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FOURIER-LEGENDRE SPECTRAL METHOD FOR THE UNSTEADY NAVIER-STOKES EQUATIONS*

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Abstract

Fourier-Legendre spectral approximation for the unsteady Navier-Stokes equations is analyzed. The generalized stability and convergence are proved respectively.

1. Introduction

The numerical methods of the Navier-Stokes equations can be found in [1-4]. Specific algorithms in [5-8] have been devoted to the semi-periodic cases which describe channel flow, parallel boundary layers, curved channel flow and cylindrical Couette flow. In this paper, we consider the mixed Fourier-Legendre spectral approximation for the unsteady Navier-Stokes equations. We use Fourier spectral approximation in the periodic directions and Legendre spectral approximation in the non-periodic one. For approximating continuity equation, we adopt small parameter technique[9]. This method has better stability and higher accuracy.

Let $x=(x_1,\cdots,x_n)^T \quad (n=2 \text{ or } 3)$ and $\Omega=I\times Q$ where $I=\{x_1/-1<x_1<1\}$, $\quad Q=\{y=(x_2,\cdots,x_n)^T/-\pi<x_q<\pi, 2\le q\le n\}$. We denote by $U(x,t)$ and $P(x,t)$ the speed and the pressure. $\nu>0$ is the kinetic viscosity. $U_0(x), P_0(x)$ and $f(x,t)$ are given functions. We consider the Navier-Stokes equations as follows

$$\begin{cases} \frac{\partial U}{\partial t}+(U\cdot\nabla)U-\nu\nabla^2U+\nabla P=f, & \text{in} \quad \Omega\times(0,T], \\ \nabla\cdot U=0, \qquad \text{in} \quad \Omega\times(0,T], \\ U(x,0)=U_0(x), \quad P(x,0)=P_0(x), & \text{in} \quad \Omega. \end{cases} \tag{1.1}$$

Assume that all functions have the period 2π for the variable y. In addition, we also suppose that U satisfies the homogeneous boundary conditions in the x_1-direction

$$U(-1,y,t)=U(1,y,t)=0, \quad \forall y\in Q.$$

To fix $P(x,t)$, we require

$$\mu(P)\equiv\int_\Omega P(x,t)dx=0, \quad \forall t\in[0,T].$$

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We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the usual inner product and norm of $L^2(\Omega)$. $|\cdot|_1$ denotes the semi-norm of $H^1(\Omega)$. Let $C^\infty_{0,p}(\Omega)$ be the subset of $C^\infty(\Omega)$, whose elements vanish at $x_1 = \pm 1$ and have the period 2π for $y \in Q$. $H^1_{0,p}(\Omega)$ denotes the closure of $C^\infty_{0,p}(\Omega)$ in $H^1(\Omega)$. Note that the solution of (1.1) satisfies the energy conservation

$$\|U(t)\|^2 + 2\nu \int_0^t |U(t')|_1^2 \, dt' = \|U_0\|^2 + 2\int_0^t (U(t'), f(t'))dt'.$$

One of the both important hands for approximating solutions is an appropriate choice of two discrete spaces for the speed and the pressure. Another is suitable to simulate the conservation.

2. The Scheme

Let M and N be positive integers. Suppose that there exist positive constants d_1 and d_2 such that

$$d_1 N \le M \le d_2 N.$$

$\mathcal{P}_M(I)$ denotes the space of all polynomials with degree $\le M$. Define

$$V_M = \{v(x_1) \in \mathcal{P}_M(I)/v(-1) = v(1) = 0\}.$$

Let $l = (l_2, \cdots, l_n)$, l_q being integers. Set $|l|_\infty = \max\limits_{2\le q\le n} |l_q|$, $|l| = (l_2^2 + \cdots + l_n^2)^{\frac{1}{2}}$, $ly = l_2 x_2 + \cdots + l_n x_n$ and

$$\tilde{V}_N = \text{Span}\,\{e^{ily}/\,|l|_\infty \le N\}.$$

Let V_N be the subset of $\tilde{V}_N$, containing all real-valued functions. Define

$$V_{M,N} = (V_M \times V_N)^n, \quad S_{M-1,N} = \{v \in \mathcal{P}_{M-1}(I) \times V_N/\mu(v) = 0\}.$$

Let $P^1_{M,N} : (H^1_{0,p}(\Omega))^n \longrightarrow V_{M,N}$ be the projection operator such that for any $u \in (H^1_{0,p}(\Omega))^n$,

$$(\nabla(u - P^1_{M,N}u), \nabla v) = 0, \quad \forall v \in V_{M,N}.$$

While $P_{M-1,N} : L^2(\Omega) \longrightarrow \mathcal{P}_{M-1}(I) \times V_N$ is the orthogonal projection such that for any $u \in L^2(\Omega)$,

$$(u - P_{M-1,N}u, v) = 0, \quad \forall v \in \mathcal{P}_{M-1}(I) \times V_N.$$

Obviously, if $u \in L^2(\Omega)$ and $\mu(u) = 0$, then $\mu(P_{M-1,N}u) = 0$.

For continuity equation, we use small parameter technique. Then the incompressible condition is approximated by

$$\beta \frac{\partial P}{\partial t} + \nabla \cdot U = 0, \quad \beta > 0.$$

To approximate the nonlinear term, we define

$$d(u,v) = \frac{1}{2}\sum_{j=1}^n v^{(j)} \frac{\partial u}{\partial x_j} + \frac{1}{2}\sum_{j=1}^n \frac{\partial}{\partial x_j}(v^{(j)}u)$$

where $v^{(j)}$ denotes the component of v.

