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THE STABILITY ANALYSIS OF THE θ-METHODS FOR DELAY DIFFERENTIAL EQUATIONS*

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Abstract

This paper deals with the stability analysis of $\theta-$methods for the numerical solution of delay differential equations (DDEs). We focus on the behaviour of such methods in the solution of the linear test equation $y'(t) = a(t)y(t) + b(t)y(t-\tau)$, where $\tau > 0$, $a(t)$ and $b(t)$ are functions from R to C. It is proved that the linear $\theta-$method and the one-leg $\theta-$method are TGP-stable if and only if $\theta = 1$.

1. Introduction

This paper deals with the numerical solution of the following initial- value problems

$$\begin{cases} y'(t) &= f(t, y(t), y(t-\tau(t))) \qquad t \geq 0, \\ y(t) &= \phi(t) \qquad t \leq 0, \end{cases} \tag{1.1}$$

where $y: R \to C$, $\tau(t) \geq 0$ is the delay term , $\phi(t): R \to C$ is the initial function, whereas $y(t)$ is unknown for $t > 0$.

Let us consider the following linear delay differential equation:

$$\begin{cases} y'(t) &= ay(t) + by(t-\tau) \qquad t \geq 0 \\ y(t) &= \varphi(t) \qquad -\tau \leq t \leq 0, \end{cases} \tag{1.2}$$

where $y: R \to C$, a, b are complex, $\tau > 0$ is a constant delay. $\varphi(t)$ denotes a given function on $[-\tau, 0]$.

It is well-known that (see[1,2]), if $\varphi(t)$ is continuous and if

$$|b| < -\mathrm{Re}(a), \tag{1.3}$$

then the solution $y(t)$ to (1.2) tends to zero as $t \to \infty$ for every $\tau > 0$. In this case the solution $y(t)$ to (1.2) is called asymptotically stable.

Concerning numerical solution of (1.2), let's recall Barwell's (see[3]) definitions of P- and GP-stability .

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Definition 1.1. *A numerical method for DDEs is called P-stable if, for all coefficients a, b satisfying* (1.3)*, the numerical solution $y_n \sim y(t_n)$ of* (1.2) *at the mesh points $t_n = nh$, $n \geq 0$, satisfies*

$$y_n \to 0, \quad as \quad n \to \infty,$$

for every stepsize h such that $h = \tau/m$, m is a positive integer.

Definition 1.2. *A numerical method for DDEs is called GP-stable if, under condition* (1.3)*, $y_n \to 0$, as $n \to \infty$ for every stepsize $h > 0$.*

Consider the following linear test equation which was introduced in [9]:

$$\begin{cases} y'(t) &= a(t)y(t) + b(t)y(t-\tau) \qquad t \geq 0, \\ y(t) &= \phi(t) \qquad -\tau \leq t \leq 0, \end{cases} \tag{1.4}$$

where $y : [-\tau, +\infty) \to C$, $a, b : [0, +\infty) \to C$ and $\tau > 0$, and the solution $y(t)$ of (1.4) is bounded by $max_{-\tau \leq t \leq 0} |\phi(t)|$, provided that, for every $t \geq 0$,

$$|b(t)| \leq -\mathrm{Re}(a(t)). \tag{1.5}$$

In [9], Torelli introduced two definitions of stability based on the test equation (1.4) as follows:

Definition 1.3. *A numerical method for DDEs is said to be PN-stable if, under the condition* (1.5)*, the numerical solution y_n of* (1.4) *is such that*

$$|y_n| \leq \max_{-\tau \leq t \leq 0} |\phi(t)| \tag{1.6}$$

for every $n \geq 0$ and for every stepsize $h = \tau/m$, where m is a positive integer.

Definition 1.4. *A numerical method for DDEs is called GPN-stable if, under the condition* (1.5)*, the numerical solution of* (1.4) *satisfies* (1.6) *for every $n \geq 0$ and for every stepsize $h > 0$.*

The numerical stability of θ-methods and Runge-Kutta methods have been widely investigated in [5,7,8,12]. The numerical stability of the $\theta-$ methods with respect to the linear test equation (1.2) have been carefully studied in [7]. In [9], Torelli has dealt with numerical stability based on Definition 1.3 and 1.4 of the $\theta-$methods with respect to the linear test equation (1.4).

