Journal of Computational Mathematics, Vol.11, No.3, 1993, 276–288.

L^∞ ERROR ESTIMATES OF NONCONFORMING FINITE ELEMENTS FOR THE BIHARMONIC EQUATION*1)

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Abstract

The paper considers the L^∞ convergence for nonconforming finite elements, such as Morley element, Adini element and De Veubeke element, solving the boundary value problem of the biharmonic equation. The nearly optimal order L^∞ estimates are given.

§1. Introduction

The L^∞ convergence of finite element methods is an interested topic. For second order partial differential equations, there are many papers discussing this problem, but for the boundary value problem of the biharmonic equation, only Morley element was discussed in [6]. Alrough the result given in [6] does not hold, the way of the proof is meaningful. This paper will consider the nonconforming finite elements and give the nearly optimal L^∞ estimates for them.

Let Ω be a convex polygonal domain. The Dirichlet boundary value problem of the biharmonic equation is the following

$$\begin{cases} \triangle^2 u = f, & \text{in } \Omega, \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial N}\Big|_{\partial\Omega} = 0 \end{cases} \tag{1.1}$$

where $N = (N_x, N_y)$ is the unit normal of $\partial\Omega$.

For $p \in [1, \infty]$ and $m \geq 0$, let $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ be the usual Sobolev spaces, and $\|\cdot\|_{m,p,\Omega}$ and $|\cdot|_{m,p,\Omega}$ be the Sobolev norm and semi-norm respectively. When $p = 2$, denote them by $H^m(\Omega)$, $H_0^m(\Omega)$, $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$ respectively. Let $H^{-m}(\Omega)$ be the dual space of $H_0^m(\Omega)$ with norm $\|\cdot\|_{-m,\Omega}$.

It is known that $\forall f \in H^{-1}(\Omega)$, problem (1.1) has a unique solution $u \in H_0^2(\Omega) \cap H^3(\Omega)$, such that

$$\|u\|_{3,\Omega} \leq C\|f\|_{-1,\Omega}, \tag{1.2}$$

with C a positive constant.

* Received April 11, 1992.

1) The Project Supported by National Natural Science Foundation of China.

Define, $\forall u, v \in H^2(\Omega)$,

$$a(u,v) = \int_\Omega \left(\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial y^2} \right) dxdy. \tag{1.3}$$

Let $f \in L^2(\Omega)$. The variational form of problem (1.1) is to find $u \in H_0^2(\Omega)$, such that,

$$a(u,v) = (f,v), \qquad \forall v \in H_0^2(\Omega), \tag{1.4}$$

where $(\cdot,\cdot)$ is the L^2 product.

For $h \in (0, h_0)$ with $h_0 \in (0,1)$, let $\mathcal{T}_h$ be a subdivision of Ω by triangles or rectangles. Let h_T =diam$T$ and ρ_T the largest of the diameters of all circles contained in T. Assume that there exists a positive constant η, independent of h, such that $\eta h < \rho_T < h_T \le h$ for all $T \in \mathcal{T}_h$. Let $V_h \subset L^2(\Omega)$ be a finite element space associated with $\mathcal{T}_h$.

Define $a_h(\cdot,\cdot)$, for $v, w \in H^2(\Omega) + V_h$, as follows;

$$a_h(v,w) = \sum_{T\in\mathcal{T}_h} \int_T \left(\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial^2 v}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2}\frac{\partial^2 w}{\partial y^2} \right) dxdy. \tag{1.5}$$

The finite element approximation to problem (1.1) is to find $u_h \in V_h$, such that

$$a_h(u_h, v) = (f,v), \qquad \forall v \in V_h. \tag{1.6}$$

For $w \in L^2(\Omega)$ and $w|_T \in H^m(T)$ for all $T \in \mathcal{T}_h$, define

$$|w|_{m,h} = \left(\sum_{T\in\mathcal{T}_h} |w|_{m,T}^2 \right)^{1/2}. \tag{1.7}$$

For $w \in L^\infty(\Omega)$ and $w|_T \in W^{m,\infty}(T)$ for all $T \in \mathcal{T}_h$, define

$$|w|_{m,\infty,h} = \max_{T\in\mathcal{T}_h} |w|_{m,\infty,T}\,. \tag{1.8}$$

This paper will show that the estimate of $|u-u_h|_{1,\infty,h}$ is $\mathcal{O}(h^2|\ln h|^{5/4})$ for Morley element, Adini element and De Veubeke element.

The remainder of the paper is arranged as follows. Section 2 will give the L^∞ estimates for Morley element and its properties. Section 3 will give the proof of the L^∞ estimate for Morley element. The last two sections will consider the case of Adini element and De Veubeke element.

§2. Morley Element

From now on, let $\mathcal{T}_h$ be a subdivision of Ω by triangles and $V_h \subset L^2(\Omega)$ be a Morley finite element space associated with $\mathcal{T}_h$. Then $v \in V_h$ if and only if it has the following properties:

1) $v|_T$ is quadratic for all $T \in \mathcal{T}_h$.
2) v is continuous at the vertices and vanishes at the vertices along $\partial\Omega$.

3) $\partial v/\partial N$ is continuous at the midpoints of interelement boundaries and vanishes at the midpoints along $\partial\Omega$.