Let τ be the step of the time, and

$$R_\tau = \{t/t = k\tau, 0 \le k \le \left[\frac{T}{\tau}\right]\}.$$

For simplicity, $u(x,t)$ is denoted by $u(t)$ or u usually. Let

$$u_t(t) = \frac{1}{\tau}(u(t+\tau) - u(t)).$$

Let u and p be the approximations to U and P respectively. The Fourier-Legendre spectral scheme for solving (1.1) is to find $u(t) \in V_{M,N}$, $p(t) \in S_{M-1,N}$ for $t \in R_\tau$, such that

$$\begin{cases} (u_t(t), v) + (d(u(t) + \delta\tau u_t(t), u(t)), v) + \nu(\nabla(u(t) + \sigma\tau u_t(t)), \nabla v) \\ +(\nabla(p(t) + \theta\tau p_t(t)), v) = (f(t), v), \qquad \forall v \in V_{M,N}, \\ (\beta p_t(t), v) + (\nabla \cdot (u(t) + \theta\tau u_t(t)), v) = 0, \quad \forall v \in S_{M-1,N}, \\ u(0) = P^1_{M,N} U_0, \quad p(0) = P_{M-1,N} P_0, \end{cases} \tag{2.1}$$

where $0 \le \delta, \sigma, \theta \le 1$ are parameters.

If we take $\delta = \sigma = \theta = \frac{1}{2}$, then we can verify the discrete conservation of the numerical solution

$$\begin{aligned} &\|u(t)\|^2 + \beta\|p(t)\|^2 + \frac{\nu\tau}{2} \sum_{t' \in R_\tau, t' < t} |u(t') + u(t'+\tau)|_1^2 \\ &= \|u(0)\|^2 + \beta\|p(0)\|^2 + \tau \sum_{t' \in R_\tau, t' < t} (u(t') + u(t'+\tau), f(t')). \end{aligned}$$

So it simulates the energy conservation.

3. Notations and Lemmas

We first introduce some notations. For any integer $r \ge 0$, let $H^r(I)$ be the Hilbertian Sobolev space with the usual norm $\|\cdot\|_{r,I}$ and semi-norm $|\cdot|_{r,I}$, and $H^r_0(I)$ be the closure of $C^\infty_0(I)$ in $H^r(I)$. For any real $r > 0$, $H^r(I)$ is defined by the interpolation between the spaces $H^{[r]}(I)$ and $H^{[r+1]}(I)$, etc..

Let B be a Banach space with the norm $\|\cdot\|_B$, and Λ be an interval in R or a domain in R^2. Define

$$\begin{aligned} &L^2(\Lambda, B) = \{v(z) : \Lambda \longrightarrow B/v \text{ is strongly measurable and } \|v\|_{L^2(\Lambda,B)} < \infty\}, \\ &C(\Lambda, B) = \{v(z) : \Lambda \longrightarrow B/v \text{ is strongly measurable and } \|v\|_{C(\Lambda,B)} < \infty\} \end{aligned}$$

where

$$\|v\|_{L^2(\Lambda,B)} = (\int_\Lambda \|v(z)\|_B^2 \, dz)^{\frac{1}{2}}, \quad \|v\|_{C(\Lambda,B)} = \max_{z\in\Lambda} \|v(z)\|_B .$$

For any non-negative integer α, let

$$H^{\alpha}(\Lambda,B)=\{v(z)\in L^2(\Lambda,B)/\left\|v\right\|_{H^{\alpha}(\Lambda,B)}<\infty\}$$

equipped with

$$\left\|v\right\|_{H^{\alpha}(\Lambda,B)}=(\sum_{k=0}^{\alpha}\left\|\frac{\partial^k v}{\partial z^k}\right\|^2_{L^2(\Lambda,B)})^{\frac{1}{2}}.$$

For real $\alpha>0$, $H^{\alpha}(\Lambda,B)$ is defined by the interpolation between $H^{[\alpha]}(\Lambda,B)$ and $H^{[\alpha+1]}(\Lambda,B)$. Let

$$\begin{aligned}
&H^{r,s}(\Omega)=L^2(Q,H^r(I))\bigcap H^s(Q,L^2(I)),\quad r,s\geq 0,\\
&M^{r,s}(\Omega)=H^{r,s}(\Omega)\bigcap H^1(Q,H^{r-1}(I))\bigcap H^{s-1}(Q,H^1(I)),\quad r,s\geq 1,\\
&X^{r,s}(\Omega)=H^s(Q,H^{r+1}(I))\bigcap H^{s+1}(Q,H^r(I)),\quad r,s\geq 0,
\end{aligned}$$

with the following norms

$$\begin{aligned}
&\left\|v\right\|_{H^{r,s}(\Omega)}=(\left\|v\right\|^2_{L^2(Q,H^r(I))}+\left\|v\right\|^2_{H^s(Q,L^2(I))})^{\frac{1}{2}},\\
&\left\|v\right\|_{M^{r,s}(\Omega)}=(\left\|v\right\|^2_{H^{r,s}(\Omega)}+\left\|v\right\|^2_{H^1(Q,H^{r-1}(I))}+\left\|v\right\|^2_{H^{s-1}(Q,H^1(I))})^{\frac{1}{2}},\\
&\left\|v\right\|_{X^{r,s}(\Omega)}=(\left\|v\right\|^2_{H^s(Q,H^{r+1}(I))}+\left\|v\right\|^2_{H^{s+1}(Q,H^r(I))})^{\frac{1}{2}}.
\end{aligned}$$

We denote by $H^{r,s}_{0,p}(\Omega)$ and $M^{r,s}_{0,p}(\Omega)$ the closures of $C^{\infty}_{0,p}(\Omega)$ in $H^{r,s}(\Omega)$ and $M^{r,s}(\Omega)$, etc.. Let $\left\|\left|v\right|\right\|_{q,\infty}=\max\limits_{t\in R_\tau}\left\|v(t)\right\|_{q,\infty}$.