It is the purpose of this paper to investigate the asymptotic stability behaviour of the theoretical solution and the numerical solution of (1.4). In section 2 , we derive a sufficient condition for (1.4) such that the solution to (1.4) is asymptotically stable. In section 3 and section 4, it is proven that the linear $\theta-$method and the one-leg $\theta-$method are TGP-stable if and only if $\theta = 1$.

2. Asymptotic Stability of the Theoretic Solution of DDEs

First of all, let us consider the following nonlinear systems

$$\begin{cases} y'(t) &= f(t, y(t), y(t-\tau)) \qquad t \geq 0, \\ y(t) &= \phi(t) \qquad t \leq 0, \end{cases} \tag{2.1}$$

and

$$\begin{cases} z'(t) &= f(t, z(t), z(t-\tau)) \qquad t \geq 0, \\ z(t) &= \psi(t) \qquad t \leq 0, \end{cases} \tag{2.2}$$

where $f : [0, \infty] \times C^s \times C^s \to C^s$, $y(t)$, $z(t) : R \to C^s, \tau > 0$.

Theorem 2.1. *Assume that $\phi(t)$ and $\psi(t)$ are continuous and*

$$\mathrm{Re} < f(t, y_1, u) - f(t, y_2, u), y_1 - y_2 > \leq \sigma(t)||y_1 - y_2||^2, \tag{2.3}$$

$$\forall t \in \bar{R}, \forall u, y_1, y_2 \in C^s,$$

$$||f(t, y, u_1) - f(t, y, u_2)|| \leq \gamma(t)||u_1 - u_2||, \tag{2.4}$$

$$\forall t \in \bar{R}, \forall y, u_1, u_2 \in C^s,$$

$\sigma(t)$ and $\gamma(t)$ are continuous and satisfy

$$\gamma(t) \leq -q\sigma(t), \quad 0 \leq q < 1, \tag{2.5}$$

and

$$\sigma(t) \leq -\beta < 0, \quad \bar{R} = [0, \infty), \quad ||x||^2 =< x, x >, \quad x \in C^s.$$

If the solutions of (2.1) *and* (2.2) *exist uniquely, then*

$$lim_{t\to\infty}||y(t) - z(t)|| = 0. \tag{2.6}$$

Remark 1. Before proving the theorem, observe that, if

$$f(t, y(t), y(t-\tau)) = ay(t) + by(t-\tau)$$

as in (1.2), then $\sigma(t) = \mathrm{Re}(a), \gamma(t) = |b|$. In this case , if $|b| < -\mathrm{Re}(a)$,by theorem 2.1, we obtain at once that $\lim_{t\to\infty} y(t) = 0$, i.e., the solution to (1.4) is asymptotically stable.

Proof of Theorem 2.1. According to the definition of the norm on C^s, we have

$$\begin{aligned} 1/2(d/dt)(||y(t) - z(t)||^2) &= \mathrm{Re} < y'(t) - z'(t), y(t) - z(t) > \\ &= \mathrm{Re} < f(t, y(t), y(t-\tau)) - f(t, z(t), z(t-\tau)), y(t) - z(t) > \\ &= \mathrm{Re} < f(t, y(t), y(t-\tau)) - f(t, z(t), y(t-\tau)), y(t) - z(t) > \\ &+ \mathrm{Re} < f(t, z(t), y(t-\tau)) - f(t, z(t), z(t-\tau)), y(t) - z(t) > . \end{aligned} \tag{2.7}$$

Application of Schwartz's inequality yields

$$\begin{aligned} 1/2(d/dt)(||y(t) - z(t)||^2) \leq\ & \sigma(t)||y(t) - z(t)||^2 \\ & + \gamma(t)||y(t) - z(t)||||y(t-\tau) - z(t-\tau)||. \end{aligned} \tag{2.8}$$

