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BOX–LINE RELAXATION SCHEMES FOR SOLVING THE STEADY INCOMPRESSIBLE NAVIER–STOKES EQUATIONS USING SECOND–ORDER UPWIND DIFFERENCING*1)

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Abstract

We extend the SCGS smoothing procedure (Symmetrical Collective Gauss–Seidel relaxation) proposed by S. P. Vanka[4], for multigrid solvers of the steady viscous incompressible Navier–Stokes equations, to corresponding line–wise versions. The resulting relaxation schemes are integrated into the multigrid solver based on second–order upwind differencing presented in [5]. Numerical comparisons on the efficiency of point–wise and line–wise relaxations are presented.

1. Introduction

The convection–diffusion behaviour of the viscous incompressible Navier–Stokes equations is a main source of difficulties in the numerical solution. When discretizing the equations using finite difference schemes, upwind or hybrid schemes are usually used on the convection terms for ensuring the stability of the discrete system [1]. The first–order upwind differencing has proved to be inadequate for the incompressible Navier–Stokes equations with large Reynolds numbers, although the resulting discrete systems are very stable and easily solved. In [5], we constructed a multigrid solver based on second–order upwind differencing and we adapted the SCGS relaxation, which was originally proposed for hybrid schemes, as the smoothing procedure. It gives good discrete solutions and the convergence rate is comparable to (even faster than) the same multigrid solver using first–order upwind differencing when the cell Reynolds number is not very large. There are two main disadvantages for the SCGS relaxation: 1) with second–order upwind differencing, it is difficult to obtain convergence for very large Reynolds numbers ($R \geq 2000$) and the convergence rate is sensitive to the relaxation factor; 2) it fails for strongly anisotropic problems, e.g., when the aspect ratio of the grid cells is not close to 1, so it is not suitable on non–uniform grids.

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In this paper, we give two line–wise extensions to the SCGS relaxation for the second–order upwind scheme and we make some numerical comparisons on the convergence rate of different relaxation methods.

2. Discretization

The dimensionless steady viscous incompressible Navier–Stokes equations in a 2D domain Ω can be formulated as follows:

$$\begin{cases} -\frac{1}{R}\Delta u + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = f_1, \\ -\frac{1}{R}\Delta v + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = f_2, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \\ + \text{Boundary conditions} \end{cases} \tag{1}$$

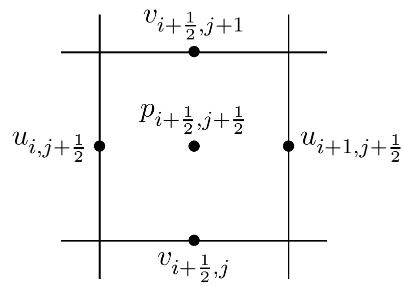

Fig. 1. Location of unknowns

where (u, v) is the velocity, p the pressure, R the Reynolds number and (f_1, f_2) denotes the external force.

We discretize Equation (1) on uniform staggered grids (MAC grid). The location of different variables and the corresponding discrete equations on the cell (i, j) is shown by Fig. 1 (in which the index (i, j) corresponds to the grid point $(i\Delta x, j\Delta y)$).

The convection terms in (1) are discretized using second–order upwind differencing. For example, the term $v\dfrac{\partial u}{\partial y}$ on the point $(i\Delta x, (j + \frac{1}{2})\Delta y)$ is discretized by:

$$\begin{cases} \dfrac{v_{i,j+\frac{1}{2}}}{2\Delta y}(3u_{i,j+\frac{1}{2}} - 4u_{i,j-\frac{1}{2}} + u_{i,j-\frac{3}{2}}), & \text{if } v_{i,j+\frac{1}{2}} \geq 0, \\ -\dfrac{v_{i,j+\frac{1}{2}}}{2\Delta y}(3u_{i,j+\frac{1}{2}} - 4u_{i,j+\frac{3}{2}} + u_{i,j+\frac{5}{2}}), & \text{if } v_{i,j+\frac{1}{2}} < 0 \end{cases}$$

where the term $v_{i,j+\frac{1}{2}}$, which is not defined on the grid points, is computed by bilinear interpolation:

$$v_{i,j+\frac{1}{2}} = \frac{1}{4}(v_{i-\frac{1}{2},j} + v_{i-\frac{1}{2},j+1} + v_{i+\frac{1}{2},j} + v_{i+\frac{1}{2},j+1}).$$

All other terms are discretized by standard central differencing. For details on the discretization and treatment near the boundaries we refer to [5].