Next, we list some lemmas. Throughout the paper, c will be a positive constant which may be different in different cases.

Lemma 1. *If $v(t)\in L^2(\Omega)$, then*

$$2(v(t),v_t(t))=(\left\|v(t)\right\|^2)_t-\tau\left\|v_t(t)\right\|^2.$$

Lemma 2. *If $v\in\mathcal{P}_M\times V_N$, then*

$$|v|_1^2\leq(2M^4+(n-1)N^2)\left\|v\right\|^2.$$

Lemma 3. (Theorem A.1 of [8]). *If $v\in H^{r,s}_p(\Omega)$ and $r,s\geq 0$, then*

$$\left\|v-P_{M-1,N}v\right\|\leq c(M^{-r}+N^{-s})\left\|v\right\|_{H^{r,s}(\Omega)}.$$

Lemma 4. *If $v\in(H^1_{0,p}(\Omega)\bigcap M^{r,s}(\Omega))^n$ and $r,s\geq 1$, then*

$$\left\|v-P^1_{M,N}v\right\|_{\alpha}\leq c(M^{\alpha-r}+N^{\alpha-s})\left\|v\right\|_{M^{r,s}(\Omega)},\quad\alpha=0,1.$$

This lemma comes from the proof of Theorem A.2 in [8].

Lemma 5. *If $v\in(H^1_{0,p}(\Omega)\bigcap X^{r,s}(\Omega))^n$ with $r>\frac{3}{2}$, $s>\frac{n-1}{2}$, then*

$$\left\|P^1_{M,N}v\right\|_{1,\infty}\leq c\left\|v\right\|_{X^{r,s}(\Omega)}.$$

Proof. Let

$$\begin{aligned} v_l(x_1) &= \int_Q v(x_1,y)e^{-ily}dy, \\ P^1_{M,N}v &= \sum_{|l|_\infty \le N} v_l^*(x_1)e^{ily}, \\ a_l(u,w) &= \Big(\frac{\partial u}{\partial x_1}, \frac{\partial w}{\partial x_1}\Big)_{L^2(I)} + |l|^2\,(u,w)_{L^2(I)}, \quad |l|_\infty \le N. \end{aligned}$$

Then $v_l \in H_0^1(I), v_l^* \in V_M$ and

$$a_l(v_l - v_l^*, w) = 0, \quad \forall w \in V_M.$$

Obviously,

$$|v_l - v_l^*|^2_{1,I} + |l|^2\, \|v_l - v_l^*\|^2_{0,I} = a_l(v_l - v_l^*, v_l - v_l^*) = a_l(v_l - v_l^*, v_l - w), \quad \forall w \in V_M.$$

By Schwarz inequality and Poincare inequality,

$$\|v_l - v_l^*\|^2_{1,I} + |l|^2\, \|v_l - v_l^*\|^2_{0,I} \le c(\|v_l - w\|^2_{1,I} + |l|^2\, \|v_l - w\|^2_{0,I}), \quad \forall w \in V_M.$$

Let $P^1_M : H_0^1(I) \longrightarrow V_M$ be the projection operator, i.e., for all $u \in H_0^1(I)$,

$$\Big(\frac{d}{dx_1}(u - P^1_M u), \frac{dw}{dx_1}\Big) = 0, \quad \forall w \in V_M.$$

By Theorem 1.6 of [10], we get for any $u \in H_0^1(I)\bigcap H^r(I), r \ge 1$,

$$\left\|u - P^1_M u\right\|_{\alpha,I} \le cM^{\alpha - r}\,\|u\|_{r,I}\,, \quad 0 \le \alpha \le 1.$$

Then

$$\|v_l - v_l^*\|^2_{1,I} + |l|^2\, \|v_l - v_l^*\|^2_{0,I} \le c\Big(\left\|v_l - P^1_M v_l\right\|^2_{1,I} + |l|^2 \left\|v_l - P^1_M v_l\right\|^2_{0,I}\Big) \le cM^{2-2r}\,\|v_l\|^2_{r,I}\,.$$

Using duality technique, we have

$$\|v_l - v_l^*\|_{\alpha,I} \le cM^{\alpha - r}\,\|v_l\|_{r,I}\,, \quad \alpha = 0, 1.$$

Moreover,

$$\left\|P^1_{M,N}v\right\|_{1,\infty} \le \sum_{|l|_\infty \le N} (|v_l^*|_{1,\infty,I} + (1 + |l|)\,\|v_l^*\|_{\infty,I}).$$

Let $P_C : C(\bar{I}) \longrightarrow \mathcal{P}_M$ be the interpolation operator whose interpolation points are the nodes of Gauss-Lobatto integration formulas, i.e.,

$$x_1^{(0)} = -1, \quad x_1^{(N)} = 1, \quad x_1^{(j)} \quad (j = 1, \cdots, M-1) \text{ zeroes of } L'_M,$$

L_M being Legendre polynomial of degree M. Then

$$|v_l^*|_{1,\infty,I} \le \left\|\frac{dv_l^*}{dx_1} - P_C\frac{dv_l}{dx_1}\right\|_{\infty,I} + \left\|\frac{dv_l}{dx_1} - P_C\frac{dv_l}{dx_1}\right\|_{\infty,I} + \left\|\frac{dv_l}{dx_1}\right\|_{\infty,I}.$$