Let $Y(t) = ||y(t) - z(t)||$. Then

$$1/2(d/dt)(Y(t)^2) \leq \sigma(t)Y(t)^2 + \gamma(t)Y(t)Y(t-\tau),$$

which implies

$$Y(t)Y'(t) \le \sigma(t)Y(t)^2 + \gamma(t)Y(t-\tau)Y(t)$$

(Note that $Y(t) > 0$ for every $t > 0$ because we assume that the function f is such that (2.1) has only a unique solution for every initial condition. See [9]). Then

$$\begin{cases} Y'(t) & \le \sigma(t)Y(t) + \gamma(t)Y(t-\tau) \qquad t \ge 0, \\ Y(t) & = ||\phi(t) - \psi(t)|| = \Phi(t) \qquad t \le 0. \end{cases} \tag{2.9}$$

Consider the following differential equation

$$\begin{cases} \bar{Y}'(t) & = \sigma(t)\bar{Y}(t) + \gamma(t)\bar{Y}(t-\tau) \qquad t \ge 0, \\ \bar{Y}(t) & = \Phi(t) \qquad t \le 0. \end{cases} \tag{2.10}$$

When $t \in [0, \tau]$, then (2.10) reads

$$\begin{cases} \bar{Y}'(t) & = \sigma(t)\bar{Y}(t) + \gamma(t)\Phi(t-\tau) \qquad t \in [0, \tau], \\ \bar{Y}(t) & = \Phi(0), \end{cases} \tag{2.10'}$$

The solution of (2.10') is

$$\bar{Y}(t) = e^{A_0(t)}\Phi(0) + e^{A_0(t)} \int_o^t e^{-A_0(x)}\gamma(x)\Phi(x-\tau)dx, \tag{2.11}$$

where

$$A_i(t) = \int_{i\tau}^t \sigma(x)dx, t \in [i\tau, (i+1)\tau], i = 0, 1, 2, \cdots.$$

Since

$$\gamma(t) \le -q\sigma(t), 0 \le q < 1, \qquad \sigma(t) \le -\beta < 0,$$

we have

$$\bar{Y}(t) \le (e^{-\beta t} + (1 - e^{-\beta t})q)M = G_0(t)M \tag{2.12}$$

where

$$M = \max_{-\tau \le t \le 0} \quad \Phi(t), \quad G_0(t) = e^{-\beta t} + (1 - e^{-\beta t})q.$$

When $t \in [\tau, 2\tau]$, then (2.10') reads

$$\begin{cases} \bar{Y}'(t) & = \sigma(t)\bar{Y}(t) + \gamma(t)\bar{Y}(t-\tau) \qquad t \in [\tau, 2\tau], \\ \bar{Y}(\tau) & = \bar{Y}(\tau). \end{cases} \tag{2.13}$$

Then the solution of (2.13) is

$$\begin{aligned}
\bar{Y}(t) &= e^{A_1(t)}\bar{Y}(\tau) + e^{A_1(t)}\int_{\tau}^{t} e^{-A_1(x)}\gamma(x)\bar{Y}(x-\tau)dx \\
&\le \{e^{A_1(t)}G_0(\tau) + e^{A_1(t)}\int_{\tau}^{t} e^{-A_1(x)}\gamma(x)G_0(x-\tau)dx\}M \\
&\le \{e^{A_1(t)}G_0(\tau) + qe^{A_1(t)}G_0(\xi_0)\int_{\tau}^{t} e^{-A_1(x)}(-\sigma(x))dx\}M \qquad (\xi_0 \in [0,\tau]) \\
&\le \{e^{A_1(t)}G_0(\tau) + qG_0(0)(1-e^{A_1(t)})\}M \\
&\le \{e^{-\beta(t-\tau)}G_0(\tau) + qG_0(0)(1-e^{-\beta(t-\tau)})\}M \\
&= \{e^{-\beta(t-\tau)}G_0(\tau) + q(1-e^{-\beta(t-\tau)})\}M \\
&= G_1(t-\tau)M, \qquad (2.14)
\end{aligned}$$

where $G_1(t) = e^{-\beta t}G_0(\tau) + q(1-e^{-\beta t})$.

