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ON NUMERICAL SOLUTION OF QUASILINEAR BOUNDARY VALUE PROBLEMS WITH TWO SMALL PARAMETERS*

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Abstract

We consider the singular perturbation problem

$$-\varepsilon^2 u'' + \mu b(x,u)u' + c(x,u) = 0, \quad u(0), u(1) \text{ given},$$

with two small parameters ε and μ, $\mu = \varepsilon^{1+p}, p > 0$. The problem is solved numerically by using finite difference schemes on the mesh which is dense in the boundary layers. The convergence uniform in ε is proved in the discrete L^1 norm. Some convergence results are given in the maximum norm as well.

§1. Introduction

Consider the following singularly perturbed boundary value problem:

$$Tu := -\varepsilon^2 u'' + \mu b(x,u)u' + c(x,u) = 0, \quad x \in I := [0,1], \tag{1.1a}$$

$$Bu := (u(0), u(1)) = (U_0, U_1), \tag{1.1b}$$

where ε is a small parameter:

$$0 < \varepsilon \le \varepsilon^* \ll 1,$$

and

$$\mu = \varepsilon^{1+p}, \quad p > 0.$$

U_0, U_1 are given numbers. We suppose that functions b and c are sufficiently smooth and

$$c_u(x,u) > c_* > 0, \quad x \in I, \quad u \in R. \tag{1.2}$$

This implies that there exist numbers u^* and u_* such that

$$c(x,u_*) < 0 < c(x,u^*), \quad x \in I, \quad u \in R, \tag{1.3a}$$

$$u_* \le U_j \le u^*, \quad j = 0,1. \tag{1.3b}$$

This means that u^* and u_* are upper and lower solutions, respectively, to problem (1.1). Hence (1.1) has a solution, which will be denoted by u_ε. Moreover,

$$u_\varepsilon(x) \in W := [u_*, u^*], \quad x \in I. \tag{1.4}$$

We shall use the conservation form of equation (1.1a):

$$Tu = -\varepsilon^2 u'' + \mu f(x,u)' + g(x,u) = 0, \quad x \in I, \tag{1.5}$$

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where

$$f(x,u) = \int_{u_*}^{u} b(x,s)ds, \quad g(x,u) = c(x,u) - \mu f_x(x,u).$$

Throughout the paper we shall assume that ε^* is sufficiently small. Then,

$$g_u(x,u) \geq g_* > 0, \quad x \in I, \quad u \in W.$$

Hence, because of the inverse monotonicity of the operator (T, B), u_ε is the unique solution satisfying (1.4); see [4].

Problems of type (1.1) belong to the class of two-parameter problems. The asymptotic behaviour of linear two-parameter problems was investigated in [7], and semilinear (i.e. $b = b(x)$) problems were treated numerically in [1], [2, p.251], [10]. These problems represent models of different phenomena arising in chemistry or biology; see [1], [2].

On the other hand, numerical methods for quasilinear singular perturbation problems with $\mu = 1$ were considered in [4]–[6], [8], [12]–[14], just to mention some of the papers.

In this paper our aim is to solve (1.1) numerically by using the approach from [9]–[12], [14], [3]. First, in Section 2, we derive estimates of the derivatives of u_ε. As we may expect, they show boundary layer behaviour of u_ε at $x = 0$ and $x = 1$. (Note that (1.2) guarantees the unique solvability of the reduced problem

