi(t), u-\chi) dt \le \int_0^T \|u-\chi\|_\infty \|\delta_h^{z_0}\|_{0,1} \phi(t) dt \le C \max_{t\in J} \|u-\chi\|_\infty. \tag{3.9}$$

The estimates for I_3 and I_4 can be done along the same line. Thus, we have

$$I_3 \le Ch \log \frac{1}{h} \max_{t\in J} \|\nabla w(t)\|_\infty \tag{3.10}$$

and

$$I_4 \le C \log \frac{1}{h} \max_{t\in J} \|w(t)\|_\infty, \tag{3.11}$$

since $\|G\|_{2,1} \le C \log \frac{1}{h}$ holds. Now combining (3.7) with (3.8), (3.9), (3.10), and (3.11) gives

$$\int_0^T \rho(z_0, t)\phi(t) dt \le C \log \frac{1}{h} (|||u-\chi|||_h + |||w|||_h). \tag{3.12}$$

Since ϕ and $z_0 \in \Omega$ can be arbitrary, we have from (3.12) that

$$\|\rho\|_\infty \le C \log \frac{1}{h} (|||u-\chi|||_h + |||w|||_h), \tag{3.13}$$

which, together with (3.4), demonstrates (3.2).

Theorem 3.1. *Under the assumptions of Lemma* 3.1*, there exists a constant* C *such that*

$$\|u - V_h u\|_\infty \le C \log \frac{1}{h} \inf_{\chi \in S_h} |||u - \chi|||_h. \tag{3.14}$$

4. Applications

We intend to apply the result derived in § 3 to some time dependent problems. The Ritz-Volterra type operator is the key structure of those problems. The first problem we are considering is the parabolic integro-differential equation.

4.1. Parabolic Integro-differential equation

Consider the finite element approximation of the problem (1.1) with an operator $V(t)$ defined by (1.2). The problem is termed as a parabolic integro-differential equation with homogeneous Dirichlet boundary condition. For the sake of convenience, we restate the problem as follows. For each $t \in J$, find $u(t) \in H_0^1$ such that

$$\begin{aligned} &(u_t, v) + A(t; u, v) + \int_0^t B(t, s; u(s), v) ds = (f, v), \quad v \in H_0^1 \\ &u(0) = u_0, \quad \text{in } \Omega. \end{aligned} \tag{4.1}$$

It is clear that formally the problem (4.1) is similar to the heat equation with homogeneous Dirichlet boundary condition, except the memory term charactrized by the integration on the bilinear form $B(t; \cdot, \cdot)$. In fact, these two problems share many properties that are used in the analysis in both theoretical and computational aspects (e.g., energy and Gronwall argument etc.) A semi-discrete finite element approximation for (4.1) is defined by seeking $u_h(t) \in S_h$ for each $t \in J$ such that

$$(u_{h,t}, \chi) + V(t; u_h(t), \chi) = (f(t), \chi), \quad \chi \in S_h \tag{4.2}$$

with an initial data $u_h(0) = u_0^h$, where u_0^h is an approximation to the initial value u_0 in the finite element subspace S_h. Our object is to establish a similar analysis to the heat equation of the error in L^∞-norm for this problem. For any function $v(x,t) \in W^{2,\infty}$, denote by $|||v|||_{2,\infty}$ the norm

$$|||v|||_{2,\infty} = \max_{t \in J} \|v\|_{2,\infty}.$$

Then, our first result concerning this method can be stated as follows.

Theorem 4.1. *Let* $u(t)$ *be the exact solution of the parabolic integro-differential equation* (4.1)*, and* $u_h(t)$ *the finite element approximation in* S_h *defined by* (4.2)*. Assume that* $u(t) \in W^{2,\infty}$ *for each* $t \in J$*. Then, there exists a constant* C*, independent of* h *and* u*, such that*

$$\|u(t) - u_h(t)\|_\infty \le C h^2 \log \frac{1}{h} \Big(|||u|||_{2,\infty} + \Big(\int_0^t \|u_t(\tau)\|_2^2 d\tau \Big)^{1/2} \Big), \tag{4.3}$$

provided that the initial approximation u_0^h is the Ritz projection on S_h associated with the operator $A_0 \equiv A(0)$.

