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TIME DISCRETIZATION SCHEMES FOR AN INTEGRO-DIFFERENTIAL EQUATION OF PARABOLIC TYPE*1)

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Abstract

In this paper a new approach for time discretization of an integro-differential equation of parabolic type is proposed. The methods are based on the backward-Euler and Crank-Nicolson Schemes but the memory and computational requirements are greatly reduced without assuming more regularities on the solution u.

1. Introduction

We consider the time discretization of the equation

$$\begin{cases} u_t + Au = \int_0^T b(t,s)Bu(s)ds + f(t), \quad 0 < t < T, \\ u(0) = v, \end{cases} \tag{1.1}$$

where A is an unbounded positive definite self-adjoint operator with dense domain $D(A)$ in a Hilbert space H and B is another operator with domain $D(B) \supset D(A)$. The kernel $b(t,s)$ is assumed to be a smooth real-valued function of both t and s for $0 \le s \le t$ and $f(t) \in H$ is a smooth function.

This type of problem occurs in applications such as heat conduction in material with memory, compression of poro-viscoelastic media, nuclear reactor dynamics, etc. The numerical solution by means of spatial discretization by finite differences and finite element methods has been studied by several authors; see V. Thomee [2] and the references cited there.

In this paper, we shall restrict our attention to the time discretization of such problems. A standard way of time discretization is to employ the quadrature formula

$$\int_0^{t_n} g(s)ds \approx \sum_{j=0}^{n-1} \omega_{nj} g(jk), \tag{1.2}$$

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where k denotes the time step, e.g., the left rectangle rule and the trapezoidal rule are simple quadrature rules which are consist with $O(k)$ accuracy of the backward-Euler scheme and with $O(k^2)$ accuracy of the Crank-Nicolson scheme , respectively.

Let $t_n = nk$ and U^n be the approximation of $u(t_n)$ and $f^n = f(t_n)$. Also we define the backward difference operator by

$$\bar{\partial}U^n = \frac{U^n - U^{n-1}}{k}. \tag{1.3}$$

Let $\sigma_1^n(g) = k\sum_{j=0}^{n-1} g(t_j)$ and $\sigma_2^n(g) = \frac{1}{2}kg(0) + \sum_{j=0}^{n-1} g(t_j)$ be the left rectangle rule and the trapezoidal rule respectively. Then, the standard backward Euler and Crank-Nicolson schemes are

$$BE : \begin{cases} \bar{\partial}U^n + AU^n = \sigma_1^n(b(t_n, s)BU) + f^n, & n = 1, 2, \cdots, \\ U^0 = V; \end{cases} \tag{1.4}$$

$$CN : \begin{cases} \bar{\partial}U^n + A\Big(\dfrac{U^n + U^{n-1}}{2}\Big) = \sigma_2^n(b(t_{n-\frac{1}{2}}, s)BU) + f^{n-\frac{1}{2}}, & n = 1, 2, \cdots, \\ U^0 = V, \end{cases} \tag{1.5}$$

where $\sigma_1^n(b(t_n, s)BU) = k\sum_{j=0}^{n-1} b(t_n, t_j)BU^j$ and σ_2^n is similar.

A practical difficulty of these methods is that all U^j need to be stored as they all enter the subsequent equations; hence the number of U^j which have to be stored is of order $O(\frac{1}{k})$ per unit time.

In order to reduce the memory requirement, Sloan and Thomee[1] proposed more economical schemes by using quadrature rules with higher order truncation errors. For example, in order to retain the accuracy of the backward Euler scheme, they used the trapezoidal rule with mesh size $k_1 = O(\sqrt{k})$ on $[0, \bar{t_{jn}}]$ and the rectangle rule with mesh size k on the remaining small part $[\bar{t_{jn}}, t_n]$, where $\bar{t_{jn}} = \max\{jk_1\}$ $(jk_1 \le t_{n-1})$. For this scheme, the storage requirements are reduced from $O(\frac{1}{k})$ to $O(\frac{1}{\sqrt{k}})$ per unit time. Likewise, a combination of Simpson's rule and the trapezoidal rule preserves the accuracy of the Crank-Nicolson scheme. Because of using higher order quadratures, the regularity requirement of the solution u is very severe.