Let u be the solution of problem (1.1) and u_h the solution of problem (1.6). Then from [8], the following estimates are true:

$$|u-u_h|_{m,h} \le Ch^{3-m}\left(|u|_{3,\Omega}+h\|f\|_{0,\Omega}\right), \quad m=1,2. \tag{2.1}$$

Throughout the paper, C always denotes the positive constant independent of h, with different values at different places. For L^∞ estimates, we have

Theorem 1. *Let V_h be a Morley finite element space, u the solution of problem (1.1) and u_h the solution of problem (1.6). If $u \in W^{3,\infty}(\Omega)$ and $f \in L^\infty(\Omega)$, then*

$$|u-u_h|_{1,\infty,h} \le Ch^2|\ln h|^{5/4}\left(|u|_{3,\infty,\Omega}+h\|f\|_{0,\infty,\Omega}\right). \tag{2.2}$$

The proof of Theorem 1 will be given in Section 3. Now we list some properties of the Morley element space.

For $T \in \mathcal{T}_h$, let Π_T be the interpolation operator of Morley element. For $v \in H^3(\Omega)$, let $\Pi_h v|_T = \Pi_T v$ for all $T \in \mathcal{T}_h$. The following results are well known:

$$|v-\Pi_T v|_{m,T} \le Ch^{3-m}|v|_{m,T}, \quad 0 \le m \le 3, v \in H^3(T), T \in \mathcal{T}_h. \tag{2.3}$$

From [10,11], the following inequalities are true for $v \in V_h$:

$$\sum_{i=0}^{2}|v|_{i,h}+|v|_{0,\infty,\Omega} \le C|v|_{2,h}, \tag{2.4}$$

$$|v|_{0,\infty,\Omega} \le C|\ln h|^{1/2}|v|_{1,h}, \tag{2.5}$$

$$|v|_{1,\infty,h} \le C|\ln h|^{1/2}|v|_{2,h}. \tag{2.6}$$

Theorem 2. *Let V_h be a Morley finite element space, u the solution of problem (1.1) and u_h the solution of problem (1.6). If $f \in L^2(\Omega)$, then*

$$|u-u_h|_{0,\infty,\Omega} \le Ch^2|\ln h|^{1/2}\left(|u|_{3,\Omega}+h\|f\|_{0,\Omega}\right). \tag{2.7}$$

Proof. From the interpolation theory(see [1]), we have

$$|u-\Pi_h u|_{0,\infty,\Omega} \le Ch^2|u|_{3,\Omega}.$$

By the above inequality, inequalities (2.1), (2.3) and (2.5), we have

$$\begin{aligned}
|u-u_h|_{0,\infty,\Omega} &\le |u-\Pi_h u|_{0,\infty,\Omega}+|\Pi_h u-u_h|_{0,\infty,\Omega}\\
&\le C\left(h^2|u|_{3,\Omega}+|\ln h|^{1/2}|\Pi_h u-u_h|_{1,h}\right)\\
&\le C\left[h^2|u|_{3,\Omega}+|\ln h|^{1/2}\left(|\Pi_h u-u|_{1,h}+|u-u_h|_{1,h}\right)\right]\\
&\le Ch^2|\ln h|^{1/2}\left(|u|_{3,\Omega}+h\|f\|_{0,\Omega}\right).
\end{aligned}$$

The estimate (2.7) is proved.

Let $P_h : L^2(\Omega) \to V_h$ be the $L^2(\Omega)$ orthogonal projection operator, i.e., for $\forall w \in L^2(\Omega)$,

$$(P_h w, v) = (w, v), \quad \forall v \in V_h. \tag{2.8}$$

Lemma 1. *If $0 \le m \le 2$, then for $w \in H_0^m(\Omega)$,*

$$|w - P_h w|_{0,\Omega} \le Ch^m |w|_{m,\Omega}, \tag{2.9}$$

$$|P_h w|_{m,h} \le C|w|_{m,\Omega}. \tag{2.10}$$

Proof. The following inequality is trivial:

$$\|w - P_h w\|_{0,\Omega} \le 2\|w\|_{0,\Omega}, \quad \forall w \in L^2(\Omega). \tag{2.11}$$

From (2.3), we have

$$\|w - P_h w\|_{0,\Omega} \le Ch^3\|w\|_{3,\Omega}, \quad \forall w \in H^3(\Omega) \cap H_0^2(\Omega). \tag{2.12}$$

By the interpolation theory of Hilbert spaces([4]), we have

$$\|w - P_h w\|_{0,\Omega} \le Ch^m\|w\|_{m,\Omega}, \quad \forall w \in H_0^m(\Omega).$$

Then inequality (2.9) follows. For each T in $\mathcal{T}^h$, denote by $P_m(T)$ the space of polynomials with degree not greater than m, and define $P_T^m : L^2(T) \to P_m(T)$ as the $L^2(T)$ orthogonal projection operator. Then

$$\sum_{i=0}^{m} h^i|v - P_T^m v|_{i,T} \le Ch^m|v|_{m,T}, \quad \forall w \in H^m(T). \tag{2.13}$$

From inequalities (2.9) and (2.13) and the inverse inequality, for $w \in H_0^m(\Omega)$, we have

$$\begin{aligned}
|P_h w|_{m,h}^2 &= \sum_{T\in\mathcal{T}_h} |P_h w|_{m,T}^2 \le C \sum_{T\in\mathcal{T}_h} (|P_h w - P_T^m w|_{m,T}^2 + |P_T^m w|_{m,T}^2) \\
&\le C \sum_{T\in\mathcal{T}_h} (h^{-2m}|P_h w - P_T^m w|_{0,T}^2 + |w|_{m,T}^2) \\
&\le C \sum_{T\in\mathcal{T}_h} [h^{-2m}(|P_h w - w|_{0,T}^2 + |w - P_T^m w|_{0,T}^2) + |w|_{m,T}^2] \\
&\le C[h^{-2m}(|P_h w - w|_{0,\Omega}^2 + h^{2m} \sum_{T\in\mathcal{T}_h} |w|_{m,T}^2) + |w|_{m,\Omega}^2] \\
&\le C|w|_{m,\Omega}^2.
\end{aligned}$$

It follows that (2.10) is true.