3. Relaxation Schemes

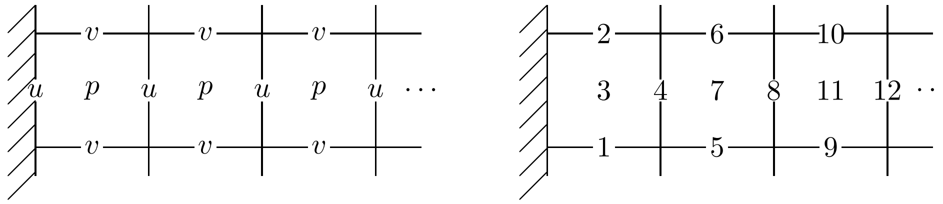

Fig. 2. Numbering of the unknowns in an x-line

The point–wise SCGS relaxation consists of updating cell by cell corresponding variables. In each grid cell (also called a "box"), there are five unknowns (see Fig. 1). They are updated simultaneously using the five corresponding discrete equations (four momentum equations and one continuity equation). The momentum equations are linearized by replacing the coefficients of the convection terms by their current values, and diagonalized by keeping only the main velocity component (which is defined at the same point as the discrete equation) and the pressure component defined at the center of the current cell as unknowns. This leads to the solution of a linear system of equations with 5 unknowns in the following form, on each grid cell:

$$\begin{pmatrix} * & 0 & 0 & 0 & * \\ 0 & * & 0 & 0 & * \\ 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & * & * \\ * & * & * & * & 0 \end{pmatrix} \begin{pmatrix} * \\ * \\ * \\ * \\ * \end{pmatrix} = \begin{pmatrix} * \\ * \\ * \\ * \\ * \end{pmatrix}.$$

Now we extend the above procedure to line-wise relaxations. Let us first consider the case of "x-line" relaxation. Instead of updating each time variables defined on a single grid cell, as in the SCGS scheme, we consider all cells located on a same horizontal line (x-line) and update simultaneously the variables defined in these cells using corresponding linearized discrete equations, so we have to solve a system of linear equations for each x-line.

To minimize the band width of the linear system to be solved, the velocity and the pressure components are numbered in a mixed way, as shown in Fig. 2, and we write the linear system in the following form:

$$\mathbf{L}\mathbf{x} = \mathbf{b} \tag{2}$$

where $\mathbf{x}$ is the vector of unknowns, $\mathbf{b}$ a known vector and $\mathbf{L}$ a banded matrix.

The two relaxation schemes that we will consider only differ from the treatment of the convection terms (with less or more implicitness):

Scheme 1. All variables defined on the current x–line are treated as unknowns.

For example, the discrete momentum equation on the point $(i, j+\frac{1}{2})$ is written as:

$$\frac{1}{R\Delta x^2}\Big(4u_{i,j+\frac{1}{2}} - u_{i+1,j+\frac{1}{2}} - u_{i-1,j+\frac{1}{2}} - u^{\rm old}_{i,j+\frac{3}{2}} - u^{\rm old}_{i,j-\frac{1}{2}})\Big) + \frac{1}{\Delta x}(p_{i+\frac{1}{2},j+\frac{1}{2}} - p_{i-\frac{1}{2},j+\frac{1}{2}})$$

$$\pm\frac{1}{2\Delta x}u^{\rm old}_{i,j+\frac{1}{2}}(3u_{i,j+\frac{1}{2}} - 4u_{i\mp 1,j+\frac{1}{2}} + u_{i\mp 2,j+\frac{1}{2}})$$

$$\pm\frac{1}{2\Delta y}v^{\rm old}_{i,j+\frac{1}{2}}(3u_{i,j+\frac{1}{2}} - 4u^{\rm old}_{i,j+\frac{1}{2}\mp 1} + u^{\rm old}_{i,j+\frac{1}{2}\mp 2}) = f_1(i\Delta x, (j+\tfrac{1}{2})\Delta y)$$

where $u^{\rm old}_{i,j+\frac{3}{2}}$, $u^{\rm old}_{i,j-\frac{1}{2}}$, etc., denote the current known values of the corresponding variables. The band width of $\mathbf{L}$ is 17 for this scheme.