For $t \in [2\tau, 3\tau]$,we can see that

$$\begin{aligned}
\bar{Y}(t) &= e^{A_2(t)}\bar{Y}(2\tau) + e^{A_2(t)}\int_{2\tau}^{t} e^{-A_2(x)}\gamma(x)\bar{Y}(x-\tau)dx \\
&\le \{e^{A_2(t)}G_1(\tau) + qG_1(\xi_1)(1-e^{A_2(t)})\}M \qquad (\xi_1 \in [0,\tau]) \\
&\le \{e^{A_2(t)}G_0(\tau) + qG_0(\tau)(1-e^{A_2(t)})\}M \\
&\le G_0(\tau)\{e^{A_2(t)} + q(1-e^{A_2(t)})\}M \\
&\le G_0(\tau)\{e^{-\beta(t-2\tau)} + q(1-e^{-\beta(t-2\tau)})\}M \\
&\le G_0(\tau)G_0(t-2\tau)M. \qquad (2.15)
\end{aligned}$$

When $t \in [3\tau, 4\tau]$, we have

$$\bar{Y}(t) \le G_0(\tau)G_1(t-3\tau)M. \qquad (2.16)$$

By induction, we obtain

$$Y(t) \le \bar{Y}(t) \le [G_0(\tau)]^k G_0(t-2k\tau)M \quad \text{for} \ \ t \in [2k\tau, (2k+1)\tau],$$

$$Y(t) \le \bar{Y}(t) \le [G_0(\tau)]^k G_1(t-(2k+1)\tau)M \ \ \text{for} \ \ t \in [(2k+1)\tau, 2(k+1)\tau].$$

Therefore we have $\lim_{t\to\infty} Y(t) = \lim_{t\to\infty} ||y(t)-z(t)|| = 0$.

This completes the proof of this theorem.

As a special case of (2.1), in (1.4), $\sigma(t) = \mathrm{Re}(a(t)), \gamma(t) = |b(t)|$, then (2.5) becomes (2.5'), where $\mathrm{Re}(a(t))$ and $|\ b(t)|$ are continuous and

$$|b(t)| \le -q\mathrm{Re}(a(t)), \quad 0 \le q < 1, \quad \mathrm{Re}(a(t)) \le -\beta < 0. \qquad (2.5')$$

Corollary 2.2. *Suppose $a(t)$ and $b(t)$ satisfy* (2.5') *and $\phi(t)$ is continuous, then the solution to* (1.4) *is asymptotically stable.*

We now introduce some new definitions of stability based on the linear test equation (1.4).

Definition 2.3. *A numerical method for DDEs is called TP-stable if, under the condition* (2.5'), *the numerical solution* y_n *of* (1.4) *satisfies*

$$\lim_{n\to\infty} y_n = 0 \tag{2.17}$$

for every stepsize h *such that* $h = \tau/m$, *where* $m \geq 1$ *is a positive integer.*

Definition 2.4. *A numerical method for DDEs is called TGP-stable if, under the condition* (2.5'), *the numerical solution* y_n *of* (1.4) *satisfies* (2.17) *for every stepsize* $h > 0$.

From Definition 2.3 and 2.4 , one can see at once that TGP-stability implies TP-stability.

3. Stability Analysis of the Linear θ-Method

Consider the following method called the linear $\theta-$method :

$$\begin{aligned} y_{n+1} = y_n &+ h\theta f(t_{n+1}, y_{n+1}, y^h(t_{n+1} - \tau(t_{n+1}))) \\ &+ h(1-\theta) f(t_n, y_n, y^h(t_n - \tau(t_n))) \end{aligned} \tag{3.1}$$

for $n = 0, 1, 2, \cdots$, here θ is a parameter with $0 \leq \theta \leq 1$, $h > 0$ is the stepsize. $y_o = \phi(0)$, $y^h(t) = \phi(t)$ for $t \leq 0$, and $y^h(t)$ with $t \geq 0$ is defined by piecewise linear interpolation, i.e.,

$$y^h(t) = \frac{t - nh}{h} y_{n+1} + \frac{(n+1)h - t}{h} y_n, \text{ for } nh \leq t \leq (n+1)h, n = 0, 1, 2, \cdots \tag{3.2}$$