Proof. Let $\theta = u_h - V_h u$ and $\eta = V_h u - u$ with $V_h u$ being the Ritz-Volterra projection of u defined by (1.3). It is easy to see that

$$u_h - u = (u_h - V_h u) + (V_h u - u) = \theta + \eta. \tag{4.4}$$

As shown in [14], the function θ can be estimated as follows

$$\|\nabla\theta\|_0 \le Ch^2\Big(\|v\|_2 + \Big(\int_0^t \|u_t(\tau)\|_2^2 d\tau\Big)^{1/2}\Big).$$

Thus,

$$\|\theta(t)\|_\infty \le C\Big(\log\frac{1}{h}\Big)^{1/2}\|\nabla\theta\|_0 \le Ch^2\Big(\log\frac{1}{h}\Big)^{1/2}\Big(\|v\|_2 + \Big(\int_0^t \|u_t(\tau)\|_2^2 d\tau\Big)^{1/2}\Big), \tag{4.5}$$

since $\theta \in S_h$. To estimate $\eta = V_h u - u$ in the L^∞-norm, one can apply (3.14) combined with the interpolation theory to get

$$\|\eta\|_\infty \le Ch^2 \log\frac{1}{h}|||u|||_{2,\infty}. \tag{4.6}$$

Thus, combining (4.4) with (4.5) and (4.6) yields the conclusion of Theorem 3.1.

Our next goal in this section is to relax the assumption on the approximation u_0^h of the initial value u_0. We intend to derive a sharp estimate for this method as long as the initial data is approximated with a certain accuracy compatible to the interpolation. To be more precise, let the differential operator A be time independent. Assume that the initial value satisfies the following approximation properties

$$\|u_0 - u_0^h\|_0 \le Ch^2\|u_0\|_2, \tag{4.7}$$
$$\|u_0 - u_0^h\|_{i,\infty} \le Ch^{2-i}\|u_0\|_{2,\infty}, \quad i = 0, 1. \tag{4.8}$$

Then, a point-wise error estimate for this method can be given as follows.

Theorem 4.2. *Assume that the solution of* (4.1) $u(t) \in W^{2,\infty}$ *for each* $t \in J$. *Then, there exists a constant* C *such that*

$$\|u(t) - u_h(t)\|_\infty \le Ch^2 \log\frac{1}{h}\Big(\|u_0\|_{2,\infty} + |||u_t|||_{2,\infty}\Big). \tag{4.9}$$

The proof of Theorem 4.2 will be given along a similar idea employed in [20, 21] (see also [1] [25]) for the heat equation. But we need to do a little preparation before presenting the proof. Let $A_h : S_h \to S_h$ be a linear operator defined by

$$(A_h\phi, \psi) = A(\phi, \psi), \quad \phi, \psi \in S_h.$$

Similarly, one could define an operator $B_h(t,\tau)$ from the bilinear form $B(t,\tau;\cdot,\cdot)$. Here we would like to recall that $A(\cdot,\cdot)$ and $B(t,\tau;\cdot,\cdot)$ are respectively the bilinear forms associated with operators A and $B(t,\tau)$. Next, let $T_h : L^2(\Omega) \to S_h$ be the approximation operator of $T = A^{-1}$ defined by

$$A(T_h f, \chi) = (f, \chi), \quad \chi \in S_h.$$

It is easy to see that $T_h = A_h^{-1}$ on S_h and

$$\|(T_h - T)f\|_0 + h\|\nabla(T_h - T)f\|_0 \le Ch^2\|f\|_0. \tag{4.10}$$

Lemma 4.1. *There exists a constant C such that*

$$\|T_h B_h \chi\|_i \le C\|\chi\|_i, \quad \chi \in S_h \tag{4.11}$$

for $i = 0, 1$.