If $w \in \mathcal{P}_M$, then

$$\|w\|_{\infty,I} \le cM \|w\|_{L^2(I)}.$$

By Theorem 3.2 of [11], we have that for $0 \le m \le r$ and $r > \frac{1}{2}$,

$$\|v_l - P_C v_l\|_{m,I} \le cM^{2m-r+\frac{1}{2}} \|v_l\|_{r,I}.$$

So

$$\begin{aligned}\left\|\frac{dv_l^*}{dx_1} - P_C\frac{dv_l}{dx_1}\right\|_{\infty,I} &\le cM\left\|\frac{dv_l^*}{dx_1} - P_C\frac{dv_l}{dx_1}\right\|_{0,I} \\ &\le cM\Big(|v_l - v_l^*|_{1,I} + \left\|\frac{dv_l}{dx_1} - P_C\frac{dv_l}{dx_1}\right\|_{0,I}\Big) \\ &\le cM\,|v_l - v_l^*|_{1,I} + cM^{\frac{3}{2}-r}\|v_l\|_{r+1,I}, \quad r > \frac{1}{2}.\end{aligned}$$

On the other hand,

$$\|w - P_C w\|_{\infty,I} \le \left\|w - \Pi_{\frac{1}{2}+\epsilon,M} w\right\|_{\infty,I} + \left\|P_C(w - \Pi_{\frac{1}{2}+\epsilon,M} w)\right\|_{\infty,I}, \quad \epsilon > 0,$$

where $\Pi_{\frac{1}{2}+\epsilon,M}$ is defined in Lemma 3.3 of [11]. We get from the proof of Theorem 3.2 in [11] that

$$\begin{aligned}\left\|P_C(w - \Pi_{\frac{1}{2}+\epsilon,M} w)\right\|_{\infty,I} &\le cM\left\|P_C(w - \Pi_{\frac{1}{2}+\epsilon,M} w)\right\|_{0,I} \le cM\left\|w - \Pi_{\frac{1}{2}+\epsilon,M} w\right\|_{\infty,I}, \\ \left\|w - \Pi_{\frac{1}{2}+\epsilon,M} w\right\|_{\infty,I} &\le c(\left\|w - \Pi_{\frac{1}{2}+\epsilon,M} w\right\|_{H^{\frac{1}{2}+\epsilon}(I)}\left\|w - \Pi_{\frac{1}{2}+\epsilon,M} w\right\|_{H^{\frac{1}{2}-\epsilon}(I)})^{\frac{1}{2}}.\end{aligned}$$

By (3.10) of [11],

$$\|w - P_C w\|_{\infty,I} \le cM^{\frac{3}{2}-r}\|w\|_{r,I}, \quad r > \frac{1}{2}.$$

So

$$\left\|\frac{dv_l}{dx_1} - P_C\frac{dv_l}{dx_1}\right\|_{\infty,I} \le cM^{\frac{3}{2}-r}\|v_l\|_{r+1,I}.$$

Since imbedding theorem, we have for $r > \frac{1}{2}$,

$$|v_l|_{1,\infty,I} \le c\|v_l\|_{r+1,I}.$$

Thus we get for $r > \frac{3}{2}, s > \frac{n-1}{2}$,

$$\begin{aligned}\sum_{|l|_\infty \le N} |v_l^*|_{1,\infty,I} &\le cM\sum_{|l|_\infty \le N}|v_l - v_l^*|_{1,I} + c\sum_{|l|_\infty \le N}\|v_l\|_{r+1,I} \\ &\le (cM^{1-r} + c)\sum_{|l|_\infty \le N}\|v_l\|_{r+1,I} \\ &\le c(\sum_{|l|_\infty \le N}(1+|l|^2)^{-s})^{\frac{1}{2}}(\sum_{|l|_\infty \le N}(1+|l|^2)^s\|v_l\|^2_{r+1,I})^{\frac{1}{2}} \\ &\le c\|v\|_{X^{r,s}(\Omega)}.\end{aligned}$$

We can estimate $\sum\limits_{|l|_\infty \le N} (1+|l|)\, \|v_l^*\|_{\infty,I}$ in similar way. This completes the proof.

Lemma 6. *If $v, w \in \mathcal{P}_M \times V_N$, then*

$$\|vw\|^2 \le cM^2 N^{n-1} \|v\|^2 \|w\|^2 .$$

Proof. Let

$$A(v) = \int_Q \sup_{x_1 \in I} v^2(x_1, y) dy, \quad B(w) = \int_{-1}^{1} \sup_{y \in Q} w^2(x_1, y) dx_1.$$

Then

$$\|vw\|^2 \le A(v)B(w).$$

Set

$$v = \sum_{k=0}^{M} v_k(y) L_k(x_1), \quad w = \sum_{|l|_\infty \le N} w_l(x_1) e^{ily}.$$

Since $|L_k(x_1)| \le 1$, we have

$$\sup_{x_1 \in I} v^2(x_1, y) \le \Big(\sum_{k=0}^{M} |v_k(y)| \Big)^2 \le \frac{1}{2} \sum_{k=0}^{M} (2k+1) \sum_{k=0}^{M} \frac{2 v_k^2(y)}{2k+1} \le cM^2 \|v\|_{L^2(I)}^2 .$$

On the other hand,

$$\sup_{y \in Q} w^2(x_1, y) \le (\sum_{|l|_\infty \le N} |w_l(x_1)|)^2 \le cN^{n-1} \|w\|_{L^2(Q)}^2 .$$

Therefore,

$$A(v) \le cM^2 \|v\|^2 , \quad B(w) \le cN^{n-1} \|w\|^2 .$$

Then the proof is completed.

Lemma 7. *If $v, w \in V_{M,N}$, then*

$$\begin{aligned} \|d(v,w)\| &\le c \|v\|_{1,\infty} (\|w\| + |w|_1), \\ \|d(v,w)\| &\le c \|w\|_{1,\infty} (\|v\| + |v|_1), \\ \|d(v,w)\| &\le cMN^{\frac{n-1}{2}} (\|v\| \, |w|_1 + |v|_1 \|w\|). \end{aligned}$$

The first and second conclusions are obvious. The third one is obtained from Lemma 6.