Applying (3.1) and (3.2) to (1.4), we arrive at the following recurrence relation

$$\begin{aligned} y_{n+1} = y_n &+ h\theta[a(t_{n+1})y_{n+1} + b(t_{n+1})(\delta y_{n-m+2} + (1-\delta) y_{n-m+1})] \\ &+ (1-\theta) h[a(t_n) y_n + b(t_n)(\delta y_{n-m+1} + (1-\delta) y_{n-m})]. \end{aligned} \tag{3.3}$$

Here $n \geq m$, m is the smallest integer with $\tau h^{-1} \leq m, \delta = m - \tau h^{-1}$, $\delta \in [0, 1)$.

Definition 3.1. *Let* $\delta \in [0,1)$ *and* $a(t)$ *and* $b(t) : R \to C$. *Then a numerical method for DDEs is called* $T\delta-$*stable at* $(a(t), b(t))$, *if any application of the method to* (1.4) *yields approximation* $y_n \to 0$ *as* $n \to \infty$, *whenever* h *is given with* $h = (m-\delta)^{-1}\tau$. *The set consisting of all* $(a(t), b(t))$ *at which the method is* $T\delta$*-stable is called* $T\delta-$*stability region. For the linear* $\theta-$*method we denote it by* $S_{\theta,\delta}$. *The stability region* S_θ *of the linear* $\theta-$*method is defined by*

$$S_\theta = \bigcap_{o \leq \delta < 1} S_{\theta,\delta}. \tag{3.4}$$

Define $H = \{(a(t), b(t))$: $a(t)$ *and* $b(t)$ *satisfy* (2.5') $\}$.

At once we have the following lemma.

Lemma 3.2. (i) *A numerical method for DDEs is TP-stable if and only if*

$$H \subseteq S_{\theta,0}.$$

(ii) *A numerical method for DDEs is TGP-stable if and only if*

$$H \subseteq S_{\theta}.$$

Lemma 3.3.[11] *Let y be either root of a real quadratic equation $x^2 - bx + c = 0$. If b and c are real, then $|y| < 1$ if and only if $|c| < 1$and $|b| < 1 + c$.*

Theorem 3.4. *Let $0 \le \theta \le 1$. Then the linear $\theta-$method is TGP-stable if and only if $\theta = 1$.*

Proof. (I) $0 \le \theta < \frac{1}{2}$. Consider the linear test equation (1.2), from Remark 1 and [7], we can obtain at once that the linear $\theta-$method is not TGP-stable.

(II) $\frac{1}{2} \le \theta < 1$. Consider the following special DDE

$$\begin{cases} y'(t) = -a(t)y(t) - \theta a(t)y(t-1), & t \ge 0, \\ y(t) = \phi(t), \quad t \le 0, \end{cases} \tag{3.5}$$

where $a(t) \ge \frac{1-\theta}{4}$ is a real continuous function, $\phi(t)$ is continuous. Then the condition (2.5') is satisfied.

Let stepsize $h = 1$ and hence $\delta = 0$. Then (3.3) reads

$$y_{n+1} = \frac{1-(1-\theta)a(t_n) - \theta^2 a(t_{n+1})}{1+\theta a(t_{n+1})} y_n - \frac{\theta(1-\theta)a(t_n)}{1+\theta a(t_{n+1})} y_{n-1},$$

where $n = 0, 1, 2, \ldots$. Now we choose a sequence $\{a(t_n)\}$ such that it is 2-periodic with $a(t_o) = a(t_2) = \ldots = e = \frac{1-\theta}{4}$, $a(t_1) = a(t_3) = \ldots = f = \frac{4}{1-\theta}$. Define $Y_k = (y_k, y_{k-1})^T$. Hence equation (3.6) is equivalent to

$$Y_{n+1} = A_n Y_n, \tag{3.7}$$

where $n = 0, 1, \cdots$, and

$$A_n = \begin{pmatrix} \frac{1-(1-\theta)a(t_n)-\theta^2 a(t_{n+1})}{1+\theta a(t_{n+1})} & \frac{-\theta(1-\theta)a(t_n)}{1+\theta a(t_{n+1})} \\ 1 & 0 \end{pmatrix}$$