Proof. We shall prove the case $i = 0$ only, since the proof for $i = 1$ is similar. For any $\psi \in L^2$ we have from (4.10) and the definition of T_h and $B_h(t,\tau)$ that

$$\begin{aligned}(T_h B_h \chi, \psi) = (B_h\chi, T_h\psi) &= B(t,\tau;\chi, T_h\psi) = B(t,\tau;\chi,(T_h - T)\psi) + B(t,\tau;\chi,T\psi)\\ &= C\|\chi\|_1\|(T_h - T)\psi\|_1 - (\chi, B^*(t,\tau)T\psi)\\ &\le Ch^{-1}\|\chi\|_0 h\|\psi\|_0 + C\|\chi\|_0 \ \|\psi\|_0 \le C\|\chi\|_0 \ \|\psi\|_0.\end{aligned}$$

Thus, (4.10) follows.

Remark 4.1. *It follows from* (4.10) *and the weak Sobolev inequality* [17] *that*

$$\|T_h B_h \chi\|_\infty \le C\Big(\log\frac{1}{h}\Big)^{1/2}\|\chi\|_1. \tag{4.12}$$

Let $E_h(t)$ be the semi-group generated by the operator A_h. For our purpose, we would like to cite some estimates regarding this operator. A complete analysis can be found from [21].

Lemma 4.2. *There exists a constant C such that for any $\chi \in S_h$,*

$$\|E_h(t)\chi\|_0 \le C\|\chi\|_0, \quad \Big\|\frac{d}{dt}E_h(t)\chi\Big\|_0 \le \frac{C}{t+h^2}\|\chi\|_0, \tag{4.13}$$

$$\|E_h(t)\chi\|_\infty \le C\Big(\log\frac{1}{h}\Big)^{1/2}\|\chi\|_\infty. \tag{4.14}$$

Now we are ready to prove Theorem 4.2.

Proof of Theorem 4.2: *Note that the error $u_h - u$ has been decomposed into θ and η as in* (4.4)*. The component η can be estimated by* (4.6)*. Thus, we need only to deal with the estimate for θ. It is not hard to see that θ satisfies the following equation*

$$(\theta_t, \chi) + V(t;\theta,\chi) = -(\eta_t, \chi), \quad \chi \in S_h$$

or in other words

$$\theta_t + A_h\theta + \int_0^t B_h(t,\tau)\theta(\tau)d\tau = -P_0\eta_t, \tag{4.15}$$

where P_0 is the L^2 projection onto S_h. Thus, by applying Duhamel's principle we get

$$\begin{aligned}\theta(t) &= E_h(t)\theta(0) - \int_0^t E_h(t-\tau)P_0\eta_t(\tau)d\tau - \int_0^t E_h(t-\tau)\int_0^\tau B_h(\tau,s)\theta(s)dsd\tau\\ &= K_1 - K_2 - K_3.\end{aligned} \tag{4.16}$$

The estimates for $K_i's$ can be given as follows. It follows from (4.14) *and* (3.14) *that*

$$\begin{aligned}\|K_1\|_\infty &\le C\log\frac{1}{h}\ \|\theta(0)\|_\infty \le C\log\frac{1}{h}(\|u_0^h - V_h u(0)\|_\infty)\\ &\le C\log\frac{1}{h}(\|\eta(0)\|_\infty + \|u_0 - u_0^h\|_\infty) \le Ch^2\Big(\log\frac{1}{h}\Big)^2\|u_0\|_{2,\infty},\end{aligned} \tag{4.17}$$

where the approximation assumption (4.8) *has been used as well. To estimate K_2, assume for the moment that there exists a constant C such that*

$$\|\eta_t\|_\infty \le Ch^2\log\frac{1}{h}(\|u_0\|_{2,\infty} + |||u_t|||_{2,\infty}). \tag{4.18}$$

