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DELTA-WAVE IN A SIMPLE 1-D 2×2 HYPERBOLIC PROBLEM*1)

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Abstract

A simple one-dimensional 2×2 hyperbolic system is considered in the paper. The model contains a linear hyperbolic equation, as well as a hyperbolic equation of which the coefficients are about the solution of the linear one. The exact solution is presented and discussed, then numerical experiments are given by TVD (or MmB) type schemes for Riemann problems. From the results, we know that the solutions do have $\delta-$waves for some suitable initial data.

1. Introduction

The classical solution structure has been studied from general hyperbolic system in conservation laws. Mathematicians focused on the existences and uniqueness of solutions and looked for a way to study for practical problems, and with the development of computer sciences, the study has got a great success by using of computational methods, such as finite difference, finite element and spectral methods, to describe natural phenomena.

In recent years, a new singular phenomenon has been discovered, called $\delta-$wave. The first result was presented by Korchinski[1] in 1977, he proved that no classical solutions exist for the following 2×2 system.

$$\begin{cases} u_t + (u^2/2)_x = 0 \\ v_t + (uv)_x = 0 \ . \end{cases} \tag{1.1}$$

In 1993, Joseph proved that the viscosity solution contains delta-measures for Riemann problem of the above problem [2].

In 1989, our group began to study for 2-D Riemann problem of 2-D 2×2 nonlinear hyperbolic system in conservation laws on both theoretical analyses and numerical computations,

$$\begin{cases} u_t + (u^2)_x + (uv)_y = 0 \\ v_t + (uv)_x + (v^2)_y = 0 \end{cases} \tag{1.2}$$

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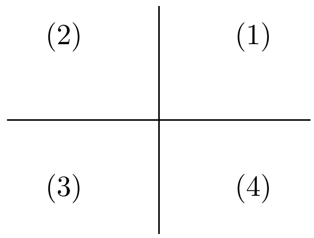
with the 2-D Riemann data

$$(u,v)|_{t=0} = (u_0^i, v_0^i), \quad (i) = 1,2,3,4 \tag{1.3}$$

where (i)-states are described to



Here (1.2) is called 2-D inviscid Burger's equations.

In the processes of the study on the solution structure of the problem, then firstly, we found a new kind of phenomenon by numerical computations for some distributions of Riemann initial data. The phenomenon in numerical results performed that there is a narrow region near shock waves that solution may produce infinity even though the initial data are bounded[3]. The theoretical analyses for (1.2) and (1.3) are given in [4,5]. From the 2-D model, we go back to some 1-D cases, then consider the 1-D 2×2 nonlinear hyperbolic system in conservation laws[4,6], with initial data

$$(u,v)|_{t=0} = (u_0(x), v_0(x)). \tag{1.5}$$

The general model was first presented in [4]. The several special system were proposed and studied in [7] and some results that solutions may produce $\delta-$waves were presented in [8] for 1-D and 2-D hyperbolic systems.

In this paper, we consider the special case of system (1.1). The exact solution is presented and discussed for the Riemann initial data in section 2, then in section 3 the numerical experiments are given for the corresponding Riemann data by TVD (or MmB) schemes. From the exact solutions and numerical solutions, $\delta-$wave do exist in some cases.

2. Exact Solutions

Here we consider the following simple 2×2 hyperbolic system,

$$\begin{cases} u_t + au_x = 0 \\ v_t + (uv)_x = 0 \end{cases} \tag{2.1}$$

with initial data

$$(u,v)|_{t=0} = (u_0(x), v_0(x)). \tag{2.2}$$

For the first equation of (2.1) and (2.2), the solution can easily be obtained,

$$u(x,t) = u_0(x - at). \tag{2.3}$$

We first consider the Riemann problem for the system,

$$\begin{cases} u_t = 0 \\ v_t + (uv)_x = 0. \end{cases} \tag{2.4}$$

and Riemann data

$$(u,v)|_{t=0} = \begin{cases} (u_l, v_l), & x < 0 \\ (u_r, v_r), & x > 0 \end{cases} \tag{2.5}$$