The periodicity of the sequence $\{a(t_n)\}$ yields

$$Y_{n+2} = BY_n, \tag{3.8}$$

where $B = A_{n+1}A_n$, $n = 0, 2, 4, \cdots$.

We obtain

$$B = \begin{pmatrix} c_1 d_1 + c_2 & c_1 d_2 \\ d_1 & d_2 \end{pmatrix}$$

where

$$c_1 = \frac{1-(1-\theta)e-\theta^2 f}{1+\theta f}, \quad c_2 = \frac{-\theta(1-\theta)e}{1+\theta f}, \quad d_1 = \frac{1-(1-\theta)f-\theta^2 e}{1+\theta e},$$

and

$$d_2 = \frac{-\theta(1-\theta)f}{1+\theta e}.$$

Thus

$$\begin{aligned} det(\lambda I - B) &= \lambda^2 - (d_2 + c_1 d_1 + c_2)\lambda + d_2 c_2 \\ &= \lambda^2 - \frac{1-e-f+(1-\theta)^2+\theta^4}{(1+\theta e)(1+\theta f)}\lambda + \frac{\theta^2(1-\theta)^2}{(1+\theta e)(1+\theta f)}. \end{aligned} \tag{3.9}$$

Since

$$\Big|\frac{1-e+f+(1-\theta)^2+\theta^4}{(1+\theta e)(1+\theta f)}\Big| > 1 + \Big|\frac{\theta^2(1-\theta)^2}{(1+\theta e)(1+\theta f)}\Big|,$$

then from Lemma 3.3, it follows that $\rho(B) \geq 1$. Hence the linear $\theta-$method yields a numerical solution y_n which cann't tend to zero as $n \to \infty$ for some continuous $\phi(t)$ ($t \in [-\tau, 0]$). Thus we can conclude that the linear $\theta-$method is not TP-stable and hence not TGP-stable.

(III) $\theta = 1$. Let $\delta \in [0, 1)$ and let $(a(t), b(t)) \in H$. Then the recurrence relation (3.3) becomes

$$(1 - ha(t_{n+1}))y_{n+1} = y_n + hb(t_{n+1})(\delta y_{n+2-m} + (1-\delta)y_{n+1-m}), \tag{3.10}$$

here $n = 0, 1, 2, \cdots$.

(a) $m > 1$.

When $n = 0$, then

$$y_1 = \frac{1}{1-ha(t_1)}y_o + \frac{hb(t_1)}{1-ha(t_1)}(\delta y_{-m+2} + (1-\delta)y_{-m+1}).$$

Since

$$\Big|\frac{1-qh\mathrm{Re}(a(t))}{1-ha(t)}\Big| \leq \Big|\frac{1-qh\mathrm{Re}(a(t))}{1-h\mathrm{Re}(a(t))}\Big| \leq \Big|\frac{1+qh\beta}{1+h\beta}\Big| =: p < 1,$$

then

$$|y_1| \leq \Big|\frac{1-qh\mathrm{Re}(a(t_1))}{1-ha(t_1)}\Big| \leq p\max_{-\tau\leq t\leq 0}|\phi(t)|.$$

One easily shows by induction that

$$|y_{n+1}| \leq p\max_{-\tau\leq t\leq 0}|\phi(t)| \tag{3.11}$$

for all $n \leq m - 1$.

