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THE BIVARIATE SPLINE APPROXIMATE SOLUTION TO THE HYPERBOLIC EQUATIONS WITH VARIABLE COEFFICIENTS*

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§1. Introduction

We are interested in constructing a continuously differentiable surface of approximate solution to the linear hyperbolic equations with variable coefficients. To this end, in Section 2 we develop a finite difference scheme by overlapping three finite difference schemes, which approximate the exact solution and its two partial derivatives. By using this scheme we can conveniently obtain a continuously differentiable surface in the space of bivariate spline functions. In fact, this scheme is determined "uniquely" by the spline space.

In Section 3 we show that the scheme is stable and convergent in L_2-norm. The interesting fact is that, making use of the spline approximation theory, we can estimate the error by using appropriate moduli of smoothness. This leads to the fact that the spline approximate solution and its partial derivatives are convergent to the exact solution and its partial derivatives respectively when the exact solution is in C^2. In fact, in Section 5 we will prove the following result:

Under the hypotheses of Theorem 3.1, if the exact solution v of the hyperbolic equation (2.1) is in $C^2(D)$, then the following estimates hold:

$$\|v-u\|_{D'} \le K\alpha\omega(D^2v,\alpha,D'),$$

$$\left.\begin{array}{l}\|\partial_x(v-u)\|_{D'}\\ \|\partial_t(v-u)\|_{D'}\end{array}\right\} \le K\omega(D^2v,\alpha,D'),$$

where u is a spline approximate solution and $\alpha=\sqrt{h^2+k^2}$, with h and k as steps in the directions x and t respectively.

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The convergence can be shown even if the exact solution is in C^1.

In this paper we handle only the single equation, but it is not difficult to generalize our method to the hyperbolic system. In Sections 4 and 5 we introduce some definitions and results about the space of bivariate spline functions and the bivariate approximation theory. For more details, the readers are referred to [1] and [3]. All constants appearing in this paper are denoted by K although they may be different.

§2. The Finite Difference Scheme

Consider the first order hyperbolic equation of the form

$$\partial_t v = a(x,t)\partial_x v \tag{2.1.1}$$

where $a(x,t) < 0$ for all $(x,t) \in D = [0,+\infty) \times [0,+\infty)$, with the initial-boundary conditions

$$V(x,0) = f(x), \qquad 0 \le x < +\infty; \tag{2.1.2}$$

$$V(0,t) = g(t), \qquad 0 \le t < +\infty. \tag{2.1.3}$$

Let u_i^n, $(\partial_t u)_i^n$ and $(\partial_x u)_i^n$ be three grid functions which approximate the exact solution v, $\partial_t v$ and $\partial_x v$ at the points (ih, nk) respectively, where k is the time step and h is the grid spacing. Those grid functions are determined by the following relations:

$$u_i^n = (1 - a_i^n\lambda)^{-1}[u_i^{n-1} + \frac{k}{2}(\partial_t u_t)_i^{n-1} - \frac{k}{2}a_i^n(\partial_x u)_{i-1}^n - \lambda a_i^n u_{i-1}^n], \tag{2.2.1}$$

$$(\partial_t u)_i^n = -(\partial_t u)_i^{n-1} + 2k^{-1}(u_i^n - u_i^{n-1}), \tag{2.2.2}$$

$$(\partial_x u))_i^n = -(\partial_x u)_{i-1}^n + 2h^{-1}(u_i^n - u_{i-1}^n), \quad i,n = 1,2,3,\cdots, \tag{2.2.3}$$

where $\lambda = k/h$. In order to implement this scheme, let

$$u_0^0 = f(0) = g(0), \tag{2.3.1}$$

$$(\partial_x u)_i^0 = f'(ih), \quad (\partial_t u)_i^0 = a_i^0 f'(ih), \quad i = 0,1,2,\cdots, \tag{2.3.2}$$

$$(\partial_t u)_0^n = g'(nk), \quad (\partial_x u)_0^n = (a_0^n)^{-1}g'(nk), \quad n = 0,1,2,\cdots. \tag{2.3.3}$$

The following lemma shows where the relation (2.2.1) comes from.

Lemma 2.1: *The three relations in (2.2) are equivalent to the relations (2.2.2), (2.2.3) and the following relation*

$$(\partial_t u)_i^n = a_i^n(\partial_x u)_i^n, \quad i,n = 0,1,2,3,\cdots. \tag{2.2.1'}$$

This lemma can be shown by substituting (2.2.2) and (2.2.3) into (2.2.1'). (2.2.2) and (2.2.3) are necessary in constructing a continuously differentiable spline funtion in the space of $S_2^1(\Delta)$ by using the grid functions u_i^n, $(\partial_t u)_i^n$ and $(\partial_x u)_i^n$. The work of the second author in [1] is referred to.

Lemma 2.2. *For fixed i and n, the following relations hold:*

$$u_i^n = \frac{1+a_i^{n-1}\lambda}{1-a_i^n\lambda}u_i^{n-1} - \frac{1+a_i^n\lambda}{1-a_i^n\lambda}u_{i-1}^n + \frac{1-a_i^{n-1}\lambda}{1-a_i^n\lambda}u_{i-1}^{n-1}$$
$$+\frac{a_{i-1}^{n-1}-a_i^{n-1}}{1-a_i^n\lambda}\cdot\frac{k}{2}(\partial_x u)_{i-1}^{n-1} + \frac{a_{i-1}^n - a_i^n}{1-a_i^n\lambda}\cdot\frac{k}{2}(\partial_x u)_{i-1}^n, \tag{2.4.1}$$

$$(\partial_x u)_i^n = \frac{1+a_i^{n-1}\lambda}{1-a_i^n\lambda}(\partial_x u)_i^{n-1} - \frac{1+a_{i-1}^n\lambda}{1-a_i^n\lambda}(\partial_x u)_{i-1}^n + \frac{1-a_{i-1}^{n-1}\lambda}{1-a_i^n\lambda}(\partial_x u)_{i-1}^{n-1}, \tag{2.4.2}$$

$$(\partial_t u)_i^n = -\frac{1+(a_i^{n-1}\lambda)^{-1}}{1-(a_i^n\lambda)^{-1}}(\partial_t u)_i^{n-1} + \frac{1+(a_{i-1}^n\lambda)^{-1}}{1-(a_i^n\lambda)^{-1}}(\partial_t u)_{i-1}^n$$
$$+\frac{1-(a_{i-1}^{n-1}\lambda)^{-1}}{1-(a_i^n\lambda)^{-1}}(\partial_t u)_{i-1}^{n-1}. \tag{2.4.3}$$

Proof. By (2.2.1), we have

$$(1-a_i^n\lambda)u_i^n = u_i^{n-1} + \frac{k}{2}(\partial_t u)_i^{n-1} - \frac{k}{2}a_i^n(\partial_x u)_{i-1}^n - a_i^n\lambda u_{i-1}^n$$
$$= u_i^{n-1} + \frac{k}{2}a_i^{n-1}(\partial_x u)_i^{n-1} - \frac{k}{2}a_i^n(\partial_x u)_{i-1}^n - a_i^n\lambda u_{i-1}^n$$
$$= u_i^{n-1} + \frac{k}{2}a_i^{n-1}[-(\partial_x u)_{i-1}^{n-1} + \frac{2}{h}(u_i^{n-1} - u_{i-1}^{n-1})] - \frac{k}{2}a_i^n(\partial_x u)_{i-1}^n - a_i^n\lambda u_{i-1}^n$$
$$= (1+a_i^{n-1}\lambda)u_i^{n-1} - (1+a_i^n\lambda)u_{i-1}^n + (1-a_i^{n-1}\lambda)u_{i-1}^{n-1} +$$
$$+(a_{i-1}^{n-1} - a_i^{n-1})\frac{k}{2}(\partial_x u)_{i-1}^{n-1} + (a_{i-1}^n - a_i^n)\frac{k}{2}(\partial_x u)_{i-1}^n.$$

This leads to (2.4.1). Similarly, we have

$$k(\partial_x u)_i^n = \lambda h(\partial_x u)_i^n = -k(\partial_x u)_{i-1}^n + 2\lambda(u_i^n - u_{i-1}^n) = a_i^n\lambda k(\partial_x u)_i^n$$
$$+(1+a_i^{n-1}\lambda)k(\partial_x u)_i^{n-1} - (1+a_{i-1}^n\lambda)k(\partial_x u)_{i-1}^n + (1-a_{i-1}^{n-1}\lambda)k(\partial_x u)_{i-1}^{n-1}.$$

This gives (2.4.2). In the same way (2.4.3) can be shown.