Then, the L^∞-stability of the L^2 projection P_0 implies that

$$\begin{aligned}\|K_2\|_\infty &\le C\log\frac{1}{h}\int_0^t \|P_0\eta_t\|_\infty d\tau \le C\log\frac{1}{h}\int_0^t\|\eta_t\|_\infty d\tau\\ &\le Ch^2\Big(\log\frac{1}{h}\Big)^2(\|u_0\|_{2,\infty} + |||u_t|||_{2,\infty}).\end{aligned} \tag{4.19}$$

As far as K_3 was concerned, we see from integration by parts that

$$\begin{aligned}K_3 = &-\int_0^t T_hB_h(t,s)\theta(s)ds + \int_0^t E_h(t-\tau)T_hB_h(\tau,\tau)\theta(\tau)d\tau\\ &+\int_0^t E_h(t-\tau)\int_0^\tau T_hB_{h,\tau}(\tau,s)\theta(s)dsd\tau,\end{aligned}$$

so that by Lemmas 4.1 *and* 4.2

$$\|K_3\|_\infty \le C\Big(\log\frac{1}{h}\Big)^{1/2}\int_0^t\|\nabla\theta\|_0 d\tau.$$

Recall from [14] *that*

$$\int_0^t \|\nabla\theta\|_0 d\tau \le Ch^2\Big(\|u\|_2 + \int_0^t\|u_t\|_2 d\tau\Big).$$

Thus,

$$\|K_3\|_\infty \le Ch^2\Big(\log\frac{1}{h}\Big)^{1/2}\Big(\|u\|_2 + \int_0^t\|u_t\|_2 d\tau\Big). \tag{4.20}$$

Combining (4.16) *with* (4.17), (4.19) *and* (4.20) *yields*

$$\|\theta\|_\infty \le Ch^2\Big(\log\frac{1}{h}\Big)^2(\|u_0\|_{2,\infty} + \|u\|_2 + |||u_t|||_{2,\infty}),$$

which, along with the estimate for η, demonstrates Theorem 4.2.

It now remains to prove (4.18). Clearly,

$$A(\eta,\chi) + \int_0^t B(t,\tau;\eta(\tau),\chi)d\tau = 0, \quad \chi\in S_h.$$

Thus, by taking differentiation we get

$$A(\eta_t, \chi) + B(t, t; \eta, \chi) + \int_0^t B_t(t, \tau; \eta(\tau), \chi) d\tau = 0, \quad \chi \in S_h. \tag{4.21}$$

Set $w = u_t$ and $w_h = u_{h,t}$. Then,

$$u = \int_0^t w(\tau) d\tau + u_0, \quad u_h = \int_0^t w_h(\tau) d\tau + u_0^h$$

Substituting the last two equations back to (4.21) gives

$$W(t; w - w_h, \chi) = D(t; u_0^h - u_0, \chi), \tag{4.22}$$

where the bilinear forms $W(t; \cdot, \cdot)$ and $D(t; \cdot, \cdot)$ are defined respectively by

$$\begin{aligned} W(t; \phi, \psi) &= A(\phi, \psi) + \int_0^t B(t, t; \phi(\tau), \psi) d\tau + \int_0^t \int_0^\tau B_t(t, \tau; \phi(s), \psi) ds d\tau \\ D(t; \phi, \psi) &= B(t, t; \phi, \psi) + \int_0^t B_t(t, \tau; \phi(\tau), \psi) d\tau. \end{aligned}$$

The bilinear form $W(t; \cdot, \cdot)$ is of Ritz-Volterra type. Thus, the Lemma 3.1 is applicable to the problem (4.22). Thus,

$$\|w - w_h\|_\infty \leq C \log \frac{1}{h} \Big(\inf_{\chi \in S_h} |||w - \chi|||_h + |||u_0 - u_0^h|||_h \Big). \tag{4.23}$$

Now (4.18) follows from (4.23).

We shall now show the following type error estimates in which there is no time derivatives involved [21].