In order to get the exact solution of (2.4)(2.5), smoothe u_0 by linear interpolation, then we have the function u_0^ε,

$$u_0^\varepsilon = \begin{cases} u_l, & x < -\varepsilon \\ \dfrac{u_l(\varepsilon - x) + u_r(x+\varepsilon)}{2\varepsilon}, & -\varepsilon \le x \le \varepsilon \\ u_r, & x > \varepsilon \end{cases}$$

and $u^\varepsilon(x,t) = u_0^\varepsilon(x)$ for (2.4)(2.5).

Now consider the second equation of (2.4),

$$v_t + uv_x + vu_x = 0.$$

From the function $u^\varepsilon(x,t)$, we have,

$$u_x^\varepsilon(x,t) = \begin{cases} 0, & x \in [-\varepsilon, \varepsilon] \\ \dfrac{u_r - u_l}{2\varepsilon}, & x \in (-\varepsilon, \varepsilon) \end{cases}$$

then we divide the equation into three parts of the equation:

$$\begin{aligned} & v_t + u_l v_x = 0, \quad \infty < x < -\varepsilon \\ & v_t + u_r v_x = 0, \quad \varepsilon < x < \infty \\ & v_t + \left(\frac{u_l + u_r}{2} + \frac{(u_r - u_l)x}{2\varepsilon}\right) v_x = 0, \quad x \in (-\varepsilon, \varepsilon). \end{aligned}$$

Obviously, the solution for the former two equations is v_l and v_r; for the last equation, we consider the case $u_l = -u_r$ and have the solution,

$$v(x,t) = \begin{cases} \dfrac{-v_0\varepsilon}{x}, & x \in (-\varepsilon, -\varepsilon \exp(\frac{u_r - u_l}{2\varepsilon} t)) \\ \dfrac{v_0\varepsilon}{x}, & x \in (\varepsilon \exp(\frac{u_r - u_l}{2\varepsilon})t, \varepsilon) \\ v_0 \exp(-\frac{u_r - u_l}{2\varepsilon} t), & x \in (-\varepsilon \exp(\frac{u_r - u_l}{2\varepsilon})t, \varepsilon \exp(\frac{u_r - u_l}{2\varepsilon} t)). \end{cases}$$

By the approximation in the sense of distribution for $\varepsilon \to 0$, we obtain the exact solution of (2.4)(2.5) as follows,

$$v(x,t) = \begin{cases} v_0, & x \neq 0 \\ v_0 - v_0(u_r - u_l)\delta_{x=0} t, & x = 0. \end{cases}$$

For (2.1) and (2.5), we may do the transformations, $\tau = t, \quad y = x - at$ then we have

$$\begin{cases} u_\tau = 0 \\ v_\tau + ((u-a)v)_y = 0. \end{cases}$$

For the Riemann data, the solution is described as,

$$\begin{cases} u(x,t) = u_0(x - at), \\ v(x,t) = v_0 - v_0(u_r - u_l)\delta_{x=at}t. \end{cases}$$

Remark. In the process of getting the exact solution, we assume that $u_l = -u_r$. If not, the term $\frac{u_l + u_r}{2} \neq 0$, then the solution $v(x,t)$ will travel by speed $a + \frac{u_l + u_r}{2}$, that is, we should have that $v(x,t) = v(x - (a + \frac{u_l + u_r}{2})(t - t_0), t_0)$, so we guess that there is no the solution structure in the form of $\delta-$wave.