In the case that the advection equation (2.4.1) is very simple, suppose that $a(x,t) = a < 0$, where a is a constant. It follows from (2.4.1) that

$$u_i^n = \mu u_i^{n-1} - \mu u_{i-1}^n + u_{i-1}^{n-1}$$

where $\mu = \dfrac{1+a\lambda}{1-a\lambda}$, or that

$$\frac{1}{2}\left(\frac{u_i^n - u_i^{n-1}}{k} + \frac{u_{i-1}^n - u_{i-1}^{n-1}}{k}\right) = \frac{a}{2}\left(\frac{u_i^n - u_{i-1}^n}{h} + \frac{u_i^{n-1} - u_{i-1}^{n-1}}{h}\right),$$

which shows how we discretize equation (2.1.1). By using Taylor's formula we can show that the order of the trunction error is $O(k^2) + O(h^2)$. The symbol functions [2] are constant functions, $\rho(\xi) = 1$, and the von Neumann condition is satisfied. Hence our method is unconditionally stable in L_2-norm and convergent in L_2-norm. But in the case of (2.1), that is, the hyperbolic equation is with variable coefficients, the symbol functions are too complicated to analyse. In the next section we shall show the stability and the convergence in L_2-norm by using the energy method.

§3. The Convergence and the Stability in L_2-norm

We rewrite the grid functions u_i^n, $(\partial_t u)_i^n$ and $(\partial_x u)_i^n$ in the form of vectors in the space of dimension n:

$$u^n = (u_0^n, u_1^{n-1}, \cdots, u_n^0),$$
$$(\partial_t u)^n = ((\partial_t u)_0^n, (\partial_t u)_1^{n-1}, \cdots, (\partial_t u)_n^0),$$
$$(\partial_x u)^n = ((\partial_x u)_0^n, (\partial_x u)_1^{n-1}, \cdots, (\partial_x u)_n^0),$$

where $n = 0, 1, 2, \cdots$. The L_2-norm is defined to be

$$\|u^n\| = \left(\sum_{j=0}^{n} (u_j^{n-j})^2 \sqrt{h^2+k^2}\right)^{\frac{1}{2}}.$$

Let $M^{(n)}$ be the norm of the initial-boundary value, denoted by

$$M^{(n)} = \left[|f(0)|^2 + \sum_{i=0}^{n} (|f'(ih)|^2 + |g'(ik)|^2)\sqrt{h^2+k^2}\right]^{\frac{1}{2}}. \tag{3.1}$$

Theorem 3.1. *The scheme (2.2) is unconditionally stable, provided that there are two constants m_1 and m_2 such that*

$$m_1 \le a(x,t) \le m_2 < 0, \quad \text{for all} \quad (x,t) \in R^2,$$

and that the function $a(x,t)$ has Lipschitz continuity, i.e., there is a positive constant A such that for any (x_1,t_1) and (x_2,t_2) in the domain $D = [0,+\infty) \times [0,+\infty)$,

$$|a(x_1,t_1) - a(x_2,t_2)| \le A(|x_1 - x_2| + |t_1 - t_2|).$$

Proof. Let $\alpha = \sqrt{h^2+k^2}$. For $n \geq 2$, we have

$$\begin{aligned}\|(\partial_x u)^{n+1}\|^2 - \|(\partial_x u)^{n-1}\|^2 &= |(\partial_x u)_0^{n+1}|^2\alpha + |(\partial_x u)_{n+1}^0|^2\alpha \\ &+ \sum_{j=0}^{n-1}[(\partial_x u)_{j+1}^{n-j} - (\partial_x u)_j^{n-1-j}][(\partial_x u)_{j+1}^{n-j} + (\partial_x u)_j^{n-1-j}]\alpha \\ &\doteq |(\partial_x u)_0^{n+1}|^2\alpha + |(\partial_x u)_{n+1}^0|^2\alpha + I.\end{aligned} \tag{3.2}$$

Here $(\partial_x u)_{j+1}^{n-j}$ can be replaced by (2.4.2). It follows that

$$\begin{aligned}I = \sum_{j=0}^{n-1}\alpha[(\partial_x u)_{j+1}^{n-j} + (\partial_x u)_j^{n-j-1}]\Big[&\frac{1+a_{j+1}^{n-j-1}\lambda}{1-a_{j+1}^{n-j}\lambda}(\partial_x u)_{j+1}^{n-j-1} - \frac{1+a_j^{n-j}\lambda}{1-a_{j+1}^{n-j}\lambda}(\partial_x u)_j^{n-j} \\ &+ \frac{(a_{j+1}^{n-j} - a_j^{n-j-1})\lambda}{1-a_{j+1}^{n-j}\lambda}(\partial_x u)_j^{n-j-1}\Big].\end{aligned}$$

In order to use the Abelian summation by parts, we rewrite the coefficients by making use of the Lipschitz continuity and the uniform bound of the negative function $a(x,t)$. For any four points $(i_j h, n_j k)$, $j = 1,2,3,4$, we have

$$\frac{1+a_{i_1}^{n_1}\lambda}{1-a_{i_2}^{n_2}\lambda} - \frac{1+a_{i_3}^{n_3}\lambda}{1-a_{i_4}^{n_4}\lambda} = \frac{(a_{i_1}^{n_1} - a_{i_4}^{n_4})\lambda + (a_{i_2}^{n_2} - a_{i_3}^{n_3})\lambda + (a_{i_2}^{n_2}a_{i_3}^{n_3} - a_{i_1}^{n_1}a_{i_4}^{n_4})\lambda^2}{(1-a_{i_2}^{n_2}\lambda)(1-a_{i_4}^{n_4}\lambda)} = O(\delta),$$

where $\delta = \max\{h|i_l - i_j|,\ k|n_l - n_j| \,\big|\, l \neq j;\ l,j = 1,2,3,4\}$. Let $\dfrac{1+a_j^m\lambda}{1-a_j^m\lambda}$ be denoted by A_j^m. This gives

$$\begin{aligned}I &= \sum_{j=0}^{n-1}(\partial_x u)_{j+1}^{n-j}[A_{j+1}^{n-j-1}(\partial_x u)_{j+1}^{n-j-1} - A_j^{n-j}(\partial_x u)_j^{n-j}]\alpha \\ &+ \sum_{j=0}^{n-1}A_j^{n-j-1}(\partial_x u)_j^{n-j-1}[(\partial_x u)_{j+1}^{n-j-1} - (\partial_x u)_j^{n-j}]\alpha \\ &+ \sum_{j=0}^{n-1}O(\delta)[(\partial_x u)_{j+1}^{n-j} + (\partial_x u)_j^{n-j-1}][(\partial_x u)_{j+1}^{n-j-1} - (\partial_x u)_j^{n-j} + (\partial_x u)_j^{n-j-1}]\alpha.\end{aligned} \tag{3.3}$$

By using the Abelian summation by parts, we have

$$\begin{aligned}\sum_{j=0}^{n-1}A_j^{n-j-1}(\partial_x u)_j^{n-j-1}[(\partial_x u)_{j+1}^{n-j-1} - (\partial_x u)_j^{n-j}]\alpha &= [A_{n-1}^0(\partial_x u)_{n-1}^0(\partial_x u)_n^0 \\ -A_0^{n-1}(\partial_x u)_0^{n-1}(\partial_x u)_0^n]\alpha - \sum_{j=1}^{n-1}(\partial_x u)_j^{n-j}&[A_j^{n-j-1}(\partial_x u)_j^{n-j-1} - A_{j-1}^{n-j}(\partial_x u)_{j-1}^{n-j}].\end{aligned} \tag{3.4}$$

$(\partial_x u)_{j+1}^{n-j}$ in the third sum in (3.3) can be replaced by (2.4.2) again. It follows that

$$\sum_{j=0}^{n-1} O(\delta)\,[(\partial_x u)_{j+1}^{n-j} + (\partial_x u)_j^{n-j-1}][(\partial_x u)_{j+1}^{n-j-1} - (\partial_x u)_j^{n-j} + (\partial_x u)_j^{n-j-1}]\alpha$$

$$= O(\delta)(\sum_{j=0}^{n-1} |(\partial_x u)_j^{n-j-1}|^2\alpha + \sum_{j=0}^{n} |(\partial_x u)_j^{n-j}|^2\alpha). \tag{3.5}$$

The last equality follows from the fact that $|ab| \leq \frac{a^2+b^2}{2}$ and $a(x,t)$ is a uniformly bounded negative function.