Theorem 4.3. *There exists a constant $C > 0$ such that*

$$\|u(t) - u_h(t)\|_0 \leq Ch^2 \Big(1 + \log\Big(1 + \frac{t}{h^2}\Big)\Big) \sup_{0 \leq s \leq t} \|u(s)\|_2. \tag{4.24}$$

Proof. First of all, we see from [14] that

$$\|\eta\|_0 \leq Ch^2 \Big(\|u\|_2 + \int_0^t \|u\|_2 d\tau \Big) \leq Ch^2 \sup_{0 \leq s \leq t} \|u(s)\|_2.$$

Furthermore, θ can be expressed as

$$\begin{aligned} \theta(t) =& E_h(t)\theta(0) - E_h(t) P_0 \eta(0) - P_0 \eta(t) - \int_0^t \frac{d}{dt} E_h(t - \tau) P_0 \eta(\tau) d\tau \\ &+ \int_0^t T_h B_h(t, s) \theta(s) ds - \int_0^t E_h(t - \tau) T_h B_h(\tau, \tau) \theta(\tau) d\tau \\ &- \int_0^t E_h(t - \tau) \int_0^\tau T_h B_{h,\tau}(\tau, s) \theta(s) ds d\tau \end{aligned} \tag{4.25}$$

Thus, we obtain by Lemmas 4.1 and 4.2 that

$$
\begin{aligned}
|\theta(t)\|_0 &\leq C\{\|\theta(0)\|_0 + \|\eta(0)\|_0 + \|\eta\|_0\} + C\int_0^t \Big(\|\theta\|_0 + \frac{\|\eta(s)\|_0}{s+h^2} ds\Big) \\
&\leq Ch^2\Big(1 + \log\Big(1 + \frac{t}{h^2}\Big)\Big) \sup_{0\leq s\leq t} \|u(s)\|_2 + C\int_0^t \|\theta\|_0 d\tau.
\end{aligned}
$$

An argument of Gronwall's lemma will yield

$$
\|\theta\|_0 \leq Ch^2\Big(1 + \log\Big(1 + \frac{t}{h^2}\Big)\Big) \sup_{0\leq s\leq t} \|u(s)\|_2, \tag{4.26}
$$

which concludes Theorem 4.3.

4.2. Sobolev Equation

Consider following problem for the Sobolev equation. Find $u(t)$ for each $t \in J$ such that

$$
\begin{aligned}
A(t)u_t + B(t)u(t) =& f(t), && \text{in } \Omega, \\
u(x,0) =& v(x), && \text{in } \Omega, \\
u(x,t) =& 0, && \text{on } \partial\Omega,
\end{aligned} \tag{4.27}
$$

where $A(t)$ is a symmetric positive definite elliptic operator and $B(t)$ an arbitrary differential operator of second order with smooth coefficients. Assume that $f(t) \in L^2$ for each $t \in J$. A finite element method for (4.27) can be defined by seeking $u_h(t) \in S_h$ for each $t \in J$ such that

$$
A(t; u_{h,t}, \chi) + B(t; u_h, \chi) = (f, \chi),\ \ \chi \in S_h,\ \ t > 0 \tag{4.28}
$$

with initial value $u_h(0) = v_h$. Here $A(t;\cdot,\cdot)$ and $B(t;\cdot,\cdot)$ are the bilinear forms associated with the operators $A(t)$ and $B(t)$, respectively, and v_h is an approximation of the initial value v in S_h.

Set $w(t) = u_t(t)$ and $w_h(t) = u_{h,t}(t)$. Then,

$$
u(t) = \int_0^t w(\tau)d\tau + v, \quad u_h(t) = \int_0^t w_h(\tau)d\tau + v_h. \tag{4.29}
$$

Thus, substituting (4.29) back into (4.28) yields

$$
A(t; w(t) - w_h(t), \chi) + \int_0^t B(t; w(\tau) - w_h(\tau), \chi)d\tau = B(t; v_h - v, \chi),\ \ \chi \in S_h, \tag{4.30}
$$

where a corresponding weak form for (4.27) has been used as well. It is clear that the equation (4.30) is of form (3.1). Thus, the result of Lemma 3.1 could be used to estimate $w - w_h$. The estimate is then summarized as follows.