§3. The proof of Theorem 1

In this section, we will prove Theorem 1. Let the assumption of Theorem 1 be true. By the interpolation result (see [1]), we have

$$\begin{aligned}
|u - u_h|_{1,\infty,h} &\le |u - \Pi_h u|_{1,\infty,h} + |\Pi_h u - u_h|_{1,\infty,h} \\
&\le Ch^2|u|_{3,\infty,\Omega} + |\Pi_h u - u_h|_{1,\infty,h}.
\end{aligned} \tag{3.1}$$

So we must estimate $|\Pi_h u - u_h|_{1,\infty,h}$. Let $T' \in \mathcal{T}_h$ be the element such that $|\Pi_h u - u_h|_{1,\infty,h} = |\Pi_h u - u_h|_{1,\infty,T'}$. Without loss of generality, suppose that

$$|\Pi_h u - u_h|_{1,\infty,T'} = \left|\frac{\partial(\Pi_h u - u_h)}{\partial x}\right|_{0,\infty,T'}.$$

Let $(x_0, y_0) \in T'$ be the point such that

$$\left|\frac{\partial(\Pi_h u - u_h)}{\partial x}\right|_{0,\infty,T'} = \left|\frac{\partial(\Pi_h u - u_h)}{\partial x}(x_0, y_0)\right|.$$

To prove Theorem 1, we need some results about the weight function and the regular Green function. For (x_0, y_0), define the weight function ρ as

$$\rho(x,y) = (x - x_0)^2 + (y - y_0)^2 + \lambda^2 h^2,$$

with λ a fixed positive number. For integer α and $v \in H^m(T)$ and $T \in \mathcal{T}_h$, define

$$|v|_{m,(\alpha),T} = \left(\sum_{i+j=m} \int_T \rho^{-\alpha} \left|\frac{\partial^m v}{\partial x^i \partial y^j}\right|^2 dxdy\right)^{1/2}. \tag{3.2}$$

For $v \in H^m(\Omega) + V_h$, define

$$|v|_{m,(\alpha)} = \left(\sum_{T\in\mathcal{T}_h} |v|^2_{m,(\alpha),T}\right)^{1/2}.$$

For the weight function, the following inequalities are true:

$$|v|_{m,(\gamma)} \le (\lambda h)^{-(\gamma-\alpha)} |v|_{m,(\alpha)}, \quad \gamma > \alpha, v \in H^m(\Omega) + V_h, \tag{3.3}$$

$$|v|_{0,(1)} \le C|\ln h|^{1/2} \|v\|_{0,\infty,\Omega}, \qquad \forall v \in L^\infty(\Omega), \tag{3.4}$$

$$\left|\int_\Omega vw\, dxdy\right| \le |v|_{0,(\alpha)} |v|_{0,(-\alpha)}, \quad v, w \in L^2(\Omega), \tag{3.5}$$

$$|v - \Pi_T v|_{k,(\alpha),T} \le Ch^{3-k} |v|_{3,(\alpha),T}, \quad 0 \le k \le 3, v \in H^3(T), T \in \mathcal{T}_h. \tag{3.6}$$

Lemma 2. *There exists a constant C such that, for $v \in H_0^2(\Omega) \cap H^3(\Omega)$, $\triangle^2 v \in L^2(\Omega)$ and $w \in H_0^2(\Omega), v_h \in V_h$, the following inequality is true:*

$$|a_h(v, w - v_h) - (\triangle^2 v, w - v_h)| \le Ch(|v|_{3,(\alpha)} + h|\triangle^2 v|_{0,(\alpha)})|w - v_h|_{2,(-\alpha)}. \tag{3.7}$$

Proof. By the approach used in [8], we can derive

$$|a_h(v, w - v_h) - (\triangle^2 v, w - v_h)| \le Ch \sum_{T\in\mathcal{T}_h} (|v|_{3,T} + h\|\triangle^2 v\|_{0,T})|w - v_h|_{2,T}. \tag{3.8}$$

For $\forall T \in \mathcal{T}_h$, we have

$$\max_{(x,y)\in T} \rho(x,y) \le C \min_{(x,y)\in T} \rho(x,y).$$

Inequality (3.7) follows from the above two inequalities.