By combining (3.2) with (3.3), (3.4) and (3.5), we have

$$\begin{aligned}
\|(\partial_x u)^{n+1}\|^2 - \|(\partial_x u)^{n-1}\|^2 &= \sum_{j=0}^{n-1} (\partial_x u)_{j+1}^{n-j}[A_{j+1}^{n-j-1}(\partial_x u)_{j+1}^{n-j-1} - A_j^{n-j}(\partial_x u)_j^{n-j}]\alpha \\
&- \sum_{j=1}^{n-1} (\partial_x u)_j^{n-j}[A_j^{n-j-1}(\partial_x u)_j^{n-j-1} - A_{j-1}^{n-j}(\partial_x u)_{j-1}^{n-j}]\alpha \\
&+ O(\delta)(\|(\partial_x u)^{n-1}\|^2 + \|(\partial_x u)^n\|^2) + |(\partial_x u)_0^{n+1}|^2\alpha + |(\partial_x u)_{n+1}^0|^2\alpha \\
&+ [A_{n-1}^0(\partial_x u)_{n-1}^0(\partial_x u)_n^0 - A_0^{n-1}(\partial_x u)_0^{n-1}(\partial_x u)_0^n]\alpha.
\end{aligned} \tag{3.6}$$

Let

$$\begin{aligned}
[(\partial_x u)^m : (\partial_x u)^{m-1}] &:= \|(\partial_x u)^m\|^2 + \|(\partial_x u)^{m-1}\|^2 \\
&- \sum_{j=1}^{m-1} (\partial_x u)_j^{m-j}[A_j^{m-j-1}(\partial_x u)_j^{m-j-1} - A_{j-1}^{m-j}(\partial_x u)_{j-1}^{m-j}]\alpha.
\end{aligned}$$

The function $f(a) = \frac{1+a\lambda}{1-a\lambda}$ is increasing in $\{a \leq 0\}$. This gives

$$|A_j^n| \leq \frac{1+\tilde{m}\lambda}{1-\tilde{m}\lambda} < 1.$$

Let $\mu = 1 - \frac{1+\tilde{m}\lambda}{1-\tilde{m}\lambda} = \frac{-2\tilde{m}\lambda}{1-\tilde{m}\lambda}$. It follows from $|ab| \leq \frac{a^2+b^2}{2}$ that

$$0 < \mu(\|(\partial_x u)^{n+1}\|^2 + \|(\partial_x u)^n\|^2) \leq [(\partial_x u)^{n+1} : (\partial_x u)^n], \quad \text{for any } n.$$

Hence from (3.6) we can have

$$\mu(\|(\partial_x u)^{n+1}\|^2 + \|(\partial_x u)^n\|^2) \leq K[\|(\partial_x u)^2\|^2 + \sum_{i=0}^{n+1}(|(\partial_x u)_0^i|^2 + |(\partial_x u)_i^0|^2)\alpha]. \tag{3.7}$$

By using (2.4.2), we have

$$\|(\partial_x u)^2\|^2 \leq K[|(\partial_x u)_0^1|^2 + |(\partial_x u)_0^0|^2 + |(\partial_x u)_1^0|^2 + |(\partial_x u)_2^0|^2 + |(\partial_x u)_0^2|^2]\alpha. \tag{3.8}$$

(3.7) and (3.8) lead to

$$\|(\partial_x u)^n\| \le K\sum_{i=0}^{n}(|(\partial_x u)_0^i|^2 + |(\partial_x u)_i^0|^2)\alpha \le KM^{(n)}. \tag{3.9}$$

This implies that the scheme for $(\partial_x u)_i^n$ is unconditionally stable. In the same way we can show the stability of the scheme for $(\partial_t u)_i^n$, determined by (2.4.3). For u_i^n, we have

$$\|u^n\|^2 \le K\Big[\sum_{m=0}^{n}|u_0^m|^2\alpha + \sum_{j=0}^{n}|u_j^0|^2\alpha + M^{(n)}\Big], \tag{3.10}$$

where $M^{(n)}$ is defined by (3.1).

Let m_0 satisfy $|u_0^{m_0}|^2 = \max\limits_{0\le m\le n}|u_0^m|^2$. Then

$$\sum_{m=0}^{n}|u_0^m|^2\alpha \le K\max_{0\le m\le n}|u_0^m|^2 = K|u_0^{m_0}|^2 = K|u_0^{m_0-1} + \frac{k}{2}(\partial_t u)_0^{m_0} + \frac{k}{2}(\partial_t u)_0^{m_0-1}|^2$$

$$\vdots$$

$$= K|u_0^0 + \frac{k}{2}[(\partial_t u)_0^0 + (\partial_t u)_0^{m_0}] + k\sum_{i=1}^{m_0-1}(\partial_t u)_0^i|^2 \le KM^{(n)}.$$

Similarly,

$$\sum_{j=0}^{n}|u_j^0|^2\alpha \le KM^{(n)}.$$

By combining these inequalities with (3.10) we can show the stability of the scheme for u_i^n. The proof is completed.

Now we consider the convergence in L_2-norm. Let p be a function in C^3. Then, it follows from Taylor's formula that

$$k(\partial_t p)_i^n = -k(\partial_t p)_i^{n-1} + 2(p_i^n - p_i^{n-1}) - \int_{k(n-1)}^{kn}(\partial_t^3 p)(ih,s)[(n-\frac{1}{2})k - s]^2\,ds,$$

$$h(\partial_x p)_i^n = -h(\partial_x p)_{i-1}^n + 2(p_i^n - p_{i-1}^n) - \int_{(i-1)h}^{ih}(\partial_x^3 p)(s,nk)[s-(i-\frac{1}{2})h]^2\,ds.$$

The remainders are denoted by

$$R_t(p,i,n) = -\frac{1}{k}\int_{k(n-1)}^{kn}(\partial_t^3 p)(ih,s)[(n-\frac{1}{2})k - s]^2\,ds,$$

$$R_x(p,i,n) = -\frac{1}{h}\int_{(i-1)h}^{ih}(\partial_x^3 p)(s,nk)[(i-\frac{1}{2})h - s]^2\,ds.$$