Theorem 4.3. *Assume that $u(t)$ is the unique solution of* (4.27) *and $u_h(t)$ its discrete analogue defined by* (4.28). *Then, there exists a constant C such that*

$$
\|u - u_h\|_\infty + \|u_t - u_{h,t}\|_\infty \leq Ch^2 \log\frac{1}{h}(\|v\|_{2,\infty} + |||u_t|||_{2,\infty}), \tag{4.31}
$$

provided that $v, u_t \in W^{2,\infty}$ *for each* $t \in J$ *and*

$$\|v - v_h\|_\infty \le Ch^2 \log \frac{1}{h} \|v\|_{2,\infty}$$

holds.

4.3. A Diffusion Equation

Consider the following heat equation with non-local boundary condition and initial value. The problem reads to seek $u(t)$ for each $t \in J$ such that

$$\begin{aligned} u_t - \Delta u &= f, && \text{in } \Omega, \\ u(x,0) &= v(x), && x \in \Omega, \\ \frac{\partial u}{\partial \mu} + \int_0^t K(t,\tau)u(\tau)d\tau &= 0, && \text{on } \partial\Omega, \end{aligned} \tag{4.32}$$

where $\mu = (\mu_1, \mu_2)$ denotes the outer-ward normal direction on $\partial\Omega$ and K, f, and v are known functions. A weak form of (4.32) is defined by finding $u(t) \in H^1$ for each $t \in J$ such that

$$(u_t, \psi) + A(u, \psi) + \int_0^t \langle K(t,\tau)u(\tau), \psi\rangle d\tau = (f, \psi), \;\; \psi \in H^1(\Omega) \tag{4.33}$$

with initial value $u(\cdot, 0) = v$, where

$$A(u,v) = \int_\Omega \nabla u \cdot \nabla v dx, \quad \langle f, g\rangle = \int_{\partial\Omega} fg ds.$$

Thus, a finite element approximation can be defined by solving $u_h(t) \in S_h$ from the following linear system.

$$(u_{h,t}, \chi) + A(u_h, \chi) + \int_0^t \langle K(t,\tau)u_h(\tau), \chi\rangle d\tau = (f, \chi), \;\; \chi \in S_h, \;\; t > 0 \tag{4.34}$$

with initial value $u_h(0) = v_h$, where v_h is an appropriate approximation of v in the finite element subspace S_h. Here the finite element subspace S_h is the piecewise linear one associated with H^1. The method (4.34) and the corresponding estimates in L^2 and H^1 norms have been considered in [5]. There, a Ritz-Volterra type projection was introduced as follows. Find $F_h u(t) \in S_h$ for each $t \in J$ such that

$$A(u - F_h u, \chi) + \lambda(u - F_h u, \chi) + \int_0^t \langle K(t,\tau)(u(\tau) - F_h u(\tau)), \chi\rangle d\tau = 0, \;\; \chi \in S_h, \tag{4.35}$$

where λ is a sufficiently large positive constant. The purpose of introducing the λ-term is to enhance the bilinear form $A(\cdot,\cdot)$ so that the resulting one is coercive in H^1. It is not hard to see that the operator F_h is well defined for any positive λ. The error estimate for this projection when λ is large enough can also be derived easily. However, the estimate may dependent upon λ. Thus, the limiting case of $\lambda \to \infty$ is less interesting in our analysis. Actually, the projection operator F_h is of Ritz-Volterra type for sufficiently large λ and hence, the result of Lemmas 3.1 or 3.2 can be applied to F_h. This yields the existence of a constant C such that

$$\|u - F_h u\|_\infty \le Ch^2 \log \frac{1}{h} |||u|||_{2,\infty}, \tag{4.36}$$

provided that $u(t) \in W^{2,\infty}$ for each $t \in J$.

We are now ready to establish the error estimate in maximum norm for the finite element method (4.34). The result can be stated as follows.