Lemma 8 (Lemma 4.16 of [1]). *Assume that the following conditions are fulfilled*

(i) $E(t)$ and $E_1(t)$ are non-negative functions defined on R_τ;

(ii) *ρ, M_1 and M_2 are non-negative constants, and $M_3 > 0$;*

(iii) $B(z)$ is a function such that if $z \le M_3$, then $B(z) \le 0$;

(iv) for all $t \in R_\tau$,

$$E(t) \le \rho + M_1 \tau \sum_{\zeta \in R_\tau, \zeta < t} [E(\zeta) + M_2 E^2(\zeta) + B(E(\zeta)) E_1(\zeta)];$$

(v) $E(0) \le \rho$, *and for some* $t_1 \in R_\tau$,

$$\rho e^{2M_1 t_1} \le \min(\frac{1}{M_2}, M_3).$$

Then for all $t \in R_\tau, \quad t \le t_1$,

$$E(t) \le \rho e^{2M_1 t}.$$

4. Error Estimation

In this section, we first analyze the generalized stability of scheme (2.1). Assume that the initial values and the right terms of scheme (2.1) have the errors $\tilde{u}(0), \tilde{p}(0), \tilde{f}(t)$ and $\tilde{g}(t)$ respectively, which induce the errors $\tilde{u}$ and $\tilde{p}$ of u and p. They satisfy

$$\begin{cases} (\tilde{u}_t(t), v) + (d(u(t) + \delta\tau u_t(t), \tilde{u}(t)), v) + (d(\tilde{u}(t) + \delta\tau\tilde{u}_t(t), u(t) + \tilde{u}(t)), v) \\ \quad +\nu(\nabla(\tilde{u}(t) + \sigma\tau\tilde{u}_t(t)), \nabla v) + (\nabla(\tilde{p}(t) + \theta\tau\tilde{p}_t(t)), v) = (\tilde{f}(t), v), \ \forall v \in V_{M,N}, \\ (\beta\tilde{p}_t(t), v) + (\nabla\cdot(\tilde{u}(t) + \theta\tau\tilde{u}_t(t)), v) = (\tilde{g}(t), v), \quad \forall v \in S_{M-1,N}. \end{cases} \tag{4.1}$$

Let $\epsilon > 0$, and m be an undetermined positive constant. By putting $v = 2\tilde{u}(t) + m\tau\tilde{u}_t(t)$ in the first formula of (4.1), we have from Lemma 1 and integrating by parts that

$$\begin{aligned} &(\|\tilde{u}(t)\|^2)_t + \tau(m-1-\epsilon)\|\tilde{u}_t(t)\|^2 + 2\nu|\tilde{u}(t)|_1^2 + \nu\tau(\sigma + \frac{m}{2})(|\tilde{u}(t)|_1^2)_t \\ &\quad + \nu\tau^2(\sigma m - \sigma - \frac{m}{2})|\tilde{u}_t(t)|_1^2 + (\nabla(\tilde{p}(t) + \theta\tau\tilde{p}_t(t)), 2\tilde{u}(t) + m\tau\tilde{u}_t(t)) \\ &\quad + \sum_{j=1}^{3} F_j(t) \le \|\tilde{u}(t)\|^2 + (1 + \frac{\tau m^2}{4\epsilon})\left\|\tilde{f}(t)\right\|^2, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} F_1(t) &= \tau(m - 2\delta)(d(\tilde{u}(t), u(t) + \tilde{u}(t)), \tilde{u}_t(t)), \\ F_2(t) &= 2(d(u(t) + \delta\tau u_t(t), \tilde{u}(t)), \tilde{u}(t)), \\ F_3(t) &= m\tau(d(u(t) + \delta\tau u_t(t), \tilde{u}(t)), \tilde{u}_t(t)). \end{aligned}$$

By putting $v = 2\tilde{p}(t) + m\tau\tilde{p}_t(t)$ in the second formula of (4.1), we get from Lemma 1 that

$$\begin{aligned} &\beta(\|\tilde{p}(t)\|^2)_t + \beta\tau(m-1-\epsilon)\|\tilde{p}_t(t)\|^2 + (\nabla\cdot(\tilde{u}(t) + \theta\tau\tilde{u}_t(t)), 2\tilde{p}(t) + m\tau\tilde{p}_t(t)) \\ &\quad \le \beta\|\tilde{p}(t)\|^2 + (\frac{1}{\beta} + \frac{\tau m^2}{4\beta\epsilon})\|\tilde{g}(t)\|^2. \end{aligned} \tag{4.3}$$

By combining (4.2) and (4.3), we obtain

$$\begin{aligned} &(\|\tilde{u}(t)\|^2 + \beta\|\tilde{p}(t)\|^2)_t + \tau(m-1-\epsilon)(\|\tilde{u}_t\|^2 + \beta\|\tilde{p}_t(t)\|^2) + 2\nu|\tilde{u}(t)|_1^2 \\ &\quad + \nu\tau(\sigma + \frac{m}{2})(|\tilde{u}(t)|_1^2)_t + \nu\tau^2(\sigma m - \sigma - \frac{m}{2})|\tilde{u}_t(t)|_1^2 + \sum_{j=1}^{4} F_j(t) \\ &\quad \le \|\tilde{u}(t)\|^2 + \beta\|\tilde{p}(t)\|^2 + (1 + \frac{\tau m^2}{4\epsilon})(\left\|\tilde{f}(t)\right\|^2 + \frac{1}{\beta}\|\tilde{g}(t)\|^2), \end{aligned} \tag{4.4}$$

where

$$F_4(t) = (\nabla(\tilde{p}(t) + \theta\tau\tilde{p}_t(t)), 2\tilde{u}(t) + m\tau\tilde{u}_t(t)) + (\nabla\cdot(\tilde{u}(t) + \theta\tau\tilde{u}_t(t)), 2\tilde{p}(t) + m\tau\tilde{p}_t(t))$$
$$= (m-2\theta)\tau[(\nabla\cdot\tilde{u}(t), \tilde{p}_t(t)) + (\nabla\tilde{p}(t), \tilde{u}_t(t))].$$