Suppose that the exact solution v is in C^1. Let $e = v - u$, $\bar{e} = p - u$ and $\underline{e} = v - p$. Following the proof of Lemma 2.2, we obtain

$$
\begin{aligned}
(\bar{e})_i^n = & \Bigg[\frac{1+a_i^{n-1}\lambda}{1-a_i^n\lambda}(\bar{e})_i^{n-1} - \frac{1+a_i^n\lambda}{1-a_i^n\lambda}(\bar{e})_{i-1}^n + \frac{1-a_i^{n-1}\lambda}{1-a_i^n\lambda}(\bar{e})_{i-1}^{n-1} \\
& + \frac{(a_{i-1}^{n-1}-a_i^{n-1})\lambda}{1-a_i^n\lambda}\cdot\frac{k}{2}(\partial_x\bar{e})_{i-1}^{n-1} + \frac{(a_{i-1}^n-a_i^n)\lambda}{1-a_i^n\lambda}\cdot\frac{k}{2}(\partial_x\bar{e})_{i-1}^n\Bigg] \\
& + \Big[\frac{k}{2}R_t(p,i,n) - \frac{1}{2}a_i^n kR_x(p,i,n) - \frac{1}{2}a_i^{n-1}kR_x(p,i,n-1) + \frac{1}{2}kR_t(p,i-1,n)\Big] \\
& \times\frac{1}{1-a_i^n\lambda} + \Big[-\frac{k}{2}(\partial_t\underline{e})_i^{n-1} + \frac{k}{2}a_i^{n-1}(\partial_x\underline{e})_i^{n-1} + \frac{k}{2}a_{i-1}^{n-1}(\partial_x\underline{e})_{i-1}^{n-1} \\
& + \frac{k}{2}a_{i-1}^n(\partial_x\underline{e})_{i-1}^n - \frac{k}{2}(\partial_t\underline{e})_{i-1}^{n-1} - \frac{k}{2}(\partial_x\underline{e})_{i-1}^n - \frac{k}{2}(\partial_t\underline{e})_i^n + \frac{k}{2}a_i^n(\partial_x\underline{e})_i^n\Big]\cdot\frac{1}{1-a_i^n\lambda}, \\
(\partial_x\bar{e})_i^n = & \Bigg[\frac{1+a_i^{n-1}\lambda}{1-a_i^n\lambda}(\partial_x\bar{e})_i^{n-1} - \frac{1+a_{i-1}^n\lambda}{1-a_i^n\lambda}(\partial_x\bar{e})_{i-1}^n + \frac{1-a_{i-1}^{n-1}\lambda}{1-a_i^n\lambda}(\partial_x\bar{e})_{i-1}^{n-1}\Bigg] \\
& + [-R_x(p,i,n) + R_x(p,i,n-1) + \lambda R_t(p,i,n) - \lambda R_t(p,i-1,n)]\cdot\frac{1}{1-a_i^n\lambda} \\
& + [-\lambda(\partial_t\underline{e})_i^n - \lambda(\partial_t\underline{e})_i^{n-1} + \lambda a_i^{n-1}(\partial_x\underline{e})_i^{n-1} + \lambda a_i^n(\partial_x\underline{e})_i^n + \lambda(\partial_t\underline{e})_{i-1}^n \\
& + \lambda(\partial_t\underline{e})_{i-1}^{n-1} - \lambda a_{i-1}^n(\partial_x\underline{e})_{i-1}^n - \lambda a_{i-1}^{n-1}(\partial_x\underline{e})_{i-1}^{n-1}]\cdot\frac{1}{1-a_i^n\lambda}.
\end{aligned}
\tag{3.11}
$$

Denote the second and the third terms in the last equality by $I_{i,n}^1$ and $I_{i,n}^2$. Then

$$
\begin{aligned}
(\partial_x e)_i^n = & (\partial_x\bar{e})_i^n + (\partial_x\underline{e})_i^n = \frac{1+a_i^{n-1}\lambda}{1-a_i^n\lambda}(\partial_x\bar{e})_i^{n-1} - \frac{1+a_{i-1}^n\lambda}{1-a_i^n\lambda}(\partial_x\bar{e})_{i-1}^n \\
& + \frac{1-a_{i-1}^{n-1}\lambda}{1-a_i^n\lambda}(\partial_x\bar{e})_{i-1}^{n-1} + (\partial_x\underline{e})_i^n + I_{i,n}^1 + I_{i,n}^2 = \frac{1+a_i^{n-1}\lambda}{1-a_i^n\lambda}(\partial_x e)_i^{n-1} \\
& - \frac{1+a_{i-1}^n\lambda}{1-a_i^n\lambda}(\partial_x e)_{i-1}^n + \frac{1-a_{i-1}^{n-1}\lambda}{1-a_i^n\lambda}(\partial_x e)_{i-1}^{n-1} + B_{i,n},
\end{aligned}
$$

where

$$
\begin{aligned}
B_{i,n} = & -\frac{1+a_i^{n-1}\lambda}{1-a_i^n\lambda}(\partial_x\underline{e})_i^{n-1} + \frac{1+a_{i-1}^n\lambda}{1-a_i^n\lambda}(\partial_x\underline{e})_{i-1}^n \\
& - \frac{1-a_{i-1}^{n-1}\lambda}{1-a_i^n\lambda}(\partial_x\underline{e})_{i-1}^{n-1} + (\partial_x\underline{e})_i^n + I_{i,n}^1 + I_{i,n}^2.
\end{aligned}
$$

By using the method in the proof of Theorem 3.1, we have (with details omitted)

$$[(\partial_x e)^{n+1} : (\partial_x e)^n]$$
$$\leq K\left[|(\partial_x e)_1^1|^2\alpha + \left(\sum_{l=0}^{n}\|(\partial_x e)^{(n-l+1)}\|^2\alpha\right)^{\frac{1}{2}}\left(\sum_{l=0}^{n-2}\sum_{j=0}^{n-l-1} B_{j+1,n-l-j}^2\right)^{\frac{1}{2}}\right].$$

It follows from (3.11) and the initial-boundary conditions that

$$|(\partial_x e)_1^1|^2\alpha = B_{1,1}^2\alpha.$$

Let

$$b_{n+1} = \left(\sum_{l=0}^{n-2}\sum_{j=0}^{n-l-1} B_{j+1,n-l-j}^2\right)^{\frac{1}{2}}.$$

If we consider the convergence in a triangular domain:

$$D = \{(x,t) \mid t \leq -\lambda x + M\}, \tag{3.12}$$

then we have

$$\|\partial_x e\|^2 \leq K(B_{1,1}^2\alpha + b_N\|\partial_x e\|),$$

where

$$\|\partial_x e\| = \left(\sum_{n=0}^{N}\alpha\sum_{j=0}^{n}|(\partial_x e)_j^{n-j}|^2\alpha\right)^{\frac{1}{2}}$$

and N is a natural number satisfying $M = NK$. Hence

$$\|\partial_x e\| \leq \frac{1}{2}(Kb_N + \sqrt{K^2 b_N^2 + KB_{1,1}^2\alpha} \leq Kb_N. \tag{3.13}$$

Combining this inequality with (2.2.1′), we get the estimate of $(\partial_t e)$

$$\|\partial_t e\| \leq Kb_N. \tag{3.14}$$

Combining (3.13) with the first equality in (3.11) and following the proof of (3.13), we get the estimate of e. Let

$$\overline{B}_{i,n} = \frac{(a_{i-1}^{n-1} - a_i^{n-1})\lambda}{1 - a_i^n\lambda}\cdot\frac{k}{2}(\partial_x \bar{e})_{i-1}^{n-1} + \frac{(a_{i-1}^n - a_i^n)\lambda}{1 - a_i^n\lambda}\cdot\frac{k}{2}(\partial_x \bar{e})_{i-1}^n$$
$$+\frac{1}{1 - a_i^n\lambda}\Big[\frac{1}{2}kR_t(p,i,n) - \frac{1}{2}a_i^n kR_x(p,i,n) + \frac{1}{2}a_i^{n-1}kR_x(p,i,n-1)$$
$$+\frac{1}{2}kR_t(p,i-1,n)\Big] + [-\frac{k}{2}(\partial_t \underline{e})_i^{n-1} + \frac{k}{2}a_i^{n-1}(\partial_x \underline{e})_i^{n-1} + \frac{k}{2}a_{i-1}^{n-1}(\partial_x \underline{e})_{i-1}^{n-1}$$
$$+\frac{k}{2}a_{i-1}^n(\partial_x \underline{e})_{i-1}^n - \frac{k}{2}(\partial_t \underline{e})_{i-1}^{n-1} - \frac{k}{2}(\partial_x \underline{e})_{i-1}^n - (\partial_t \underline{e})_i^n + a_i^n(\partial_x \underline{e})_i^n] + (\underline{e})_i^n$$
$$-\frac{1 + a_i^{n-1}\lambda}{1 - a_i^n\lambda}(\underline{e})_i^{n-1} + \frac{1 + a_i^n\lambda}{1 - a_i^n\lambda}(\underline{e})_{i-1}^n + \frac{1 - a_i^{n-1}\lambda}{1 - a_i^n\lambda}(\underline{e})_{i-1}^{n-1}.$$

Then

$$\|e\| \leq K\left[\bar{b}_N + \left(\sum_{n=0}^{N} |e_0^n|^2\alpha\right)^{\frac{1}{2}} + \left(\sum_{n=0}^{N} |e_i^0|^2\alpha\right)^{\frac{1}{2}}\right], \tag{3.15}$$

where

$$\bar{b}_N = \left(\sum_{l=0}^{n-2}\sum_{j=0}^{n-l-1} \overline{B}_{j+1,n-l-j}^2\right)^{\frac{1}{2}}.$$

The following lemma summarizes the hypotheses and results for the convergence.