Now we turn to the regular Green function. Let $q \in P_4(T')$ satisfy

$$\int_{T'} qp\,dxdy = \frac{\partial}{\partial x}p(x_0,y_0), \qquad \forall p \in P_4(T').$$

Define $\delta_h \in L^2(\Omega)$ such that

$$\delta_h(x,y) = \begin{cases} q(x,y), & (x,y) \in T', \\ 0, & \text{otherwise}. \end{cases} \tag{3.9}$$

Lemma 3.

$$\|\delta_h\|_{0,\Omega} \le Ch^{-2}, \tag{3.10}$$

$$\|\delta_h\|_{-1,\Omega} + \|\delta_h\|_{0,(-1)} \le Ch^{-1}. \tag{3.11}$$

Proof. Inequality (3.10) is easy to show. Now we prove inequality (3.11). By (2.9), (2.10) and (3.10), we have

$$\begin{aligned}\|\delta_h\|_{-1,\Omega} &= \sup_{0\neq v\in H_0^1(\Omega)} \frac{(\delta_h, v)}{\|v\|_{1,\Omega}} \le \sup_{0\neq v\in H_0^1(\Omega)} \left(\frac{|(\delta_h, P_h v)|}{\|v\|_{1,\Omega}} + \frac{|(\delta_h, v - P_h v)|}{\|v\|_{1,\Omega}} \right) \\ &\le \sup_{\substack{v\in H_0^1(\Omega)\\ P_h v\neq 0}} \frac{|(\delta_h, P_h v)|}{\|P_h v\|_{1,h}} + Ch\|\delta_h\|_{0,\Omega} \\ &\le \sup_{\substack{v\in H_0^1(\Omega)\\ P_h v\neq 0}} \frac{|\frac{\partial}{\partial x} P_h v(x_0,y_0)|}{\|P_h v\|_{1,h}} + Ch^{-1} \le Ch^{-1}.\end{aligned}$$

From the definitions of ρ and δ_h and inequality (3.10), we have

$$\|\delta_h\|_{0,(-1)}^2 = \int_\Omega \rho\delta_h^2 dxdy = \int_{T'} \rho\delta_h^2 dxdy \le Ch^2 \int_{T'} \delta_h^2 dxdy \le Ch^2\|\delta_h\|_{0,\Omega}^2 \le Ch^{-2}.$$

Inequality (3.11) then follows.

Let g be the regular Green function determined by

$$\begin{cases} \triangle^2 g = \delta_h, \quad \text{in} \quad \Omega, \\ g|_{\partial\Omega} = \dfrac{\partial g}{\partial N}\Big|_{\partial\Omega} = 0, \end{cases} \tag{3.12}$$

and g_h be its finite element solution by Morley element, i.e.,

$$a_h(g_h, v_h) = (\delta_h, v_h), \qquad \forall v_h \in V_h. \tag{3.13}$$

From (1.2), (2.1), (3.10) and (3.11), we get

$$|g - g_h|_{0,h} + |g - g_h|_{1,h} + h|g - g_h|_{2,h} + h^2|g|_{3,\Omega} \le Ch. \tag{3.14}$$

Lemma 4.

$$\|g\|_{2,\Omega} \le C|\ln h|^{1/2}. \tag{3.15}$$

Proof. By (1.5), (2.6), (3.9) and (3.13), we get

$$|g_h|_{2,h}^2 \le a_h(g_h, g_h) = (\delta_h, g_h) = \frac{\partial}{\partial x} g_h(x_0, y_0) \le |g_h|_{1,\infty,h} \le C|\ln h|^{1/2}|g_h|_{2,h}.$$

Hence

$$|g_h|_{2,h} \le C|\ln h|^{1/2}.$$

Inequality (3.15) then follows from (3.14).

Lemma 5.

$$|g|_{3,(-1)} \le C|\ln h|^{1/2}. \tag{3.16}$$

Proof. From the definition of the weight function we have

$$|g|_{3,(-1)}^2 = \sum_{i+j=3} \int_\Omega \left|(x-x_0)\frac{\partial^3 g}{\partial x^i \partial y^j}\right|^2 dxdy + \sum_{i+j=3} \int_\Omega \left|(y-y_0)\frac{\partial^3 g}{\partial x^i \partial y^j}\right|^2 dxdy$$

$$+ \sum_{i+j=3} \int_\Omega \left|\lambda h \frac{\partial^3 g}{\partial x^i \partial y^j}\right|^2 dxdy$$

$$\le C\{|(x-x_0)g|_{3,\Omega}^2 + |(y-y_0)g|_{3,\Omega}^2 + |g|_{2,\Omega}^2 + h^2|g|_{3,\Omega}^2\}.$$

By (3.14) and (3.15), we get

$$|g|_{3,(-1)} \le C(|(x-x_0)g|_{3,\Omega} + |(y-y_0)g|_{3,\Omega} + |\ln h|^{1/2}). \tag{3.17}$$

Define $\psi_1 = \triangle^2((x-x_0)g)$, $\psi_2 = \triangle^2((y-y_0)g)$. Then

$$\psi_1 = (x-x_0)\delta_h + 4\frac{\partial^3 g}{\partial x^3} + 2\frac{\partial^3 g}{\partial x \partial y^2}, \quad \psi_2 = (y-y_0)\delta_h + 4\frac{\partial^3 g}{\partial y^3} + 2\frac{\partial^3 g}{\partial x^2 \partial y}.$$

For $\forall v \in H_0^1(\Omega)$,

$$(\psi_1, v) = ((x-x_0)\delta_h, v) + 4(\frac{\partial^3 g}{\partial x^3}, v) + 2(\frac{\partial^3 g}{\partial x \partial y^2}, v)$$

$$= ((x-x_0)\delta_h, P_h v) + ((x-x_0)\delta_h, v - P_h v) + 4(\frac{\partial^2 g}{\partial x^2}, \frac{\partial v}{\partial x}) - 2(\frac{\partial^2 g}{\partial y^2}, \frac{\partial v}{\partial x}).$$