The results here are based on the following iterative relations for the quadrature:

$$\sigma_1^n(g) = k\sum_{j=0}^{n-1} g(t_j) = \sigma_1^{n-1}(g) + kg(t_{n-1}) \left(\approx \int_0^{t_n} g(s)ds\right), \tag{1.6}$$

$$\begin{cases} \sigma_2^n(g) = \dfrac{1}{2}kg(0) + k\displaystyle\sum_{j=0}^{n-1} g(t_j) = \sigma_2^{n-1}(g) + kg(t_{n-1}) \\ \qquad \left(\approx \displaystyle\int_0^{t_{n-\frac{1}{2}}} g(s)ds\right), \\ \sigma_2^0(g) = -\dfrac{1}{2}kg(0). \end{cases} \tag{1.7}$$

2. Iterative Backward Euler (IBE) and Iterative Crank-Nicolson (ICN) Schemes

2.1. Product type kernels $b(t,s) = p(t)q(s)$

For product type kernels, IBE and ICN can be easily defined as

$$IBE: \begin{cases} \bar{\partial}U^n + AU^n = p^n\sigma_1^n(qBU) + f^n, & n = 1,2,\cdots, \\ U^0 = V; \end{cases} \tag{2.1}$$

$$ICN: \begin{cases} \bar{\partial}U^n + A(\frac{U^n+U^{n-1}}{2}) = p^{n-\frac{1}{2}}\sigma_2^n(qBU) + f^{n-\frac{1}{2}}, & n = 1,2,\cdots, \\ U^0 = V. \end{cases} \tag{2.2}$$

For these schemes, we need only to store σ^n and U^{n-1}; the previous U^j can be discarded. Hence the storage requirement is greatly reduced.

2.2. The analytic kernels

Assume that $b(t,s)$ is analytic and all the derivatives are bounded by M. Then, by the Taylor expansion of $b(t,x)$, we have

$$b(t,s) = b_m + r_m, \tag{2.3}$$

$$b_m = b(t,0) + sb_s^1(t,0) + \cdots + \frac{s^{m-1}}{(m-1)!}b_s^{m-1}(t,0) = \sum_{i=0}^{m} p_i(t)q_i(s), \tag{2.4}$$

$$|r_m| \le \frac{M}{m!}s^m \le \frac{T^m}{m!}M. \tag{2.5}$$

Choose m such that $\frac{T^m}{m!} = O(k^\alpha)$, where $\alpha = 1$ for the backward Euler scheme and $\alpha = 2$ for the Crank-Nicolson scheme. Replace the kernel by its degenerate Taylor approximation. We define:

$$IBE: \begin{cases} \bar{\partial}U^n + AU^n = \sigma_1^n(b_mBU) + f^n, & n = 1,2,\cdots, \\ U^0 = V; \end{cases} \tag{2.6}$$

$$ICN: \begin{cases} \bar{\partial}U^n + A(\frac{U^n+U^{n-1}}{2}) = \sigma_2^n(b_mBU) + f^{n-\frac{1}{2}}, & n = 1,2,\cdots. \\ U^0 = V. \end{cases} \tag{2.7}$$

Since $b_m = \sum_{i=0}^{m} p_i(t)q_i(s)$ is a sum of m functions p_iq_i with variables separated, $\sigma_1^n(b_mBU) = \sum_{i=1}^{m} p_i^n\sigma_1^n(q_iBU), \sigma_2^n(b_mBU) = \sum_{i=1}^{m} p_i^{n-\frac{1}{2}}\sigma_2^n(q_iBU)$, we need only to store U^{n-1}, $\sigma^n(q_i BU)$, $i = 1,2,\cdots,m$. It is easy to show that $\frac{m}{|\ln k|} \to 0$ as $k \to 0$. Generally, the storage requirement for $\sigma^n(q_iBU)$ is like that of U^n if space discretization is considered. Hence in our schemes, the total storage requirement is of order $|\ln k|$ per unit time, which is much smaller than $O(\frac{1}{k})$. All the computation in the quadrature can be carried out recursively.

2.3. More general case

If the kernel is not so smooth, supposing $b \in C^m$ and $||b||_{C^m} \le M$ for some fixed m, we propose the following approach.

Dividing interval $[0, T]$ into subintervals with length $k_1 = lk$ for some integer l and using the Taylor expansion of $b(t, s)$ on the subinterval, we have:

$$
\begin{aligned}
&b(t,s) = b_{mj} + r_{mj}, \quad jk_1 \le s < (j+1)k_1, \\
&b_{mj} = b(t, jk_1) + (s - jk_1)b_s^1(t, jk_1) + \cdots + \frac{s - jk_1{}^{m-1}}{(m-1)!} b_s^{m-1}(t, jk_1), \qquad (2.8) \\
&|r_{mj}| \le \frac{k_1^m}{m!} M. \qquad (2.9)
\end{aligned}
$$

Choosing k_1 such that $\frac{k_1^m}{m!} = O(k^\alpha)$ with $\alpha = 1$ for the backward Euler and $\alpha = 2$ for the Crank-Nicolson scheme. Replacing the kernel by its piecewise degenerate Taylor approximation, we can similarly define IBE and ICN as above. Now the storage requirement is of order $m \cdot k_1^{-1} = O(mk^{-\frac{\alpha}{m}})$ per unit time.