$$c(x,u) = 0, \quad x \in I,$$

whose solution, in general, does not satisfy the boundary conditions (1.1b).) Then, in Section 3, we construct a special discretization mesh which is dense in the layers. We form the discrete problem corresponding to (1.5), (1.1b) by using the Lax-Friedrichs finite difference scheme for $p < 1$, and the central scheme for $p \geq 1$. For both schemes we prove uniform (i.e. uniform in ε) stability in the discrete L^1 norm. This norm is used because of the quasilinearity of equation (1.1a) (cf. [4], [5], [8], [12], [13], [14]). In Section 4 we deal with the consistency error using the estimates from Section 2 and properties of the special mesh. As it was shown in [13], the linear uniform convergence of the numerical solution towards the restriction of u_ε on the mesh can be obtained in the discrete L^1 norm even on equidistant meshes. Here, by using the special mesh we are able to improve the L^1 convergence result, cf. [14]. Moreover, numerical results, presented in Section 5, show the pointwise uniform convergence as well. For the case $p \geq 1$ we are able to estimate the maximum error by

$$M(n^{-1} + \varepsilon^{-1}\exp(-m_0 n)),$$

where n is the number of mesh subintervals, and m_0 is a positive constant independent of ε and n. Throughout the paper M will denote any positive constant, independent of ε and n.

§2. Estimates of the Derivatives of u_ε

In this section we shall estimate $|u_\varepsilon^{(k)}(x)|$ for $k = O(1)4, x \in I$. Throughout the section we shall assume (1.2) and that ε^* is sufficiently small. We shall use the technique from [10].

We shall start by giving some rough estimates:

Lemma 2.1.

$$|u_\varepsilon^{(k)}(x)| \leq M\varepsilon^{-k}, \quad k = O(1)4, \quad x \in I.$$

Proof. Because of (1.4) the estimate for $k = 0$ is immediate. Let us now prove the estimate for $k = 1$. If $x \in [0, 1/2]$ we take

$$x^* \in (x, x + \varepsilon) \subset I,$$

such that

$$|u'_\varepsilon(x^*)| = |u_\varepsilon(x) - u_\varepsilon(x+\varepsilon)|/\varepsilon \le M/\varepsilon.$$

Integrating (1.5) from x^* to x we get

$$|u'_\varepsilon(x)| \le M[|u'_\varepsilon(x^*)| + \varepsilon^{-2}(\mu + x^* - x)] \le M/\varepsilon.$$

The proof is similar when $x \in [1/2, 1]$.

Now the estimate for $k = 2$ follows directly form (1.1a) and the other ones can be obtained after differentiation.

Theorem 2.1. *The following estimates hold:*

$$|u_\varepsilon^{(k)}(x)| \le M[1 + \varepsilon^{-k}(y_\varepsilon(x) + z_\varepsilon(x))], \quad k = 1(1)4, \quad x \in I,$$

where

$$y_\varepsilon(x) = \exp(-mx/\varepsilon), \quad z_\varepsilon(x) = \exp(m(x-1)/\varepsilon),$$

and m is a positive constant independent of ε.

Proof. Let us prove the estimate for $k = 1$. The other estimates can be proved analogously. Define the linear operator:

$$Lu := -\varepsilon^2 u'' + \mu b(x, u_\varepsilon)u' + q(x)u,$$

where

$$q(x) = c_u(x, u_\varepsilon) + \mu b_x(x, u_\varepsilon) + \mu b_u(x, u_\varepsilon)u'_\varepsilon.$$

Since from Lemma 2.1 we have

$$\mu|b_u(x, u_\varepsilon)u'_\varepsilon| \le M\varepsilon^p, \quad x \in I,$$

it follows that

$$q(x) \ge c_* > 0, \quad x \in I$$

(note that ε is assumed to be sufficiently small). Thus, the operator (L, B) is inverse monotone, and it is easy to check that there exist appropriate constants M and m, so that

$$LM[1 + \varepsilon^{-1}(y_\varepsilon + z_\varepsilon)] \ge \mp c_x(x, u_\varepsilon) = L(\pm u'_\varepsilon), \quad x \in I;$$
$$BM(1 + \varepsilon^{-1}(y_\varepsilon + z_\varepsilon)] \ge B(\pm u'_\varepsilon).$$

The inequalities above imply the estimate.