We estimate $|F_j(t)| \quad (j = 1, \cdots, 4)$. By Lemma 7, we have

$$\begin{aligned}
|F_1(t)| &\le \epsilon\tau\|\tilde{u}_t(t)\|^2 + \frac{\tau(m-2\delta)^2}{4\epsilon}\|d(\tilde{u}(t), u(t)+\tilde{u}(t))\|^2 \\
&\le \epsilon\tau\|\tilde{u}_t(t)\|^2 + \frac{c\tau(m-2\delta)^2}{\epsilon}(\|d(\tilde{u}(t), u(t))\|^2 + \|d(\tilde{u}(t), \tilde{u}(t))\|^2) \\
&\le \epsilon\tau\|\tilde{u}_t(t)\|^2 + \frac{c\tau(m-2\delta)^2}{\epsilon}\|\|u\|\|_{1,\infty}^2(\|\tilde{u}(t)\|^2 + |\tilde{u}(t)|_1^2) \\
&+ \frac{c\tau(m-2\delta)^2M^2N^{n-1}}{\epsilon}\|\tilde{u}(t)\|^2|\tilde{u}(t)|_1^2, \\
|F_2(t)| &\le c\|\|u\|\|_{1,\infty}(\|\tilde{u}(t)\|^2 + \|\tilde{u}(t)\|\,|\tilde{u}(t)|_1) \\
&\le \frac{\nu}{2}|\tilde{u}(t)|_1^2 + (c + \frac{c}{\nu}\|\|u\|\|_{1,\infty})\|\|u\|\|_{1,\infty}\|\tilde{u}(t)\|^2.
\end{aligned}$$

We get from Lemma 7 and Lemma 2 that

$$\begin{aligned}
|F_3(t)| &\le \epsilon\tau\|\tilde{u}_t(t)\|^2 + \frac{\tau m^2}{4\epsilon}\|d(u(t)+\delta\tau u_t(t), \tilde{u}(t))\|^2 \\
&\le \epsilon\tau\|\tilde{u}_t(t)\|^2 + \frac{c\tau m^2}{\epsilon}\|\|u\|\|_{1,\infty}^2(\|\tilde{u}(t)\|^2 + |\tilde{u}(t)|_1^2).
\end{aligned}$$

Moreover,

$$\begin{aligned}
|F_4(t)| \le\ & \frac{\tau b\beta}{2}\|\tilde{p}_t(t)\|^2 + \frac{\tau(m-2\theta)^2}{2b\beta}|\tilde{u}(t)|_1^2 + \frac{n\beta(m-2\theta)^2}{\nu}\|\tilde{p}(t)\|^2 \\
&+ \frac{\nu\tau^2}{\beta}|\tilde{u}_t(t)|_1^2, \quad b > 0.
\end{aligned}$$

By substituting the above estimations into (4.4), we obtain

$$\begin{aligned}
&(\|\tilde{u}(t)\|^2 + \beta\|\tilde{p}(t)\|^2)_t + \tau(m-1-3\epsilon-\frac{b}{2})(\|\tilde{u}_t(t)\|^2 + \beta\|\tilde{p}_t(t)\|^2) \\
&\quad + (\nu - \frac{\tau(m-2\theta)^2}{2b\beta})|\tilde{u}(t)|_1^2 + \nu\tau(\sigma + \frac{m}{2})(|\tilde{u}(t)|_1^2)_t \\
&\quad + \nu\tau^2(\sigma m - \sigma - \frac{m}{2} - \frac{1}{\beta})|\tilde{u}_t(t)|_1^2 \\
&\quad \le M_1(\|\tilde{u}(t)\|^2 + \beta\|\tilde{p}(t)\|^2) + B(\|\tilde{u}(t)\|)|\tilde{u}(t)|_1^2 + G_1(t),
\end{aligned} \tag{4.5}$$

where

$$M_1 = 1 + \frac{n(m-2\theta)^2}{\nu} + c\|\|u\|\|_{1,\infty} + \left(\frac{c}{\nu} + \frac{c\tau(m^2 + (m-2\delta)^2)}{\epsilon}\right)\|\|u\|\|_{1,\infty}^2,$$

$$B(\|\tilde{u}(t)\|) = -\frac{\nu}{2} + \frac{c\tau(m^2 + (m-2\delta)^2)}{\epsilon}\|\|u\|\|_{1,\infty}^2 + \frac{c\tau(m-2\delta)^2M^2N^{n-1}}{\epsilon}\|\tilde{u}(t)\|^2,$$

$$G_1(t) = (1 + \frac{\tau m^2}{4\epsilon})(\left\|\tilde{f}(t)\right\|^2 + \frac{1}{\beta}\|\tilde{g}(t)\|^2).$$

Let $\beta < 1$, and suppose that

$$\tau = o\left(\frac{1}{2M^4 + (n-1)N^2}\right), \quad \nu\tau(2M^4 + (n-1)N^2)\left|\sigma - \frac{1}{2}\right| < \frac{1}{\alpha}, \quad \alpha > 1,$$
$$\beta > \max\left(\frac{\tau(m-2\theta)^2}{b\nu}, c\nu\tau(2M^4 + (n-1)N^2)\right).$$

We take

$$m = \frac{1 + 3\epsilon + c_0 + \frac{b}{2} + (\sigma + \frac{1}{\beta})\nu\tau(2M^4 + (n-1)N^2)}{1 - \frac{1}{\alpha}}, \quad c_0 > 0.$$