Applying (3.9), (3.10), (3.15) and (2.5) and Lemma 1, we have

$$|(\psi_1, v)| \le h^2\|\delta_h\|_{0,\Omega}|P_h v|_{0,\infty,T'} + Ch^2\|\delta_h\|_{0,\Omega}|v|_{1,\Omega} + 6|g|_{2,\Omega}|v|_{1,\Omega}$$

$$\le C(1 + |\ln h|^{1/2})(|P_h v|_{1,h} + |v|_{1,\Omega}) \le C|\ln h|^{1/2}|v|_{1,\Omega}.$$

Furthermore,

$$\|\psi_1\|_{-1,\Omega} = \sup_{0 \ne v \in H_0^1(\Omega)} \frac{(\psi_1, v)}{\|v\|_{1,\Omega}} \le C|\ln h|^{1/2}.$$

Similarly,

$$\|\psi_2\|_{-1,\Omega} \le C|\ln h|^{1/2}.$$

Hence,

$$|(x-x_0)g|_{3,\Omega} \le C\|\psi_1\|_{-1,\Omega} \le C|\ln h|^{1/2}, \tag{3.18}$$

$$|(y-y_0)g|_{3,\Omega} \le C\|\psi_2\|_{-1,\Omega} \le C|\ln h|^{1/2}. \tag{3.19}$$

From (3.17) to (3.19), we get (3.16).

For $\forall v \in H^2(T)$, let $\Pi_T^1 v$ be the linear interpolation polynomial using the function values at the vertices. For $v \in L^2(\Omega)$ and $v|_T \in H^2(T), \forall T \in \mathcal{T}_h$, define $\Pi_h^1 v$ by $(\Pi_h^1 v)|_T = \Pi_T^1(v|_T), \forall T \in \mathcal{T}_h$.

For $\forall v \in H^2(T)$, let $\tilde{\Pi}_T(\rho v) \in P_2(T)$ be determined as follows. $\tilde{\Pi}_T(\rho v)$ is equal to ρv at the vertices of T, and $\frac{\partial}{\partial N}\tilde{\Pi}_T(\rho v)$ is equal to $\frac{\partial \rho}{\partial N}\Pi_T^1 v + \rho\frac{\partial v}{\partial N}$ at the midpoints of the edges of T. For $v \in L^2(\Omega)$ and $v|_T \in H^2(T), \forall T \in \mathcal{T}_h$, define $\tilde{\Pi}_h(\rho v)$ by $(\tilde{\Pi}_h(\rho v))|_T = \tilde{\Pi}_T((\rho v)|_T), \forall T \in \mathcal{T}_h$. For $v_h \in V_h$, $\tilde{\Pi}_h(\rho v_h)$ is in V_h, but $\Pi_h(\rho v_h)$ is not.

Lemma 6. *For $\forall v_h \in V_h$ and $T \in \mathcal{T}_h$,*

$$|\Pi_T(\rho v_h) - \tilde{\Pi}_T(\rho v_h)|_{2,T} \le Ch|v_h|_{2,(-1),T}. \tag{3.20}$$

Proof. Denote $\eta = \frac{\partial^2}{\partial x^2}(\Pi_T(\rho v_h) - \tilde{\Pi}_T(\rho v_h))$. Then η is a constant on T, and

$$\int_T \eta^2 dxdy = \int_{\partial T} \eta\left\{N_x^2\frac{\partial}{\partial N}(\Pi_T(\rho v_h) - \tilde{\Pi}_T(\rho v_h)) - N_xN_y\frac{\partial}{\partial s}(\Pi_T(\rho v_h) - \tilde{\Pi}_T(\rho v_h))\right\} ds,$$

by Green's formula. Let $F_i, i = 1,2,3$ be the edges of T, and let A_i be the midpoints of F_i and $|F_i|$ the length of F_i. From the definition of Π_T and $\tilde{\Pi}_T$, we have

$$\int_T \eta^2 dxdy = \sum_{i=1}^3 \eta N_x^2|F_i|\frac{\partial \rho}{\partial N}(A_i)(v_h(A_i) - \Pi_T^1 v_h(A_i)).$$

From

$$\left|\frac{\partial \rho}{\partial x}\right|^2 + \left|\frac{\partial \rho}{\partial y}\right|^2 \le 4\rho, \tag{3.21}$$

$$|v_h - \Pi_T^1 v_h|_{0,\infty,T} \le Ch|v_h|_{2,T},$$

it follows that

$$\int_T \eta^2 dxdy \le C\sum_{i=1}^3 h^2|\eta|\rho^{1/2}(A_i)|v_h|_{2,T} \le Ch|\eta|_{0,T}|v_h|_{2,(-1),T},$$

i.e.,

$$|\eta|_{0,T} \le Ch|v_h|_{2,(-1),T}\ .$$

Similarly, we can show the case of other two second order partial derivatives. Inequality (3.20) is thus proved.