Lemma 3.2. *Let D be the domain defined as in* (3.12). *Under the hypotheses of Theorem* 3.1, *if v is in $C^1(D)$ and p is in $C^3(D)$, then the estimates* (3.13), (3.14) *and* (3.15) *hold.*

Theorem 3.3. *Under the hypotheses of Theorem* 3.1, *if v is in $C^3(D)$, then*

$$\|e\| \leq O(\alpha^2), \quad \|\partial_t e\| \leq O(\alpha), \quad \|\partial_x e\| \leq O(\alpha),$$

where

$$\alpha = \sqrt{h^2 + k^2}.$$

Proof. Let $p = v$; then $\underline{e} = 0$. Hence

$$B_{i,n} = I'_{i,n} = \frac{1}{1 - a_i^n\lambda}[-R_x(v,i,n) + \lambda R_t(v,i,n) + R_x(v,i-1,n) - \lambda R_t(v,i-1,n)]$$

and

$$|R_x(v,i,n)| = |-\frac{1}{h}\int_{(i-1)h}^{ih} (\partial_x^3 p)(s,nk)[(i-\frac{1}{2})h - s]^2\,ds|$$

$$\leq K\alpha^{3/2}\left[\int_{(i-1)h}^{ih} (\partial_x^3 v)^2(s,nk)\,ds\right]^{1/2}.$$

This leads to

$$b_N \leq \left(\sum_{l=0}^{N}\sum_{j=0}^{N-l} B_{j,N-l-j}^2\right)^{1/2} \leq K\alpha\left[\left(\int_0^{M\lambda^{-1}} dx \int_0^{-\lambda x+M} (\partial_t^3 v)^2(x,t)\,dt\right)^{1/2}\right.$$

$$\left. + \left(\int_0^M dt \int_0^{-\frac{1}{\lambda}(t-M)} (\partial_x^3 v)^2(x,t)\,dx\right)^{1/2}\right]$$

$$+ K\alpha\left[-\left(\int_0^{M\lambda^{-1}} dx \int_0^{-\lambda x+M} (\partial_t^3 v)^2(x,t)\,dt\right)^{1/2}\right.$$

$$\left. + \left(\sum_{i=0}^{N} h\int_0^{-\lambda ih+M} (\partial_x^3 v)^2(ih,s)\,ds\right)^{1/2}\right]$$

$$+ K\alpha\left[- \left(\int_0^M dt \int_0^{-\frac{1}{\lambda}(t-M)} (\partial_t^3 v)^2(x,t)\, dx\right)^{1/2} \right.$$

$$\left. + \left(\sum_{j=0}^{N} k \int_0^{-\frac{1}{\lambda}(ik-M)} (\partial_x^3 v)^2(s,jk)\, ds\right)^{1/2}\right]$$

The last two terms are convergent to zero as $\alpha \to 0$. Hence

$$\|\partial_t e\| \le K b_N \le O(\alpha), \quad \|\partial_x e\| \le O(\alpha).$$

Similarly, we have

$$\bar{b}_N \le K\alpha^2 \|\partial_x e\| + K\alpha^2 \left[\left(\sum_{i=0}^{N} h \int_0^{-\lambda ih+M} (\partial_t^3 v)^2(ih,s) ds\right)^{1/2}\right.$$

$$\left. + \left(\sum_{j=0}^{N} k \int_0^{-\frac{1}{\lambda}(jk-M)} (\partial_x^3 v)^2(s,jk) ds\right)^{1/2}\right],$$

$$|e_0^n| \le k^{5/2} \sum_{j=1}^{n} \left(\int_{k(j-1)}^{kj} (\partial_t^3 v)^2(o,s) ds\right)^{1/2},$$

$$|e_i^0| \le h^{5/2} \sum_{j=1}^{i} \left(\int_{(j-1)h}^{jh} (\partial_x^3 v)^2(s,0) ds\right)^{1/2}.$$

Hence

$$\|e\| \le K\alpha^2 \left[\left(\int_0^M (\partial_t^3 g)^2(s) ds\right)^{\frac{1}{2}} + \left(\int_0^{\lambda^{-1}M} (\partial_x^3 f)^2(s) ds\right)^{\frac{1}{2}}\right]$$

$$+ \left(\int_0^M dt \int_0^{-\frac{1}{\lambda}(t-M)} (\partial_x^3 v)^2(x,t) dx\right)^{\frac{1}{2}} + \left(\int_0^{\frac{1}{\lambda}M} dx \int_0^{-\lambda x+M} \cdot (\partial_t^3 v)^2(x,t) dt\right)^{\frac{1}{2}}\Bigg]$$

$$+ O(\alpha^2).$$

The theorem has been completely shown.

It is not necessary to assume in Lemma 3.2 that p has the third partial derivatives in the whole domain D. In fact we have the following sharper result.

Lemma 3.4. *Let $\square_{ij}$ be the rectangle with vertices (i,j), $(i,j-1)$, $(i-1,j-1)$ and $(i-1,j)$. Suppose that for each (i,j), p is in $C^3(\square_{ij})$. Then, under the hypotheses* *of Lemma* 3.2, *we have*

$$\|\partial_x e\| \le K(b_N + \|\partial_x \underline{e}\|), \|\partial_t e\| \le K(b_N + \frac{1}{\alpha}\|\partial_x \underline{e}\|),$$

$$\|e\| \le K\left[\bar{b}_N + \frac{1}{\alpha}\|\underline{e}\| + \|\partial_x \underline{e}\| + \left(\sum_{n=0}^{N} |e_0^n|^2 \alpha\right)^{1/2} + \left(\sum_{i=0}^{N} |e_i^0|^2 \alpha\right)^{1/2}\right],$$

where b_N, $\bar{b}_N$, *etc. are defined as above, but*

$$(\partial_t p)_i^n = (\partial_t p_{i,n})_i^n + (\partial_t p_{i+1,n+1})_i^n + (\partial_t p_{i+1,n})_i^n + (\partial_t p_{i,n+1})_i^n,$$

$$(\partial_x p)_i^n = (\partial_x p_{i,n})_i^n + (\partial_x p_{i+1,n+1})_i^n + (\partial_x p_{i+1,n})_i^n + (\partial_x p_{i,n+1})_i^n,$$

$$p_i^n = (p_{i,n})_i^n + (p_{i+1,n+1})_i^n + (p_{i+1,n})_i^n + (p_{i,n+1})_i^n,$$

and $R_t(p,i,n)$ *and* $R_x(p,i,n)$ *are replaced by* $R_t(p_{i,n},i,n)$ *and* $R_x(p_{i,n},i,n)$, *where*

$$p_{i,n} = p|_{\square_{in}}.$$

§4. The Spline Approximate Solution

We first introduce the space of continuously differentiable bivariate spline functions of degree 2, denoted by $S_2^1(\Delta)$, which was defined and discussed by the second author in [1].