In view of practical implementation, the computation of the derivatives of the kernel is somewhat complicated. We suggest using a piecewise Lagrange interpolation of degree $m-1$ of the kernel instead of the piecewise degenerate Taylor approximation.

Divide $[0, T]$ into elements with length $k_1 = lk$ as above and then construct a piecewise polynomial subspace with degree $m - 1$ by adding some inner nodes in the elements. It is not necessary to require the inner nodes to be time step points. Let $\{\varphi_j(s)\}$ be the nodal basis of the constructed subspace which is just a conforming finite element subspace with freedom $N^* = mTk_1^{-1}$. Now the interpolation $b_m(t, s)$ can be written as

$$
b_m(t,s) = \sum_{j=1}^{N^*} b(t, t_j^*)\varphi_j(s) \qquad (2.10)
$$

which is a sum of functions with variables separated. IBE and ICN can be defined as in (2.6) and (2.7). In order to preserve the accuracy, it is required that $k_1^m = O(k^\alpha)$. Hence the storage requirement is again of order $\frac{N^*}{T} = O(mk^{-\frac{\alpha}{m}})$ per unit time. A special case is $m = \alpha$. In this case the IBE or ICN becomes their standard form separately (BE or CN).

3. Error Estimates

In this section we derive the error estimates by using the stability of standard backward Euler and Crank-Nicolson schemes. First we have the following Lemmas from [1].

Lemma 3.1. *Assume that $||A^{-1}B||$ is bounded. Then the backward Euler scheme with left rectangle quadrature yields*

$$||U^n|| \leq C(T)\Big(||v|| + k\sum_{j=1}^{n}||f^j||\Big), \quad nk \leq T.$$

Lemma 3.2. *Assume that $||A^{-1}B||$ is bounded. Then the Crank-Nicolson scheme with trapezoidal rule yields*

$$||U^n|| \leq C(T)(||v|| + k\sum_{j=1}^{n}||f^{j-\frac{1}{2}}||), \quad nk \leq T.$$

Now we can show the error estimates:

Theorem 3.1. *Assume that $||A^{-1}B||$ is bounded in H and that $||u_{tt}||, ||Bu_t||$ and $||Bu||$ are bounded. Then for IBE scheme we have that*

$$||u(t_n) - U^n|| \leq C(T,u)k, \quad t_n \leq T.$$

Proof. Let $e^n = u(t_n) - U^n$, it follows from (1.1) and (2.6) that e^n satisfies the equation

$$\bar{\partial}e^n + Ae^n = \sigma_1^n(b_m Be) + \tau_n \tag{3.1}$$

where τ_n is the truncation error

$$\tau_n = \frac{u(t_n) - u(t_{n-1})}{k} - u_t(t_n) + \int_0^{t_n} b(t_n, s)Bu(s)ds - \sigma_1^n(bBu) + \sigma_1^n((b-b_m)Bu). \tag{3.2}$$

Under the assumptions of this theorem we can easily verify that

$$||\tau_n|| \leq C(T,u)k. \tag{3.3}$$

This and Lemma 3.1 complete the proof.

Similarly we have

Theorem 3.2. *Assume that $||A^{-1}B||$ is bounded in H and that $||u_{ttt}||, ||Bu_{tt}||$ and $||Bu||$ are bounded. Then for ICN scheme, we have that*

$$||u(t_n) - U^n|| \leq C(T,u)k^2, \quad t_n \leq T.$$

From these theorems we see that the regularity requirements on the solution are much weaker than in the economical schemes of [1] where $||Bu_{tt}||$ (for backward Euler) and $||Bu_{ttt}||$ (for Crank-Nicolson) are also required to be bounded. The difference between the schemes here and the schemes suggested in [1] are the following.

First we store the integral $\int_0^{t_n} bBUds$ in parts instead of storing the individual U^j, hence we can reduce the memory requirements in several way provided $b(t,s)$ smooth.

Secondly we employ lower order quadrature rules which are just consist with the discretization instead of employing higher order quadrature rules, hence we do not need more regularities on the solution u. What we have to do is to find an approximation of kernel $b(t, s)$ such that the approximation is a sum of functions with variables separated. In general, the interpolation is a reasonable choice.

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References

[1] I. H. Sloan and V. Thomee, Time discretization of an integro-differential equation of parabolic type, *SIAM J. NUmer. Anal.*, **23** : 5 (1986).

[2] V. Thomee , On the numerical solutionof integro-diffeential equations of parabolic type, *International Series of Numerical Mathematics,* 86 (1988).