When $m \le n \le 2m-1$, we have

$$
\begin{aligned}
|y_{n+1}| &\le \Big|\frac{1}{1-ha(t_{n+1})}\Big||y_n| + \Big|\frac{hb(t_{n+1})}{1-ha(t_{n+1})}\Big||\delta y_{n+2-m} + (1-\delta)y_{n+1-m}| \\
&\le p\Big|\frac{1-hq\mathrm{Re}(a(t_{n+1}))}{1-h\mathrm{Re}(a(t_{n+1}))}\Big|\mathrm{max}_{-\tau\le t\le 0}|\phi(t)| \\
&\le p^2\mathrm{max}_{-\tau\le t\le 0}|\phi(t)|. \qquad (3.12)
\end{aligned}
$$

By induction we derive

$$|y_{n+1}| \le p^{r+1}\mathrm{max}_{-\tau\le t\le 0}|\phi(t)|, \qquad (3.13)$$

for all $rm \le n \le (r+1)m-1$.

(b) $\quad m = 1$.

In this case, (3.3) reads

$$y_{n+1} = \frac{1+(1-\delta)hb(t_{n+1})}{1-ha(t_{n+1})-\delta hb(t_{n+1})}y_n. \qquad (3.14)$$

Then

$$
\begin{aligned}
|y_{n+1}| &\le \Big|\frac{1+(1-\delta)hb(t_{n+1})}{1-ha(t_{n+1})-\delta hb(t_{n+1})}\Big||y_n| \\
&\le \Big|\frac{1+(1-\delta)hb(t_{n+1})}{|1-ha(t_{n+1})|-\delta h|b(t_{n+1})|}\Big||y_n| \\
&\le \frac{1-(1-\delta)qh\mathrm{Re}(a(t_{n+1}))}{1-h\mathrm{Re}(a(t_{n+1}))+q\delta h\mathrm{Re}(a(t_{n+1}))}|y_n| \\
&\le \frac{1+(1-\delta)qh\beta}{1+(1-q\delta)h\beta}|y_n|. \qquad (3.15)
\end{aligned}
$$

From (3.13) and (3.15), it follows that

$$\mathrm{lim}_{t\to\infty}y_n = 0,$$

which implies the linear $\theta-$method is TGP-stable.

This completes the proof of this theorem.

4. Numerical Stability of the One-leg θ-Method

Consider the following one-leg $\theta-$method:

$$y_{n+1} = y_n + hf(t_{n+\theta}, y^h(t_{n+\theta}), y^h(t_{n+\theta}-\tau(t_{n+\theta}))) \qquad (4.1)$$

for $n = 0, 1, 2, ...,$ here θ is a parameter with $0 \le \theta \le 1$, $t_{n+\theta} = (n+\theta)h$, $y_o = \phi(0)$, $y^h(t) = \phi(t)$ for $t \le 0$, and the definition of $y^h(t)$ is given by (3.2).

Concerning the numerical stability of the one-leg $\theta-$method for DDEs based on the test equation (1.4), we apply (4.1) and (3.2) to (1.4):

$$y_{n+1} = y_n + ha(t_{n+\theta})(\theta y_{n+1} + (1-\theta)y_n) + hb(t_{n+\theta})(\sigma y_{n-r+1} + (1-\sigma)y_{n-r}), \qquad (4.2)$$

where $n \geq r$, m is the smallest integer such that $\tau^{-1}h \leq m, \delta = m - \tau^{-1}h \in [0, 1)$,

$$\sigma = \theta + \delta \quad \text{and} \quad r = m, \quad \text{if} \quad 0 \leq \delta < 1 - \theta,$$

$$\sigma = \theta + \delta - 1 \quad \text{and} \quad r = m - 1, \quad \text{if} \quad 1 - \theta \leq \delta < 1.$$

We can derive the following theorem.

Theorem 4.1. *Let $0 \leq \theta \leq 1$. The one-leg $\theta-$method is TGP-stable and hence TP-stable if and only if $\theta = 1$.*

Proof. (I) $0 \leq \theta < 1$. Consider the linear test equation (1.2), from Remark 1 and [7], we conclude that the one-leg $\theta-$method is not TGP-stable.

(II) $\theta = 1$.The method (4.2) and (3.3) coincides, from Theorem 3.4, and hence the method is TGP-stable.

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