§3. The Discretization and its Stability

Let I^h be the discretization mesh with the mesh points:

$$x_i = \lambda(t_i), \quad t_i = i/n, \quad i = 0(1)n, \quad n = 2n_0, \quad n_0 \in N,$$

where

$$\lambda(t) = \begin{cases} \omega(t) := \beta\varepsilon t/(\gamma - t), & t \in [0, \alpha], \\ \pi(t) := \delta(t-\alpha)^3 + \omega''(\alpha)(t-\alpha)^2/2 + \omega'(\alpha)(t-\alpha) + \omega(\alpha), & t \in [\alpha, 1/2], \\ 1 - \lambda(1-t), & t \in [1/2, 1]. \end{cases}$$

Here $\alpha \in (0, 1/2)$ is an arbitrary parameter (independent of ε),

$$\gamma = \alpha + \varepsilon^{1/3}, \tag{3.1}$$

and δ is determined from

$$\pi(1/2) = 1/2.$$

We have

$$\lambda \in C^2[0,1/2], \quad \lambda \in C^2[1/2,1], \quad \lambda \in C^1(I),$$

and because of (3.1),

$$\omega(\alpha) = \alpha\beta\varepsilon^{2/3}, \quad \omega'(\alpha) = \beta\gamma\varepsilon^{1/3}, \quad \omega''(\alpha) = 2\beta\gamma. \tag{3.2}$$

The parameter β should satisfy

$$0 < \beta < [2\gamma(1/2-\alpha)^2]^{-1},$$

which implies

$$\delta \geq 0, \text{ i.e. } \pi^{(3)} \geq 0,$$

provided ε^* is sufficiently small (see (3.2)). Then it follows that

$$\pi^{(k)}(t) \geq \pi^{(k)}(\alpha) = \omega^{(k)}(\alpha) > 0, \quad t \in [\alpha, 1/2],$$

first for $k = 2$ and then for $k = 1$. Obviously,

$$\omega^{(k)}(t) > 0, \quad k = 0,1,2,\cdots, \quad t \in [0,\alpha],$$

and taking (3.2) into account we get

$$0 < \lambda^{(k)}(t) \leq M, \quad k = 1,2, \quad t \in [0,1/2]. \tag{3.3}$$

Furthermore, note the inequality:

$$\exp(-\omega(t)/\varepsilon) \leq M \exp(-M/(\gamma - t)), \quad t \in [0,\gamma), \tag{3.4}$$

which will be used in Section 4.

It is easy to derive analogous properties of the function λ in $[1/2,1]$.

Similar mesh generating functions were used in [9], [10], [12], [14].

Let

$$h_i = x_i - x_{i-1}, \quad i = 1(1)n;$$

$$\bar{h}_i = (h_i + h_{i+1})/2, \quad i = 1(1)n-1.$$

Let w^h denote a mesh function on $I^h \backslash \{0,1\}$, which will be identified with the vector

$$w^h = [w_1, w_2, \cdots, w_{n-1}]^T \in R^{n-1}, \quad w_i := w_i^h,$$

and let T^h be the discrete operator corresponding to (1.5), (1.1b):

$$T^h : R^{n-1} \to R^{n-1},$$

$$T^h w_i := (T^h w^h)_i = -\varepsilon D'' w_i + \mu D' f(x_i, w_i) + \mu D^0 w_i + g(x_i, w_i), \ i = 1(1)n-1,$$

where

$$D'' w_i = [(w_{i-1} - w_i)/h_i + (w_{i+1} - w_i)/h_{i+1}]/\bar{h}_i, \quad D' w_i = (w_{i+1} - w_{i-1})/(2\bar{h}_i),$$

$$D^0 w_i = Q(-w_{i-1} + 2w_i - w_{i+1})/(2\bar{h}_i), \quad Q = \begin{cases} b^* \text{ if } 0 < p < 1, \\ 0 \text{ if } p \geq 1 \end{cases}$$

and b^* is given by

$$|b(x,u)| \leq b^*, \quad x \in I, \quad u \in W. \tag{3.5}$$

In the schemes above, the quantities w_0 and w_n should be replaced by U_0 and U_1, respectively.