Therefore for the general Riemann data, we present some theoretical results in the following subcases:

i) $u_l < u_r$

In this case, due to the structure of (2.1), we know that $v(x,t)$ is rarefied as time t increases, see figure 2.1

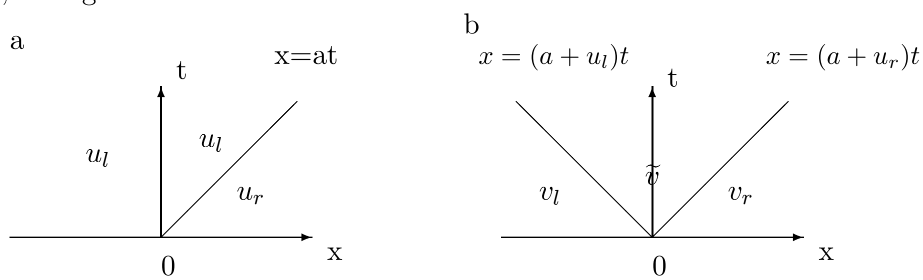


Figure 2.1. a. u(x,t) in (x,t) b. v(x,t) in (x,t)

In this case, there is no other singular solution structure in the model.

(ii) $u_l > u_r$

From (2.1), we know that $v(x,t)$ will tend to infinity along $x = at$ if $u_0(x - at) = 0$. Hence here we divide the case into three subcases to discuss.

1) $u_l > u_r > 0$, the solution structure is figured in (x,t) plane as following,

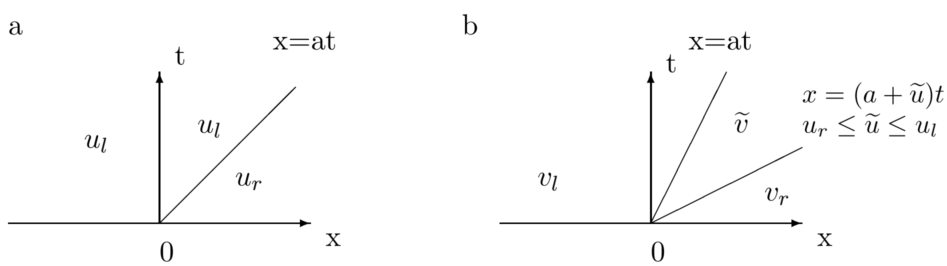


Figure 2.2. a. u(x,t) in (x,t) b. v(x,t) in (x,t)

In this subcase there is no $\delta-$wave.

2) $0 > u_l > u_r$, then we have the solution structure,

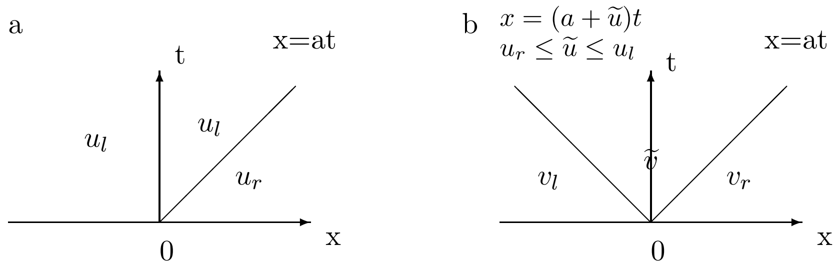


Figure 2.3. a. u(x,t) in (x,t) b. v(x,t) in (x,t)

There is also no $\delta-$wave in the subcase.

3) $u_l \ge 0 \ge u_r$, then from the data, we have $u(x,t) = 0$ along $x = at$ when $u_l = -u_r$ definitely. See Figure 2.4,

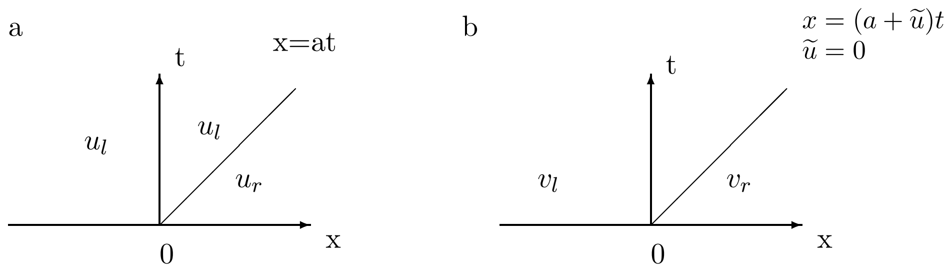


Figure 2.4. a. u(x,t) in (x,t) b. v(x,t) in (x,t)

So $v(x,t)$ will tend to infinity along $x = at$ and constant in the other region of (x,t) plane. Therefore we can say that there is $\delta-$wave in this subcase. In the following section, numerical solutions are given for the case $u_l > u_r$.