Lemma 7.

$$|g - g_h|_{2,(-1)} \le Ch|\ln h|^{3/4}. \tag{3.22}$$

Proof. By simple compution, we have

$$|g-g_h|_{2,(-1)}^2 \le \sum_{T\in\mathcal{T}_h}\int_T \rho\left[\left(\frac{\partial^2}{\partial x^2}(g-g_h)\right)^2 + 2\left(\frac{\partial^2}{\partial x\partial y}(g-g_h)\right)^2\right.$$
$$\left.+\left(\frac{\partial^2}{\partial y^2}(g-g_h)\right)^2\right]dxdy \le |a_h(g-g_h,\rho(g-g_h))|$$
$$+\Big|\sum_{T\in\mathcal{T}_h}\int_T \frac{\partial^2(g-g_h)}{\partial x^2}(g-g_h)dxdy\Big| + \Big|\sum_{T\in\mathcal{T}_h}\int_T \frac{\partial^2(g-g_h)}{\partial y^2}(g-g_h)dxdy\Big|$$
$$+C\sum_{i+j=2}\sum_{T\in\mathcal{T}_h}\int_T \Big|\frac{\partial^2}{\partial x^i\partial y^j}(g-g_h)\Big|\rho^{1/2}\Big|\frac{\partial(g-g_h)}{\partial x}\Big|dxdy$$
$$+C\sum_{i+j=2}\sum_{T\in\mathcal{T}_h}\int_T \Big|\frac{\partial^2}{\partial x^i\partial y^j}(g-g_h)\Big|\rho^{1/2}\Big|\frac{\partial(g-g_h)}{\partial y}\Big|dxdy.$$

From (3.5), we get

$$|g-g_h|_{2,(-1)}^2 \le |a_h(g-g_h,\rho(g-g_h))| + C|g-g_h|_{2,(-1)}|g-g_h|_{1,h}$$
$$+\Big|\sum_{T\in\mathcal{T}_h}\int_T \frac{\partial^2(g-g_h)}{\partial x^2}(g-g_h)dxdy\Big| + \Big|\sum_{T\in\mathcal{T}_h}\int_T \frac{\partial^2(g-g_h)}{\partial y^2}(g-g_h)dxdy\Big|, \tag{3.23}$$

$\Pi_h^1(g-g_h)\in H_0^1(\Omega)$ and

$$\sum_{T\in\mathcal{T}_h}\int_T \frac{\partial^2(g-g_h)}{\partial x^2}(g-g_h)dxdy = \sum_{T\in\mathcal{T}_h}\int_T \frac{\partial^2(g-g_h)}{\partial x^2}\Pi_h^1(g-g_h)dxdy$$
$$+\sum_{T\in\mathcal{T}_h}\int_T \frac{\partial^2(g-g_h)}{\partial x^2}(g-g_h-\Pi_h^1(g-g_h))dxdy$$
$$=\sum_{T\in\mathcal{T}_h}\left\{\int_{\partial T} \frac{\partial(g-g_h)}{\partial x}\Pi_h^1(g-g_h)N_x\,ds - \int_T \frac{\partial(g-g_h)}{\partial x}\frac{\partial \Pi_h^1(g-g_h)}{\partial x}dxdy\right\}$$
$$+\sum_{T\in\mathcal{T}_h}\int_T \frac{\partial^2(g-g_h)}{\partial x^2}(g-g_h-\Pi_h^1(g-g_h))dxdy.$$

By the interpolation theory and the properties of Morley element, we get

$$\Big|\sum_{T\in\mathcal{T}_h}\int_T \frac{\partial^2(g-g_h)}{\partial x^2}(g-g_h)dxdy\Big|$$
$$\le C(h^2|g-g_h|_{2,h}^2 + h|g-g_h|_{2,h}|\Pi_h^1(g-g_h)|_{1,\Omega} + |g-g_h|_{1,h}|\Pi_h^1(g-g_h)|_{1,\Omega})$$
$$\le C(h^2|g-g_h|_{2,h}^2 + h|g-g_h|_{2,h}|g-g_h|_{1,h} + |g-g_h|_{1,h}^2).$$

From (3.14) it follows that

$$\Big|\sum_{T\in\mathcal{T}_h}\int_T \frac{\partial^2(g-g_h)}{\partial x^2}(g-g_h)dxdy\Big| \le Ch^2. \tag{3.24}$$

Similarly,

$$\Big|\sum_{T\in\mathcal{T}_h}\int_T \frac{\partial^2(g-g_h)}{\partial y^2}(g-g_h)dxdy\Big| \le Ch^2. \tag{3.25}$$

By (3.14) and (3.23) to (3.25), we have

$$|g-g_h|^2_{2,(-1)} \le |a_h(g-g_h,\rho(g-g_h))| + Ch|g-g_h|_{2,(-1)} + Ch^2. \tag{3.26}$$

From (3.3), (3.5) and (3.6), we have

$$\begin{aligned}|a_h(g-g_h,\rho(g-\Pi_h g))| &\le C|g-g_h|_{2,(-1)}|\rho(g-\Pi_h g)|_{2,(1)}\\ &\le C|g-g_h|_{2,(-1)}(|g-\Pi_h g|_{2,(-1)} + |g-\Pi_h g|_{1,(0)} + |g-\Pi_h g|_{0,(1)})\\ &\le Ch|g-g_h|_{2,(-1)}(|g|_{3,(-1)} + h|g|_{3,\Omega} + h^2|g|_{3,(1)})\\ &\le Ch|g-g_h|_{2,(-1)}(|g|_{3,(-1)} + h|g|_{3,\Omega}).\end{aligned}$$