Scheme 2. In the convection terms, the variables which are located two grid points away from the grid point where the corresponding discrete equation is defined are treated explicitly, i.e., they are replaced by their current approximation. All other terms are treated in the same way as in Scheme 1. For example, the above equation becomes :

$$\frac{1}{R\Delta x^2}\Big(4u_{i,j+\frac{1}{2}} - u_{i+1,j+\frac{1}{2}} - u_{i-1,j+\frac{1}{2}} - u^{\rm old}_{i,j+\frac{3}{2}} - u^{\rm old}_{i,j-\frac{1}{2}})\Big) + \frac{1}{\Delta x}(p_{i+\frac{1}{2},j+\frac{1}{2}} - p_{i-\frac{1}{2},j+\frac{1}{2}})$$

$$\pm\frac{1}{2\Delta x}u^{\rm old}_{i,j+\frac{1}{2}}(3u_{i,j+\frac{1}{2}} - 4u_{i\mp 1,j+\frac{1}{2}} + u^{\rm old}_{i\mp 2,j+\frac{1}{2}})$$

$$\pm\frac{1}{2\Delta y}v^{\rm old}_{i,j+\frac{1}{2}}(3u_{i,j+\frac{1}{2}} - 4u^{\rm old}_{i,j+\frac{1}{2}\mp 1} + u^{\rm old}_{i,j+\frac{1}{2}\mp 2}) = f_1(i\Delta x, (j+\tfrac{1}{2})\Delta y).$$

In this case, the band width of $\mathbf{L}$ is reduced to 9, so the solution of System 2 needs less computation time and storage space than for Scheme 1.

In both schemes System 2 is solved by the direct Gauss method. The velocity components obtained are then underrelaxed using a relaxation factor $\beta \in (0,1)$ in the following way:

$$u := u_{\rm old} + \beta(u - u_{\rm old}), \quad v := v_{\rm old} + \beta(v - v_{\rm old}).$$

All x-lines are scanned successively in an x-line relaxation swap.

The corresponding "y-line" versions can be constructed in the same way.

We will use an alternating strategy, i.e., each relaxation swap is always composed of an x-line relaxation followed by a y-line relaxation.

4. Numerical Results

For testing the efficiency of the above relaxation schemes as smoothing procedure of multigrid solvers, we integrate them into the FAS (Full Approximation Storage) procedure presented in [5]. Our first test problem is the following problem of flows in

Table 1. Final residual obtained with different smoothing procedures ($R = 1000$)

SCGS			Scheme 1			Scheme 2		
Rel. factor	It. #	Final residual	Rel. factor	It. #	Final residual	Rel. factor	It. #	Final residual
$\beta = 0.1$	100	3.47×10^{-3}	$\beta = 0.1$	50	1.42×10^{-4}	$\beta = 0.3$	34	9.08×10^{-5}
$\beta = 0.2$	100	4.18×10^{-4}	$\beta = 0.2$	30	9.93×10^{-5}	$\beta = 0.4$	31	9.91×10^{-5}
$\beta = 0.3$	100	2.55×10^{-4}	$\beta = 0.3$	45	9.24×10^{-5}	$\beta = 0.5$	32	9.73×10^{-5}
$\beta = 0.4$	100	1.92×10^{-2}	$\beta = 0.4$	34	9.59×10^{-5}	$\beta = 0.6$	22	8.96×10^{-5}
			$\beta = 0.5$	50	4.18×10^{-2}	$\beta = 0.7$	27	8.83×10^{-5}
						$\beta = 0.8$	40	1.82×10^{-1}

a rectangular cavity:

$$\Omega = (0,3) \times (0,1),$$

$$v(0,y) = v(1,y) = v(x,0) = v(x,1) = u(0,y) = u(1,y) = u(x,0) = 0,$$

$$u(x,1) = 1, \quad f_1 = f_2 = 0.$$

This problem is solved on the 64×24 grid by the FAS procedure for different values of R. Table 1 summarizes the residual obtained after a certain number of multigrid cycles with different relaxation schemes for $R = 1000$. The CPU time for performing one FAS cycle is about 8.5 seconds with the SCGS relaxation, 35 seconds with Scheme 1 and 26 seconds with Scheme 2 (on IBM 4341).

For a better view on the efficiency of different relaxation schemes, we plot the residual as function of the CPU time elapsed for the three relaxation schemes with their optimal relaxation factor β found in Table 1.

We see that Scheme 2 has higher convergence rate and is less sensitive to the relaxation factor β than the other two schemes. This is true in all comparisons made so far. Only for small values of R, the convergence rate of the SCGS scheme is comparable to Scheme 2.