3. Numerical Experiments

Here consider numerical solutions of (2.1) and (2.5) to the corresponding three subcases for $u_l > u_r$ by TVD (or MmB) scheme[9].

1) $u_l > u_r > 0$

The initial data are taken to

$$\text{Data} \quad 1. \;\; u_l = 8,\; u_r = 1.8,\; v_l = v_r = 2$$

See Figure 3.1 and 3.2,

2) $0 > u_l > u_r$

Here the initial data are given as follows,

$$\text{Data} \quad 2. \;\; u_l = -1,\; u_r = -8,\; v_l = v_r = 2$$

See Figure 3.3 and 3.4,

data1	u=:	v=:	data2	u=:	v=:	data3	u=:	v=:
i=43	8.00	2.00	i=141	-1.00	9.00	i=69	1.00	0.10E+01
i=44	8.00	2.00	i=142	-1.00	9.00	i=70	1.00	0.32E+01
i=45	8.00	2.00	i=143	-1.00	9.00	i=71	1.00	0.12E+03
i=46	8.00	2.00	i=144	-1.00	9.00	i=72	1.00	0.71E+04
i=47	7.99	2.00	i=145	-1.00	8.99	i=73	1.00	0.40E+06
i=48	7.99	2.00	i=146	-1.00	8.98	i=74	1.00	0.22E+08
i=49	7.99	2.00	i=147	-1.01	8.93	i=75	1.00	0.12E+10
i=50	7.99	2.00	i=148	-1.05	8.77	i=76	0.99	0.73E+11
i=51	7.99	2.00	i=149	-1.18	8.25	i=77	0.99	0.35E+13
i=52	7.95	2.01	i=150	-1.64	6.86	i=78	0.90	0.22E+15
i=53	7.83	2.04	i=151	-3.13	4.45	i=79	0.31	0.36E+16
i=54	7.43	2.17	i=152	-5.44	2.82	i=80	-0.41	0.41E+16
i=55	6.10	2.70	i=153	-7.53	2.11	i=81	-0.84	0.19E+15
i=56	4.06	3.80	i=154	-7.98	2.00	i=82	-0.96	0.85E+13
i=57	2.21	11.40	i=155	-7.99	2.00	i=83	-0.99	0.32E+12
i=58	1.81	17.25	i=156	-8.00	2.00	i=84	-0.99	0.14E+11
i=59	1.80	17.50	i=157	-8.00	2.00	i=85	-1.00	0.40E+08
i=60	1.80	17.55	i=158	-8.00	2.00	i=86	-1.00	0.11E+04
i=61	1.80	17.55	i=159	-8.00	2.00	i=87	-1.00	0.10E+01

3) $u_l \geq 0 \geq u_r$

Here we take the initial data to the case $u_l = -u_r$, it is listed to

$$\text{Data} \quad 3. \;\; u_l = -u_r = 1, \;\; v_l = v_r = 1$$

The numerical results are presented in Figure 3.5 and Figure 3.6.

For the detail solution data locally, see the table before front page. It clearly gives the quantity in the region which contains $\delta-$wave.

Here M-P is mesh point, n is time step. From the table and Figures 3.5 and 3.6, we can see that $v(x,t)$ tends to inifinity along $x = at$ the region of shock wave $u(x,t)$ with time t. There is no the solution structure for the other cases. So we get the conclusion that there is $\delta-$wave solution structure for some initial data for the 2×2 hyperbolic system.

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