From (3.16), we get

$$|a_h(g-g_h,\rho(g-\Pi_h g))| \le Ch|\ln h|^{1/2}|g-g_h|_{2,(-1)}. \tag{3.27}$$

For $v\in L^2(\Omega)$, let $P_h^0 v\in L^2(\Omega)$ be determined by $(P_h^0 v)|_T = P_T^0(v|_T)$. From (3.5) and (3.6), we have

$$\begin{aligned}&|a_h(g-g_h,\rho(\Pi_h g-g_h)-\Pi_h(\rho(\Pi_h g-g_h)))|\\ &\quad=|a_h(g-g_h,(\rho-P_h^0\rho)(\Pi_h g-g_h)-\Pi_h[(\rho-P_h^0\rho)(\Pi_h g-g_h)])|\\ &\quad\le|g-g_h|_{2,(-1)}|(\rho-P_h^0\rho)(\Pi_h g-g_h)-\Pi_h[(\rho-P_h^0\rho)(\Pi_h g-g_h)]|_{2,(1)}\\ &\quad\le Ch|g-g_h|_{2,(-1)}|(\rho-P_h^0\rho)(\Pi_h g-g_h)|_{3,(1)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}|(\rho-P_h^0\rho)(\Pi_h g-g_h)|_{3,(1)} &\le C\left[h|\Pi_h g-g_h|_{3,h} + |\Pi_h g-g_h|_{2,h} + |\Pi_h g-g_h|_{1,(1)}\right]\\ &\le C\left[|\Pi_h g-g_h|_{2,h} + h^{-1}|\Pi_h g-g_h|_{1,h}\right] \le C,\end{aligned}$$

by (2.3), (3.14), (3.21) and (3.3) and the inverse inequality. Hence

$$|a_h(g-g_h,\rho(\Pi_h g-g_h)-\Pi_h(\rho(\Pi_h g-g_h)))| \le Ch|g-g_h|_{2,(-1)}. \tag{3.28}$$

From (3.20), (3.14) and (2.3), we have

$$\begin{aligned}&|a_h(g-g_h,\Pi_h(\rho(\Pi_h g-g_h))-\tilde{\Pi}_h(\rho(\Pi_h g-g_h)))|\\ &\quad\le|g-g_h|_{2,(-1)}|\Pi_h(\rho(\Pi_h g-g_h))-\tilde{\Pi}_h(\rho(\Pi_h g-g_h)))|_{2,(1)}\\ &\quad\le Ch|g-g_h|_{2,(-1)}|\Pi_h g-g_h|_{2,h} \le Ch|g-g_h|_{2,(-1)}.\end{aligned} \tag{3.29}$$

From Lemma 2 and (3.20), we have

$$
\begin{aligned}
|a_h(g-g_h,\tilde{\Pi}_h\rho(\Pi_h g-g_h))| &= |a_h(g,\tilde{\Pi}_h\rho(\Pi_h g-g_h)) - (\triangle^2 g,\tilde{\Pi}_h\rho(\Pi_h g-g_h))| \\
&\le Ch(|g|_{3,(-1)} + h|\delta_h|_{0,(-1)})|\tilde{\Pi}_h\rho(\Pi_h g-g_h)|_{2,(1)} \\
&\le Ch|\ln h|^{1/2}\left(|\Pi_h\rho(\Pi_h g-g_h) - \tilde{\Pi}_h\rho(\Pi_h g-g_h)|_{2,(1)} + |\Pi_h\rho(\Pi_h g-g_h)|_{2,(1)}\right) \\
&\le Ch|\ln h|^{1/2}(h|\Pi_h g-g_h|_{2,h} + |\Pi_h\rho(\Pi_h g-g_h)|_{2,(1)}) \\
&\le Ch|\ln h|^{1/2}(h + |\Pi_h\rho(\Pi_h g-g_h)|_{2,(1)}),
\end{aligned}
$$

and

$$
\begin{aligned}
|\Pi_h\rho(\Pi_h g-g_h)|_{2,(1)} &\le (|\rho(\Pi_h g-g_h)|_{2,(1)} + Ch|\rho(\Pi_h g-g_h)|_{3,(1)}) \le C|\rho(\Pi_h g-g_h)|_{2,(1)} \\
&\le C(|\Pi_h g-g_h|_{2,(-1)} + |\Pi_h g-g_h|_{1,h} + |\Pi_h g-g_h|_{0,(1)}) \\
&\le C(|g-g_h|_{2,(-1)} + |g-\Pi_h g|_{2,(-1)} + |\Pi_h g-g_h|_{1,h} + |\ln h|^{1/2}|\Pi_h g-g_h|_{0,\infty,\Omega}) \\
&\le C(|g-g_h|_{2,(-1)} + h|g|_{3,(-1)} + (1+|\ln h|)|\Pi_h g-g_h|_{1,h}) \\
&\le C(|g-g_h|_{2,(-1)} + h|\ln h|).
\end{aligned}
$$

That leads to

$$
|a_h(g-g_h,\tilde{\Pi}_h\rho(\Pi_h g-g_h))| \le Ch|\ln h|^{1/2}(|g-g_h|_{2,(-1)} + h|\ln h|). \tag{3.30}
$$

Combining (3.27) to (3.30), we get

$$
|a_h(g-g_h,\rho(g-g_h))| \le Ch|\ln h|^{1/2}(|g-g_h|_{2,(-1)} + h|\ln h|). \tag{3.31}
$$

By (3.26) and (3.31), we have

$$
|g-g_h|^2_{2,(-1)} \le Ch|\ln h|^{1/2}(|g-g_h|_{2,(-1)} + h|\ln h|).
$$

Inequality (3.22) then follows.