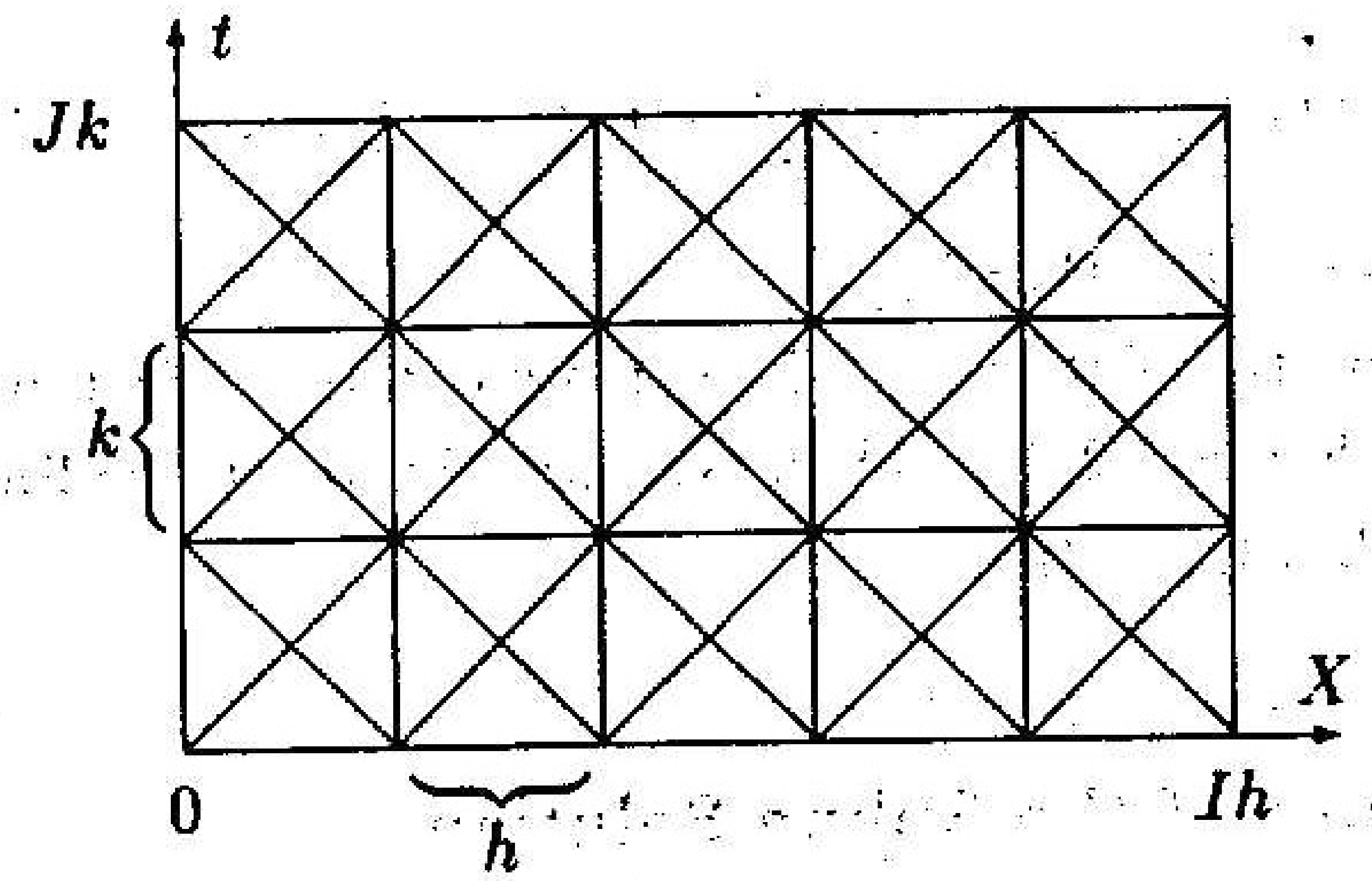

Fig. 1

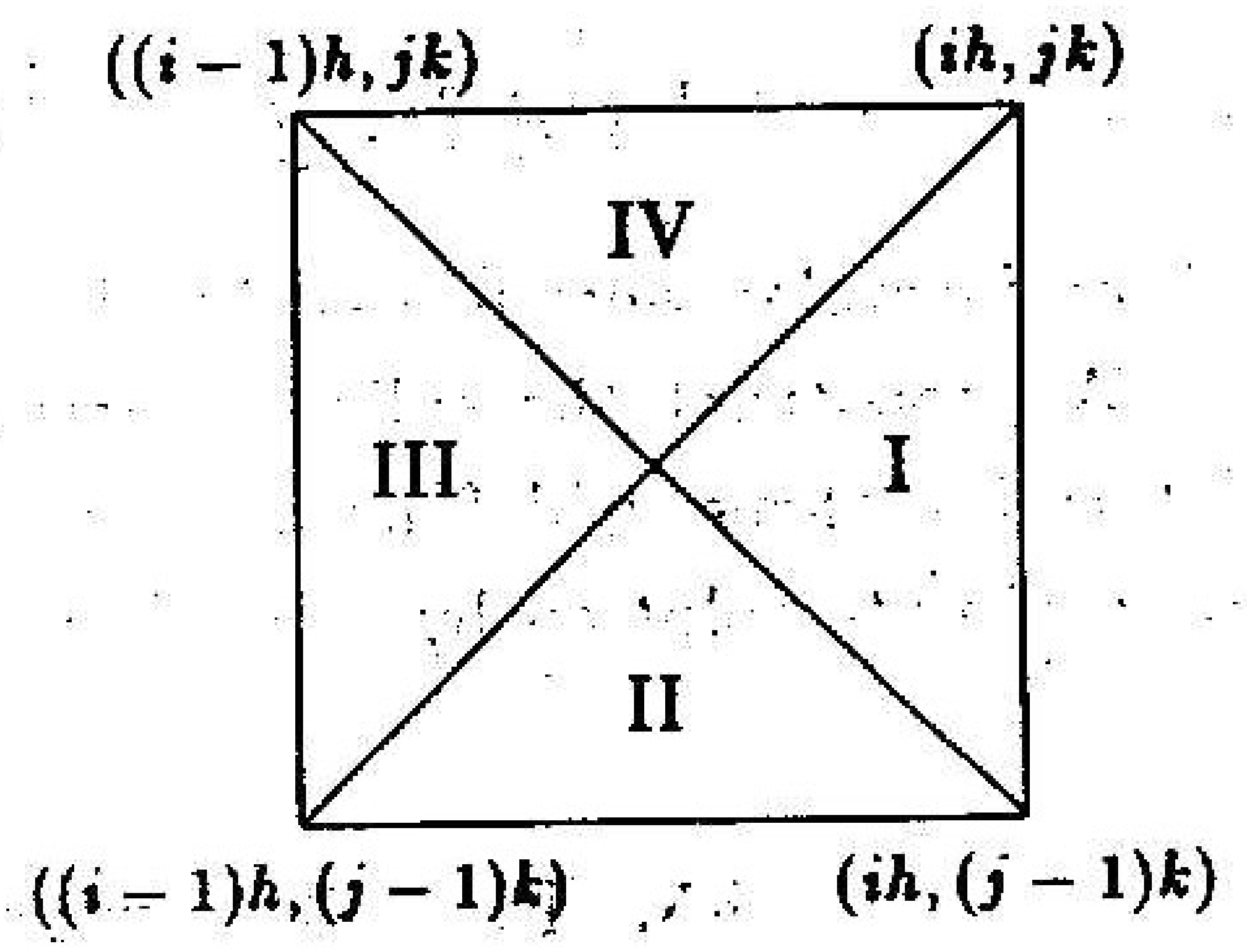

Fig. 2

The straight lines, $x = ih, i = 0,1,\cdots,I$, and $t = jk$, $j = 0,1,\cdots,J$, divide the domain $[0,Ih]\times[0,Jk]$ into IJ rectangles, where I and J are two fixed natural numbers. Two diagonal lines further divide each rectangle into four triangles. This partition is denoted by Δ; see Figure 1. Let

$$\Pi_2 = \left\{\sum_{0\le i+j\le 2} \lambda_{ij} x^i t^j \mid \lambda_{ij} \in \mathbb{R}\right\}$$

be the space of bivariate polynomials of degree 2. We define

$$S_2^1(\Delta) = \{s \in C^1[0, Ih] \times [0, JK] \mid s|_{\bar{\Delta}} \in \Pi_2, \quad \text{for any } \bar{\Delta} \in A\},$$

where A is the class of triangles, which are elements of the partition Δ. Let $\square_{ij}$ be a rectangle whose vertices are denoted by (ih, jk), $((i-1)h, jk)$, $((i-1)h, (i-l)k)$ and $(ih, (j-l)k)$. The four triangles in $\square_{ij}$ are denoted by I, II, III, and IV; see Figure 2.