Then we get from (4.5) and Lemma 2 that

$$\begin{aligned}(\|\tilde{u}(t)\|^2 + \beta\,\|\tilde{p}(t)\|^2)_t + c_0\tau(\|\tilde{u}_t(t)\|^2 + \beta\,\|\tilde{p}_t(t)\|^2) + \frac{\nu}{2}\,|\tilde{u}(t)|_1^2 + \nu\tau(\sigma + \frac{m}{2})(|\tilde{u}(t)|_1^2)_t \\ \le M_1(\|\tilde{u}(t)\|^2 + \beta\,\|\tilde{p}(t)\|^2) + B(\|\tilde{u}(t)\|)\,|\tilde{u}(t)|_1^2 + G_1(t). \qquad (4.6)\end{aligned}$$

Let

$$E(t) = \|\tilde{u}(t)\|^2 + \beta\,\|\tilde{p}(t)\|^2 + \tau \sum_{\zeta\in R_\tau,\zeta<t} (c_0\tau(\|\tilde{u}_t(\zeta)\|^2 + \beta\,\|\tilde{p}_t(\zeta)\|^2) + \frac{\nu}{2}\,|\tilde{u}(\zeta)|_1^2),$$
$$\rho(t) = \|\tilde{u}(0)\|^2 + \beta\,\|\tilde{p}(0)\|^2 + \nu\tau(\sigma + \frac{m}{2})\,|\tilde{u}(0)|_1^2 + \tau \sum_{\zeta\in R_\tau,\zeta<t} G_1(\zeta).$$

By summing (4.6) for $t \in R_\tau$,

$$E(t) \le \rho(t) + \tau \sum_{\zeta\in R_\tau,\zeta<t} [M_1 E(\zeta) + B(\|\tilde{u}(\zeta)\|)\,|\tilde{u}(\zeta)|_1^2].$$

From Lemma 8, we obtain the following result.

Theorem 1. *If the following conditions are satisfied*
(i) $\tau = o\left(\frac{1}{2M^4+(n-1)N^2}\right), \quad \nu\tau(2M^4 + (n-1)N^2)\left|\sigma - \frac{1}{2}\right| < \frac{1}{\alpha}, \quad \alpha > 1$;
(ii) $\beta > \max\left(\frac{\tau(m-2\theta)^2}{b\nu}, c\nu\tau(2M^4 + (n-1)N^2)\right)$;
(iii) $\epsilon^{-1}c\tau(m^2 + (m-2\delta)^2)\,|||u|||_{1,\infty}^2 \le \frac{\nu}{4}$;
(iv) for some $t_1 \in R_\tau$,

$$\rho(t_1)e^{2M_1 t_1} \le \frac{\nu\epsilon}{4c\tau(m-2\delta)^2 M^2 N^{n-1}}.$$

Then for all $t \in R_\tau, \quad t \le t_1$,

$$E(t) \le \rho(t)e^{2M_1 t}.$$

Next, we derive the convergence of scheme (2.1). Let

$$U^*(t) = P^1_{M,N}U(t), \quad P^*(t) = P_{M-1,N}P(t), \quad e(t) = u(t) - U^*(t), \quad \phi(t) = p(t) - P^*(t).$$

By (1.1) and (2.1), we get

$$\begin{cases} (e_t(t),v)+(d(U^*(t)+\delta\tau U_t^*(t),e(t)),v)+(d(e(t)+\delta\tau e_t(t),U^*(t)+e(t)),v) \\ \qquad +\nu(\nabla(e(t)+\sigma\tau e_t(t)),\nabla v)+(\nabla(\phi(t)+\theta\tau\phi_t(t)),v) \\ \qquad =(\sum\limits_{j=1}^{2}A_j(t),v)-\nu(A_3(t),\nabla v)-(A_4(t),\nabla\cdot v), \quad \forall v\in V_{M,N}, \\ (\beta\phi_t(t),v)+(\nabla\cdot(e(t)+\theta\tau e_t(t)),v)=-(\beta A_5(t),v)+(A_6(t),v), \quad \forall v\in S_{M-1,N}, \end{cases}$$

where

$$\begin{aligned} &A_1(t)=\frac{\partial U(t)}{\partial t}-U_t^*(t), \quad A_2(t)=(U(t)\cdot\nabla)U(t)-d(U^*(t)+\delta\tau U_t^*(t),U^*(t)), \\ &A_3(t)=\sigma\tau\nabla U_t(t), \quad A_4(t)=P(t)-P^*(t)-\theta\tau P_t^*(t), \\ &A_5(t)=P_t(t), \quad A_6(t)=\nabla\cdot(U(t)-U^*(t)-\theta\tau U_t^*(t)). \end{aligned}$$

We give the bounds of $\|A_j(t)\| \quad (j=1,\cdots,6)$. Since

$$\frac{\partial U(t)}{\partial t}-U_t(t)=-\frac{1}{\tau}\int_t^{t+\tau}(t+\tau-\zeta)\frac{\partial^2U(\zeta)}{\partial\zeta^2}d\zeta$$

and Lemma 4, we have that for any $r,s\ge 1$,

$$\begin{aligned} \|A_1(t)\|^2 &\le 2(\left\|\frac{\partial U(t)}{\partial t}-U_t(t)\right\|^2+\|U_t(t)-U_t^*(t)\|^2) \\ &\le c\tau\left\|\frac{\partial^2U}{\partial t^2}\right\|^2_{L^2(t,t+\tau;L^2(\Omega))}+c\tau^{-1}(M^{-2r}+N^{-2s})\left\|\frac{\partial U}{\partial t}\right\|^2_{L^2(t,t+\tau;M^{r,s}(\Omega))}. \end{aligned}$$