To further study the convergence property of Scheme 2 for large Reynolds numbers, we give in Fig. 4 the convergence history of Scheme 2 for $R = 5000$ and 10000, on the unit square with the same boundary condition as in the last example (this is the standard driven cavity problem), and on the 32×32 grid. It is seen that Scheme 2 indeed improves the convergence property of the SCGS scheme, since the latter does not converge on the 32×32 and 64×64 grids when $R \geq 2000$ (but it converges on the 128×128 and 256×256 grids) [7].

Although Scheme 2 improves the convergence of the SCGS scheme, the improvement is not very significant when the Reynolds number is not very large. More interesting is

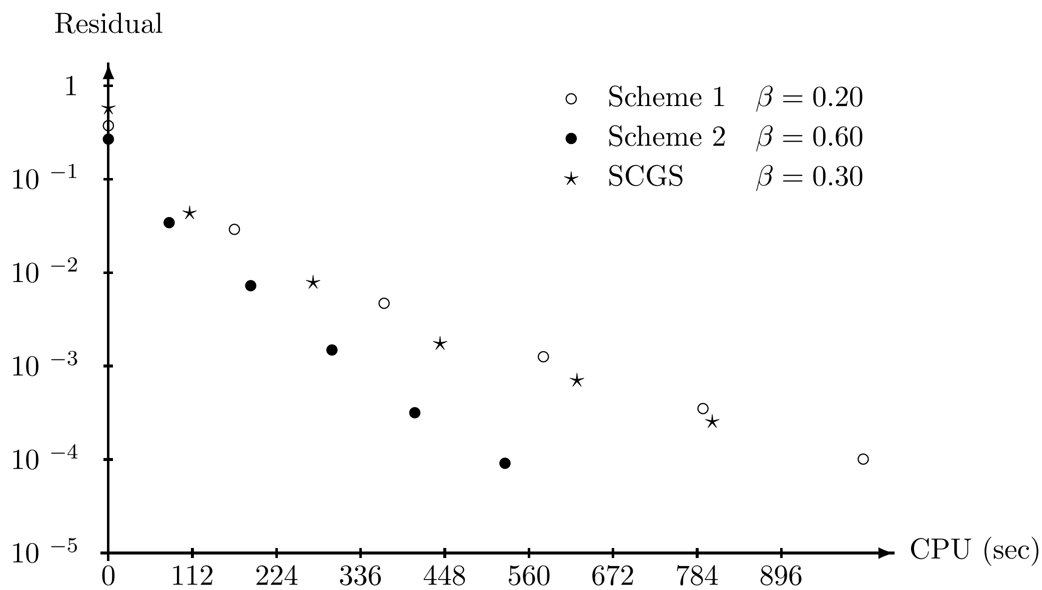

Fig.3. Evolution of the residual as function of CPU time elapsed ($R = 1000$)