Theorem 4.4. *Let u be the unique solution of* (4.32) *and u_h the finite element approximation defined by* (4.34). *Then, there exists a constant C such that*

$$\|u - u_h\|_\infty \le Ch^2 \log \frac{1}{h}\Big(|||u|||_{2,\infty} + \Big(\int_0^t \|u_t\|_2^2 d\tau \Big)^{1/2} \Big), \tag{4.37}$$

provided that the initial approximation v_h is taken to be the projection $F_h v$.

Proof. Let $\theta = u_h - F_h u$ and $\eta = F_h u - u$. Then,

$$u_h - u = \theta + \eta. \tag{4.38}$$

Because of (4.36) it suffices to estimate θ. By (4.33), (4.34) and (4.35) one obtains

$$(\theta_t, \chi) + A(\theta, \chi) + \int_0^t \langle K(t,\tau)\theta(\tau), \chi\rangle d\tau = (\eta_t, \chi) - \lambda(\eta, \chi), \quad \chi \in S_h.$$

By letting $\chi = \theta_t$,

$$\|\theta_t\|_0^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\theta\|_0^2 = (\eta_t - \lambda\eta, \theta_t) - \int_0^t \langle K(t,\tau)\theta(\tau), \theta_t(t)\rangle d\tau$$

Notice that $\theta(0) = 0$. Thus, by integration on t

$$\begin{aligned}\int_0^t \|\theta_t\|_0^2 d\tau + \frac{1}{2}\|\nabla\theta\|_0^2 \le& \int_0^t (\|\eta_t\|_0 + \lambda\|\eta\|_0)\|\theta_t\|_0 d\tau \\ &- \int_0^t \int_0^s \langle K(s,\tau)\theta(\tau), \theta_s(s)\rangle d\tau ds = H_1 + H_2.\end{aligned} \tag{4.39}$$

We have for H_1 that

$$H_1 \le \int_0^t \|\theta_t\|_0^2 d\tau + C\int_0^t (\|\eta_t\|_0^2 + \|\eta\|_0^2) d\tau. \tag{4.40}$$

By integration by parts H_2 can be rewritten as follows

$$\begin{aligned}H_2 =& -\int_0^t \langle K(t,\tau)\theta(\tau), \theta(t)\rangle d\tau \\ &+ \int_0^t \langle K(\tau,\tau)\theta(\tau), \theta(\tau)\rangle d\tau + \int_0^t \int_0^s \langle K_s(s,\tau)\theta(\tau), \theta(s)\rangle d\tau ds.\end{aligned}$$

Thus, it follows from the trace theorem that

$$H_2 \le \frac{1}{4}\|\nabla\theta\|_0^2 + C\Big(\|\theta\|_0^2 + \int_0^t \|\theta\|_1^2\Big)d\tau\Big). \tag{4.41}$$

Substituting (4.40) and (4.41) into (4.39) and using Gronwall's lemma yield

$$\|\theta\|_1^2 \le C\Big(\|\theta\|_0^2 + \int_0^t (\|\eta_t\|_0^2 + \|\theta\|_0^2 + \|\eta\|_0^2)d\tau\Big).$$

By recalling Theorem 3.1 of [5] we have that

$$\|\theta\|_0^2 + \int_0^t (\|\eta_t\|_0^2 + \|\eta\|_0^2)d\tau \le Ch^4\Big(\|v\|_2^2 + \int_0^t \|u_t\|_2^2 d\tau\Big).$$

Thus,

$$\|\theta\|_1 \le Ch^2\Big(\|v\|_2 + \Big(\int_0^t \|u_t\|_2^2 d\tau\Big)^{1/2}\Big)$$

Finally, by the weak Sobolev inequality

$$\|\theta\|_\infty \le Ch^2\Big(\log\frac{1}{h})^{1/2}\Big(\|v\|_2 + \Big(\int_0^t \|u_t\|_2^2 d\tau\Big)^{1/2}\Big). \tag{4.22}$$

Combining (4.38) with (4.36) and (4.42) gives (4.37).

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