Let s be a function in $S_2^1(\Delta)$. We can express s on $\square_{ij}$ by the parameters

$$s_m^n = s(mh, nk), \quad (\partial_x s)_m^n = \partial_x s(mh, nk), \quad (\partial_t s)_m^n = \partial_t s(mh, nk),$$

where $m = i,\ i-1$; $n = j,\ j-1$.

For example, for $(x, t) \in I$, we have

$$\begin{aligned} s(x,t) = {} & 4a_0(\frac{ih-x}{h})^2 + s_i^j\left[1 - (\frac{ih-x}{h}) - (\frac{jk-t}{k})\right]^2 + s_i^{j-1}\left[(\frac{x-ih}{h}) + (\frac{jk-t}{k})\right]^2 \\ & + 2[s_i^j + s_i^{j-1} + \frac{1}{2}(k(\partial_t s)_i^{j-1} - h(\partial_x s)_i^j](\frac{ih-x}{h})[1 - (\frac{ih-x}{h}) - (\frac{jk-t}{k})] \\ & + 2[s_{i-1}^{j-1} + s_i^{j-1} + \frac{1}{2}(k(\partial_t s)_i^{j-1} + h(\partial_x s)_{i-1}^{j-1})](\frac{ih-x}{h})(\frac{x-ih}{h} + \frac{jk-t}{k}) \\ & + [2s_i^{j-1} + k(\partial_t s)_i^{j-1}](1 - \frac{ih-x}{h} - \frac{jk-t}{k})(\frac{x-ih}{h} + \frac{jk-t}{k}), \end{aligned} \tag{4.1}$$

where

$$a_0 = \frac{1}{4}(s_i^j + s_i^{j-1} + s_{i-1}^{j-1} + s_{i-1}^j) + \frac{1}{8}[-h(\partial_x s)_j^i - k(\partial_t s)_{j-1}^{i-1} + h(\partial_x s)_{j-1}^{i-1} + k(\partial_t s)_j^{i-1}].$$

These formulas were proved by the second author in [1].

We construct the spline approximate solution to (2.1) in two steps. First we determine three grid functions by schemes (2.2). Then we can obtain the spline approximate solution by using formula (4.1), etc.

§5. The Convergence of the Spline Solution

We first introduce some definitions and properties about the two-dimensional spline approximation theory presented in [3].

Let $\|f\|_B$ denote the L_2-norm of f over the domain B, i.e.,

$$\|f\|_B = \left(\iint_B |f(x,t)|^2 dx dt\right)^{1/2}.$$

Let $\Lambda \in \{(i, n) \mid i \text{ and } n \text{ are two natural numbers}\}$ be bounded and be such that if $\alpha \in \Lambda$ and $\beta \leq \alpha$, then $\beta \in \Lambda$.

Define

$$\partial\wedge = \{\alpha \mid \alpha \notin \wedge \text{ and if } \beta < \alpha \text{ then } \beta \in \wedge\}.$$

For $\alpha = (\alpha_1, \alpha_2)$ and $h = (h_1, h_2)$, define

$$\Delta_h^\alpha f(x,t) = \Delta_{h_1}^{\alpha_1} \Delta_{h_2}^{\alpha_1} f(x,t),$$

where $\Delta_{h_1}^{\alpha_1}$ and $\Delta_{h_2}^{\alpha_2}$ are the usual α_1-th and α_2 -th forward difference of step lengths h_1 and h_2 with respect to x and t respectively. For

$$B(\alpha,h) = \{(x,t) \mid (x + s_1\alpha_1, t + s_2\alpha_2) \in B \text{ for all } (s_1,s_2) \leq h\},\ h = (h_1,h_2),$$

define

$$\omega_\alpha(f,h,B) = \sup_{0 \leq t < h} \|\Delta_t^\alpha f(\cdot,\cdot)\|_{B(\alpha,h)}$$

and

$$\omega_\wedge(f,h,B) = \sum_{\alpha\in\partial\wedge} \omega_\alpha(f,h,B).$$

Lemma 5.1. *Some simple properties of* ω_α *and* $\omega_\wedge$ *are:*

1) $\omega_\wedge(f,h,B) \to 0$, *as* $h \to 0$,

2) $\omega_{(\alpha_1,\alpha_2)}(f,h,B) \leq K \min\{\omega_{(\alpha_1-1,\alpha_2)}(\partial_x f,h,B) + \omega_{(\alpha_1,\alpha_2-1)}(\partial_t f,h,B)\}$,

3) $\omega_\alpha(f,\lambda h,B) \leq (1+\lambda)^m \omega_\alpha(f,h,B)$, $\quad m = \max\{\alpha_1,\alpha_2\}$,

4) *If* B_i, $i = 1,2,\cdots,n$, *are* n *domains with* $B_i \cap B_j = \phi, i \neq j$, *then*

$$\sum_{i=1}^n \omega_\alpha(f,h,B_i) \leq K\omega_\alpha(f,h,B),$$

where $B = \bigcup_{i=1}^n B_i$.

Lemma 5.2. *Let* $\sqrt{h^2+k^2}$. *There is a constant* $K > 0$ *such that for each* $f \in C^2$ *and each rectangle* $\square_{ij}$, *there exists a polynomial* p_{ij} *of degree 2, i.e.,*

$$p_{ij} \in \operatorname{span}\{l, x, t, xt, x^2, t^2\},$$

such that

$$|f(x,t) - p_{ij}(x,t)| \leq K\alpha\omega(D^2 f, \alpha, \sqcup_{ij}),$$

$$|\partial_x(f - p_{ij})(x,t)| \leq K\omega(D^2 f, \alpha, \sqcup_{ij}),$$

$$|\partial_t(f - p_{ij})(x,t)| \leq K\omega(D^2 f, \alpha, \sqcup_{ij}),$$

where

$$\omega\,(D^2 f, \alpha, \square_{ij}) = \sum_{0<i'+n\le 1} [\omega_{(i',n)}(\partial_x^2 f, \alpha, \square_{ij}) + \omega_{(i',n)}(\partial_x\partial_t f, \alpha, \square_{ij}) + \omega_{(i',n)}(\partial_t^2 f, \alpha, \square_{ij})].$$

Remark. We will not give the complete details for the proof of these lemmas because the proof is very similar to that used in [3].

Let $D = \{(x,t) \mid t \le -\lambda x + M\}$ be as defined in Section 3. We shall estimate the error in the domain $D' = \{(x,t) \mid \text{there exists a rectangle } \square_{ij}, 0 < i \le N,\ 0 < j \le N,$ such that $(x,t) \in \square_{ij} \subset D\}$.