Thus, the discrete problem reads

$$T^h w^h = 0. \tag{3.6}$$

Let $\|\cdot\|_\infty$ and $\|\cdot\|_1$ denote the usual vector (matrix) norms in $R^{n-1}(R^{n-1,n-1})$. Furthermore, in R^{n-1} we shall use the following discrete L^1 norm (cf. [13], [14]):

$$\|w^h\|_1^h = \sum_{i=1}^{n-1} \bar{h}_i |w_i|,$$

which can be written down in the form

$$\|w^h\|_1^h = \|Hw^h\|_1, \quad H = \text{ diag } (\bar{h}_1, \bar{h}_2, \cdots, \bar{h}_{N-1}).$$

The corresponding matrix norm is

$$\|A\|_1^h = \|HAH^{-1}\|_1, \quad A \in R^{n-1,n-1}.$$

Let

$$W^h = \{w^h \in R^{n-1} : w_i \in W, \quad i = 1(1)n-1\}.$$

Now we shall prove the stability inequality:

$$\|w^h - v^h\|_1^h \le g_*^{-1} \|T^h w^h - T^h v^h\|_1^h. \tag{3.7}$$

Theorem 3.1. *Let* (1.2) *hold and let* ε^* *be sufficiently small. Moreover, if* $p = 1$ *let* n *be sufficiently great, but independent of* ε. *Then* (3.7) *holds for any* w^h, v^h *from* W^h, *and in* W^h *there exists a unique solution to the discrete problem* (3.6).

Proof. For the technique, cf. [2], [4], [5] (equidistant meshes) and [13], [14] (non-equidistant meshes). Let

$$A = (T^h)'(w^h) \in R^{n-1,n-1}$$

be the F-derivative of the operator T^h at any $w^h \in W^h$. First we shall show that A is an L-matrix (i.e. the diagonal elements are positive and the off-diagonal elements are nonnegative). In the case $p < 1$ the D^0 term of T^h guarantees this property, since (3.5) holds. If $P \ge 1$ we have to fulfil

$$\mu b^* h_{i+1}, \mu b^* h_i \le 2\varepsilon^2, \quad i = 1(1)n-1,$$

which will follow form

$$\mu b^* h_{n_0} \le 2\varepsilon^2.$$

However, using (3.3) with $k = 1$, we can rewrite the last inequality as follows:

$$1 \le Mn\varepsilon^{1-p},$$

and this is satisfied under the assumptions of the theorem.

Now it is easy to see that

$$(HAH^{-1})^T e^h \ge g_* e^h$$

(componentwise), where

$$e^h = [1, 1, \cdots, 1]^T \in R^{n-1}.$$

This guarantees that A is nonsingular and $A^{-1} \ge 0$ (componentwise). Thus, A is an M-matrix, and we get

$$\|A^{-1}\|_1^h = \|((HAH^{-1})^T)^{-1}\|_\infty \le g_*^{-1}.$$

Now (3.7) follows from

$$w^h - v^h = ((T^h)'(\theta^h))^{-1}(T^h w^h - T^h v^h),$$

which is valid for some $\theta^h \in W^h$.