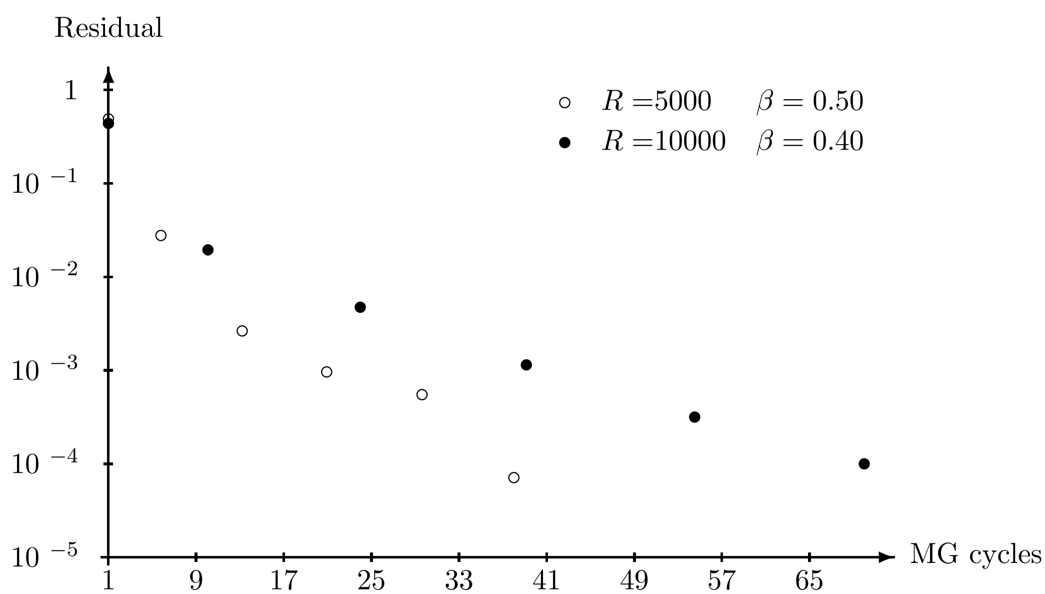

Fig. 4. Convergence history of Scheme 2 for the driven cavity problem

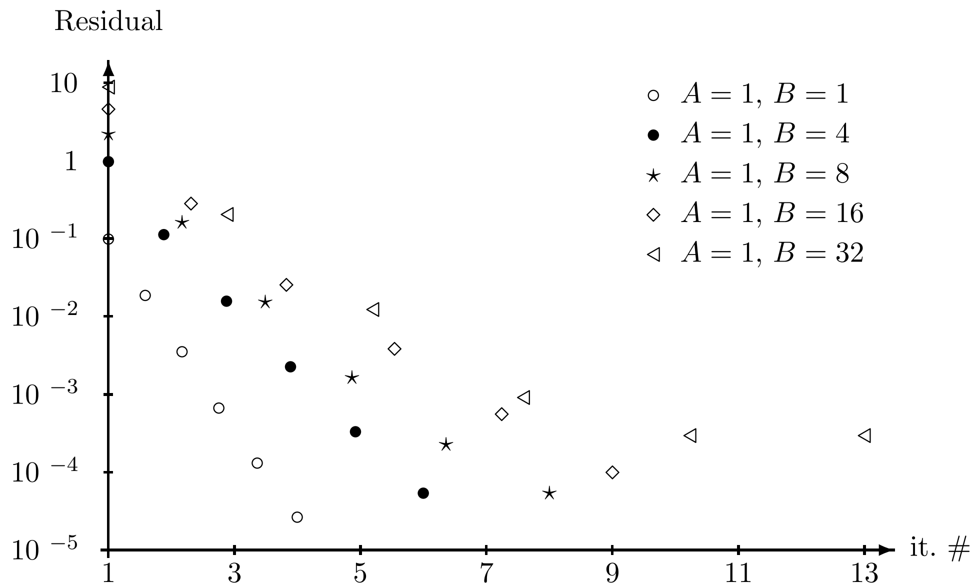

Fig. 5. Convergence of Scheme 2 for anisotropic problems

the following example of anisotropic problem, since it is generally known that alternating line–wise relaxations are much more competitive than point–wise relaxations when dealing with anisotropic problems. This test problem has been used in [6] for which the SCGS relaxation fails due to anisotropy of the grid cell, and we proposed a semi–coarsening strategy in the multigrid procedure to maintain the convergence. In this problem, the domain is the rectangle $(0, A) \times (0, B)$ and the following exact solution:

$$u(x,y) = A \sin\left(\frac{x}{A}\right) \cos\left(\frac{y}{B}\right), \quad v(x,y) = -B \cos\left(\frac{x}{A}\right) \sin\left(\frac{y}{B}\right), \quad p(x,y) = \left(\frac{x}{A}\right)\left(\frac{y}{B}\right)$$

is used to construct the boundary conditions and the right–hand side of Equation 1. The problem is solved on the 32×32 grid for $R = 100$, $A = 1$ and $B = 1$, 4, 8, 16 and 32, respectively. The aspect ratio of the grid cells is therefore 1, 4, 8, 16 and 32, respectively. Fig. 5 shows the residual as function of the number of FAS cycles performed with Scheme 2 as smoothing procedure (without semi–coarsening).

Notice that for $B = 32$, the convergence rate of the multigrid procedure is about a half of that for $B = 1$, since when $B/A \gg 1$ the residual is only reduced by x–line relaxations. Notice also that for $B = 32$, the residual ceases to descend at about 2.9×10^{-4}. This is due to the round–off errors since System 2 is ill–conditioned when $B/A \gg 1$.

5. Conclusions

We propose two line–wise relaxation schemes for solving the steady incompressible Navier–Stokes equations using second–order upwind differencing. Through out all numerical experiments we find that Scheme 2 is always the most efficient among the three schemes considered. It has good convergence rate for large Reynolds numbers and for anisotropic problems. We do not know yet why Scheme 2 has faster convergence than Scheme 1, a theoretical study of both schemes is necessary to find the reason — it will be done in the near future. All we can say now is that the multigrid solver based on second–order upwind differencing combined with Scheme 2 as smoothing procedure provides an efficient solver of the viscous incompressible Navier–Stokes equations and is readily extended to non–uniform grids.

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