Let $A_2(t)=J_1(t)+J_2(t)+J_3(t)$ where

$$\begin{aligned} &J_1(t)=(U(t)\cdot\nabla)U(t)-(U^*(t)\cdot\nabla)U^*(t), \quad J_2(t)=-\frac{1}{2}(\nabla\cdot U^*(t))U^*(t), \\ &J_3(t)=-\delta\tau(U^*(t)\cdot\nabla)U_t^*(t)-\frac{1}{2}\delta\tau(\nabla\cdot U^*(t))U_t^*(t). \end{aligned}$$

From Lemma 4 and Lemma 5, we get that for $r,s\ge 1$, $\lambda>\frac{3}{2}$, $\gamma>\frac{n-1}{2}$,

$$\begin{aligned} \|J_1(t)\|^2 &\le 2(\|(U(t)\cdot\nabla)U(t)-(U^*(t)\cdot\nabla)U(t)\|^2 \\ &\quad +\|(U^*(t)\cdot\nabla)U(t)-(U^*(t)\cdot\nabla)U^*(t)\|^2) \\ &\le c(\|U(t)\|^2_{1,\infty}\|U(t)-U^*(t)\|^2+\|U^*(t)\|^2_{\infty}\|U(t)-U^*(t)\|_1^2) \\ &\le c(M^{-2r}+N^{-2s})[\|U(t)\|^2_{1,\infty}\|U(t)\|^2_{M^{r,s}(\Omega)} \\ &\quad +\|U(t)\|^2_{X^{\lambda,\gamma}(\Omega)}\|U(t)\|^2_{M^{r+1,s+1}(\Omega)}]. \end{aligned}$$

By the second equation of (1.1), Lemma 4 and Lemma 5, we obtain that for r, $s\ge 1$, $\lambda>\frac{3}{2},\gamma>\frac{n-1}{2}$,

$$\begin{aligned} \|J_2(t)\|^2 &\le c\|U^*(t)\|^2_{\infty}\|\nabla\cdot(U(t)-U^*(t))\|^2 \\ &\le c(M^{-2r}+N^{-2s})\|U(t)\|^2_{X^{\lambda,\gamma}(\Omega)}\|U(t)\|^2_{M^{r+1,s+1}(\Omega)}. \end{aligned}$$

Similarly,

$$\begin{aligned}\|J_3(t)\|^2 &\le c\delta^2\tau^2 \|U^*(t)\|_\infty^2 (\|\nabla(U_t(t)-U_t^*(t))\|^2 + \|U_t(t)\|_1^2)\\ &\quad + c\delta^2\tau^2 \|U^*(t)\|_{1,\infty}^2 (\|U_t(t)-U_t^*(t)\|^2 + \|U_t(t)\|^2)\\ &\le c\delta^2\tau \|U(t)\|_{X^{\lambda,\gamma}(\Omega)}^2 \left\|\frac{\partial U}{\partial t}\right\|_{L^2(t,t+\tau;M^{1,1}(\Omega))}^2, \quad \lambda > \frac{3}{2}, \gamma > \frac{n-1}{2}.\end{aligned}$$

Obviously,

$$\|A_3(t)\|^2 \le c\sigma^2\tau \left\|\frac{\partial U}{\partial t}\right\|_{L^2(t,t+\tau;H^1(\Omega))}^2.$$

We have from Lemma 3 that for $r, s \ge 0$,

$$\begin{aligned}\|A_4(t)\|^2 &\le 2\|P(t)-P^*(t)\|^2 + c\theta^2\tau^2(\|P_t^*(t)-P_t(t)\|^2 + \|P_t(t)\|^2)\\ &\le c(M^{-2r}+N^{-2s})\|P(t)\|_{H^{r,s}(\Omega)}^2 + c\theta^2\tau \left\|\frac{\partial P}{\partial t}\right\|_{L^2(t,t+\tau;L^2(\Omega))}^2.\end{aligned}$$

Moreover,

$$\begin{aligned}&\|A_5(t)\|^2 \le c\tau^{-1}\left\|\frac{\partial P}{\partial t}\right\|_{L^2(t,t+\tau;L^2(\Omega))}^2,\\ &\|A_6(t)\|^2 \le c(M^{-2r}+N^{-2s})\|U(t)\|_{M^{r+1,s+1}(\Omega)}^2 + c\theta^2\tau\left\|\frac{\partial U}{\partial t}\right\|_{L^2(t,t+\tau;M^{1,1}(\Omega))}^2, \quad r,s \ge 1,\\ &e(0)=0, \quad \phi(0)=0.\end{aligned}$$

By an argument as in Theorem 1, we get the following result.

Theorem 2. *Suppose that*

(i) *conditions (i) and (ii) of Theorem 1 hold;*

(ii) *for* $r, s \ge 1, \lambda > \frac{3}{2}$ *and* $\gamma > \frac{n-1}{2}$,

$$\begin{aligned}&U \in C(0,T;M_{0,p}^{r+1,s+1}(\Omega)\bigcap X^{\lambda,\gamma}(\Omega)\bigcap W^{1,\infty}(\Omega)), \quad \frac{\partial U}{\partial t} \in L^2(0,T;M^{r,s}(\Omega)),\\ &\frac{\partial^2 U}{\partial t^2} \in L^2(0,T;L^2(\Omega)), \quad P \in C(0,T;H_p^{r,s}(\Omega)), \quad \frac{\partial P}{\partial t} \in L^2(0,T;L^2(\Omega)).\end{aligned}$$

Then for all $t \le T$,

$$\|U(t)-u(t)\|^2 \le d_3(\beta^{-1}(\tau^2 + M^{-2r} + N^{-2s}) + \beta),$$

where d_3 *is a positive constant depending only on* ν *and the norms of* U *and* P *in the spaces mentioned in the above.*

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