Let p be the function with $p|_{\square_{ij}} = p_{ij}$. It follows from Lemmas 5.1 and 5.2 that

$$\begin{aligned}
\Big(\sum_{n=0}^{N}\sum_{i=0}^{n} \alpha^2 |(v-p)_i^{n-i}|^2\Big)^{1/2} &\le K\alpha^2\Big(\sum_{n=0}^{N}\sum_{i=0}^{n} |\omega(D^2 v, \alpha, \square_{i,n-i})|^2\Big)^{1/2} \\
&\le K\alpha^2\omega(D^2 v, \alpha, D'), \\
\Big(\sum_{n=0}^{N}\sum_{i=0}^{n} \alpha^2 |(\partial_x v - \partial_x p)_i^{n-i}|^2\Big)^{1/2} &\le K\alpha\omega(D^2 v, \alpha, D'), \\
\Big(\sum_{n=0}^{N}\sum_{i=0}^{n} \alpha^2 |(\partial_t v - \partial_t p)_i^{n-i}|^2\Big)^{1/2} &\le K\alpha\omega(D^2 v, \alpha, D'), \qquad (5.1)\\
\|v-p\|_{D'} = \Big(\sum_{n=0}^{N}\sum_{i=0}^{n} \iint_{\square_{i,n-i}} |(v - p_{i,n-i})(x,t)|^2 dx dt\Big)^{1/2} & \\
\le K\alpha^2\omega(D^2 v, \alpha, D'), & \\
\|\partial_x(v-p)\|_{D'} \le K\omega(D^2 v, \alpha, D'), & \\
\|\partial_t(v-p)\|_{D'} \le K\omega(D^2 v, \alpha, D') &
\end{aligned}$$

where p_i^n, $(\partial_x p)_i^n$ and $(\partial_t p)_i^n$ are as defined in Lemma 3.4 and v is the exact solution. Since p and the spline approximate solution u are piecewise polynomials of degree 2, it follows that

$$\begin{aligned}
\|p-u\|_{D'} &= \Big(\sum_{n=0}^{N}\sum_{i=0}^{n} \iint_{\square_{i,n-i}} |(p_{i,n-i} - u)(x,t)|^2 dx dt\Big)^{1/2} \\
&\le K\Big[\Big(\sum_{n=0}^{N}\sum_{i=0}^{n} \alpha^2 |(p-u)_i^{n-i}|^2\Big)^{1/2} + \alpha\Big(\sum_{n=0}^{N}\sum_{i=0}^{n} \alpha^2 |(\partial_x p - \partial_x u)_i^{n-i}|^2\Big)^{1/2} \\
&\quad + \alpha\Big(\sum_{n=0}^{N}\sum_{i=0}^{n} \alpha^2 |(\partial_t p - \partial_t u)_i^{n-i}|^2\Big)^{1/2}\Big], \qquad (5.2)
\end{aligned}$$

and

$$\left(\sum_{n=0}^{N}\sum_{i=0}^{n}\alpha^2|(p-u)_i^{n-i}|^2\right)^{\frac{1}{2}} \le \left(\sum_{n=0}^{N}\sum_{i=0}^{n}\alpha^2|(p-v)_i^{n-i}|^2\right)^{\frac{1}{2}} + \left(\sum_{n=0}^{N}\sum_{i=0}^{n}\alpha^2|e_i^{n-i}|^2\right)^{\frac{1}{2}}$$

$$\le \left(\sum_{n=0}^{N}\sum_{i=0}^{n}\alpha^2|(p-v)_i^{n-i}|^2\right)^{\frac{1}{2}} + K\left[\bar{b}_N + \frac{1}{\alpha}\left(\sum_{n=0}^{N}\sum_{i=0}^{n}\alpha^2|(p-v)_i^{n-i}|^2\right)^{\frac{1}{2}}\right.$$

$$\left. + \left(\sum_{n=0}^{N}\sum_{i=0}^{n}\alpha^2|(\partial_x p-\partial_x v)_i^{n-i}|^2\right)^{1/2} + \left(\sum_{n=0}^{N}|e_0^n|^2\alpha\right)^{1/2} + \left(\sum_{i=0}^{N}|e_i^0|^2\alpha\right)^{1/2}\right]$$

$$\le K\alpha\omega(D^2v,\alpha,D') + K\left[\bar{b}_N + \left(\sum_{n=0}^{N}|e_0^n|^2\alpha\right)^{1/2} + \left(\sum_{i=0}^{N}|e_i^0|^2\alpha\right)^{1/2}\right], \tag{5.3}$$

where the second inequality follows from Lemma 3.4 and the third inequality follows from (5.1).

Similarly,

$$\left(\sum_{n=0}^{N}\sum_{i=0}^{n}\alpha^2|(\partial_x p-\partial_x u)_i^{n-i}|^2\right)^{1/2} \le K\omega(D^2v,\alpha,D') + Kb_N.$$

p is a piecewise polynomial of degree 2. Hence

$$R_t(p,i,n) = R_x(p,i,n) = 0,$$

namely, $I'_{i,n} = 0$. This leads to

$$|B_{i,n}| = |(\partial_x \underline{e})_i^n + I_{i,n}^2| \le K\sum_{m=-1}^{0}\sum_{j=-1}^{0}\left(|(\partial_x\underline{e})_{i+j}^{n+m}| + |(\partial_t\underline{e})_{i+j}^{n+m}|\right).$$

Hence

$$b_N = \left(\sum_{l=0}^{N}\sum_{j=0}^{N-l}B_{j,n-j}^2\right)^{1/2} \le K\alpha^{-1}\left[\left(\sum_{n=0}^{N}\sum_{i=0}^{n}\alpha^2|(\partial_x\underline{e})_i^{n-i}|^2\right)^{1/2}\right.$$

$$\left. + \left(\sum_{n=0}^{N}\sum_{i=0}^{n}\alpha^2|(\partial_t\underline{e})_i^{n-i}|^2\right)^{1/2}\right] \le K\omega(D^2v,\alpha,D').$$

This leads to

$$\left(\sum_{n=0}^{N}\sum_{i=0}^{n}\alpha^2|(\partial_x p-\partial_x u)_i^{n-i}|^2\right)^{1/2} \le K\omega(D^2v,\alpha,D').$$

We can also show that

$$\left(\sum_{n=0}^{N}\sum_{i=0}^{n}\alpha^2|(\partial_t p-\partial_t u)_i^{n-i}|^2\right)^{1/2} \le K\omega(D^2v,\alpha,D').$$

By a similar method, we have

$$\bar{b}_N \leq K\alpha\omega(D^2v, \alpha, D').$$

Combining these estimates with (5.2) and (5.3), we have

$$\|p-u\|_{D'} \leq K\left[\alpha\omega(D^2v,\alpha,D') + \left(\sum_{n=0}^{N}|e_0^n|^2\alpha\right)^{1/2} + \left(\sum_{i=0}^{N}|e_i^0|^2\alpha\right)^{1/2}\right].$$

For $0 \leq n \leq N$, we have

$$|e_0^n| \leq |(v-p)_0^n + |(p-u)_0^n| \leq |(v-p)_0^n + |(p-u)_0^{n-1}| + \frac{k}{2}|(\partial_t p - \partial_t u)_0^{n-1}|$$

$$+\frac{k}{2}|(\partial_t p - \partial_t u)_0^n| \leq \cdots \leq\leq |(v-p)_0^n| + |(p-v)_0^0| + k\sum_{l=0}^{n-1}|(\partial_t p - \partial_t v)_0^l|$$

$$\leq K\alpha\omega(D^2v, \alpha, D').$$

Similarly, for $0 \leq i \leq \lambda^{-1}M$,

$$|e_i^0| \leq K\alpha\omega(D^2v, \alpha, D').$$

This leads to

$$\|p-u\|_{D'} \leq K\alpha\omega(D^2v, \alpha, D').$$

Hence

$$\|v-u\|_{D'} \leq \|v-p\|_{D'} + \|p-u\|_{D'} \leq K\alpha\omega(D^2v, \alpha, D'). \tag{5.4}$$

By the same method we have

$$\|\partial_x(v-u)\|_{D'} \leq K\omega(D^2v, \alpha, D'). \tag{5.5}$$

and

$$\|\partial_t(v-u)\|_{D'} \leq K\omega(D^2v, \alpha, D'). \tag{5.6}$$

Now we have already shown the following theorem.

Theorem 5.3. *Under the hypotheses of Theorem* 3.1, *if the exact solution* v *is in* $C^2(D)$, *then the estimates* (5.4), (5.5) *and* (5.6) *hold.*

References

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