From (1.4), (1.6), (3.5), (3.7), (3.9) and (3.13), we have

$$
\begin{aligned}
|\Pi_h u-u_h|_{1,\infty,h} &= |(\delta_h,\Pi_h u-u_h)| = |a_h(g_h,\Pi_h u-u_h)| = |a_h(g-g_h,u-\Pi_h u) \\
&\quad + [a_h(u,g_h-g) - (\triangle^2 u,g_h-g)] + [a_h(g,\Pi_h u-u) - (\triangle^2 g,\Pi_h u-u)] \\
&\quad + (\triangle^2 g,\Pi_h u-u)| \le |g-g_h|_{2,(-1)}|u-\Pi_h u|_{2,(1)} + Ch(|u|_{3,(1)} \\
&\quad + h|f|_{0,(1)})|g-g_h|_{2,(-1)} + Ch(|g|_{3,(-1)} + h|\delta_h|_{0,(-1)})|u-\Pi_h u|_{2,(1)} \\
&\quad + |\delta_h|_{0,(-1)}|u-\Pi_h u|_{0,(1)}).
\end{aligned}
$$

By (3.4), (3.6), (3.11), (3.16) and (3.22), we get

$$
\begin{aligned}
|\Pi_h u-u_h|_{1,\infty,h} &\le Ch^2[|\ln h|^{3/4}(|u|_{3,(1)} + h|f|_{0,(1)}) + (1+|\ln h|^{1/2})|u|_{3,(1)}] \\
&\le Ch^2|\ln h|^{5/4}(|u|_{3,\infty,\Omega} + h\|f\|_{0,\infty,\Omega}).
\end{aligned} \tag{3.32}
$$

Inequalities (3.1) and (3.32) imply (2.2). Theorem 1 is finally proved.

§4. De Veubeke Element

In this section, let $\mathcal{T}_h$ be a subdivision of Ω by triangles and $V_h \subset L^2(\Omega)$ be a De Veubeke finite element space associated with $\mathcal{T}_h$. Then $v \in V_h$ if and only if it has the following properties:

1) $v|_T$ is cubic for all $T \in \mathcal{T}_h$.

2) v is continuous at the vertices and vanishes at the vertices along $\partial\Omega$.

3) $\partial v/\partial N$ is continuous at the Gaussian points of second order on interelement boundaries and vanishes at the Gaussian points of second order on the edges along $\partial\Omega$.

Let u be the solution of problem (1.1) and u_h the solution of problem (1.6) with V_h the De Veubeke finite element space. The discussion for Morley element in sections 2 and 3 is suitable for De Veubeke element. The difference is the definition of the operator $\tilde{\Pi}_T$, which is replaced by the following. For $\forall v \in H^2(T)$, $\tilde{\Pi}_T(\rho v)$ is equal to ρv at the vertices of T, and $\frac{\partial}{\partial N}\tilde{\Pi}_T(\rho v)$ is equal to $\frac{\partial \rho}{\partial N}\Pi_T^1 v + \rho\frac{\partial v}{\partial N}$ at the Gaussian points of second order on the edges of T.

For De Veubeke element, we have the following L^∞ estimates.

Theorem 3. *Let V_h be a De Veubeke finite element space, u the solution of problem (1.1) and u_h the solution of problem (1.6). Then*

$$|u-u_h|_{0,\infty,\Omega} \le Ch^2|\ln h|^{1/2}\left(|u|_{3,\Omega} + h\|f\|_{0,\Omega}\right) \tag{4.1}$$

when $f \in L^2(\Omega)$, and

$$|u-u_h|_{1,\infty,h} \le Ch^2|\ln h|^{5/4}\left(|u|_{3,\infty,\Omega} + h\|f\|_{0,\infty,\Omega}\right) \tag{4.2}$$

when $u \in W^{3,\infty}(\Omega)$ and $f \in L^\infty(\Omega)$.

§5. Adini Element

In this section, let $\mathcal{T}_h$ be a subdivision of Ω by rectangles and $V_h \subset L^2(\Omega)$ be Adini finite element space associated with $\mathcal{T}_h$. Then $v \in V_h$ if and only if it has the following properties:

1) $v|_T$ is in $P_3(T)$+span$\{x^3y, xy^3\}$ for all $T \in \mathcal{T}_h$.

2) v and its partial derivatives of first order are continuous at the vertices and vanish at the vertices along $\partial\Omega$.

Let u be a solution of problem (1.1) and u_h the solution of problem (1.6) with V_h the Adini finite element space. For Adini element, inequalities (2.1) are replaced by

$$|u-u_h|_{m,h} \le Ch^{3-m}|u|_{3,\Omega}, \qquad m = 1,2. \tag{5.1}$$

Theorem 4. *Let V_h be an Adini finite element space, u the solution of problem* (1.1) *and u_h the solution of problem* (1.6). *Then*

$$|u-u_h|_{0,\infty,\Omega} \le Ch^2|\ln h|^{1/2}|u|_{3,\Omega} \tag{5.2}$$

when $u \in H_0^2(\Omega) \cap H^3(\Omega)$, and

$$|u-u_h|_{1,\infty,h} \le Ch^2|\ln h|^{5/4}|u|_{3,\infty,\Omega} \tag{5.3}$$

when $u \in W^{3,\infty}(\Omega)$.

The proof of Theorem 4 is similar to that of Theorems 1 and 2, except that here we drop the term of $h\|f\|_{0,\Omega}$ or a similar one and do not use $\tilde{\Pi}_h$ because $\Pi_h(\rho v_h) \in V_h, \forall v_h \in V_h$, for the Adini element.

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