To prove that in W^h there exists a solution to the discrete problem (3.6), it is sufficient to show

$$T^h(u^* e^h) \geq 0 \geq T^h(u_* e^h)$$

(again, these inequalities should be understood componentwise). We shall prove the first inequality only, since the proof of the second one is analogous. For $i = 2(1)n-2$ we have

$$(T^h(u^* e^h))_i = \mu D' f(x_i, u^*) + g(x_i, u^*) \geq \mu f_x(x_i, u^*)$$
$$-M\mu/n + g(x_i, u^*) = c(x_i, u^*) - M\mu/n \geq 0$$

(because of (1.3a)). Furthermore,

$$(T^h(u^* e^h))_1 \geq c(x_1, u^*) - M\mu/n + \varepsilon^2(u^* - U_0)/(h_1\bar{h}_1) + \mu(u^* - U_0)b(0,\sigma)/(2\bar{h}_1)$$
$$+\mu Q(u^* - U_0)/(2\bar{h}_1) \geq 0, \quad \sigma \in [U_0, u^*],$$

again because of (1.3), and by using the above L-shape analysis. In the same way we can prove

$$(T^h(u^* e^h))_{n-1} \geq 0.$$

The uniqueness of the discrete problem in W^h follows from (3.7).

The unique solution to (3.6) will be denoted by $w_\varepsilon^h, w_\varepsilon^h \in W^h$, and by u_ε^h we shall denote the restriction of the continuous solution u_ε on $I^h \backslash \{0, 1\}$.

§4. The Convergence Results

The consistency-error vector is given by

$$r^h = T^h u_\varepsilon^h,$$

and its components are

$$r_i = r_i'' + r_i' + r_i^0, \quad i = 1(1)n-1, \quad r_i'' = \varepsilon^2[u_\varepsilon''(x_i) - D'' u_\varepsilon(x_i)],$$
$$r_i' = \mu[D' f(x_i, u_\varepsilon(x_i)) - f(x, u_\varepsilon(x))'_{x=x_i}], \quad r_i^0 = \mu D^0 u_\varepsilon(x_i).$$

Theorem 4.1. *Let the conditions of Theorem 3.1 hold. Then we have*

$$\|w_\varepsilon^h - u_\varepsilon^h\|_1^h \leq Md, \quad d = \begin{cases} [\varepsilon/n + \mu + \exp(-m_0 n)]/n, & 0 < p < 1, \\ [\varepsilon/n + \exp(-m_0 n)]/n, & p \geq 1 \end{cases}$$

where m_0 is a positive constant independent of ε and n.

Proof. Because of Theorem 3.1 it is sufficient to prove

$$\|r^h\|_1^h \leq Md.$$

As an illustration, we shall prove

$$\sum_{i=1}^{n_0-1} \bar{h}_i |r_i^0| \leq Md \tag{4.1}$$

for $0 < p < 1$. The other estimates can be proved analogously, cf. [9]–[12], [14].

For $i = 1(1)n_0 - 1$ we have the following two estimates:

$$\bar{h}_i|r_i^0| \le M\mu[(h_{i+1} - h_i)|u_\varepsilon'(x_i)| + h_{i+1}^2 \max_{x_i \le x \le x_{i+1}} |u_\varepsilon''(x)|], \tag{4.2}$$

and

$$\bar{h}_i|r_i^0| \le M\mu h_{i+1} \max_{x_{i-1} \le x \le x_{i+1}} |u_\varepsilon'(x)|. \tag{4.3}$$

Since we consider here the interval $[0, 1/2]$ only, the estimates from Theorem 2.1 can be simplified:

$$|u_\varepsilon^{(k)}(x)| \le M[1 + \varepsilon^{-k} y_\varepsilon(x)], \quad k = 1(1)4, \quad x \in [0, 1/2]. \tag{4.4}$$

1° Let $t_{i-1} \ge \alpha$. Then by using (4.2), (4.4), (3.2) and (3.3) we get

$$\bar{h}_i|r_i^0| \le M\mu[1 + \varepsilon^{-2} y_\varepsilon(x_{i-1})]/n^2 \le M\mu[1 + \varepsilon^{-2} y_\varepsilon(\omega(\alpha))]/n^2 \le M\mu/n^2. \tag{4.5}$$

2° Now, let $t_{i-1} < \alpha$ and $t_{i-1} \le \gamma - 3/n$, so that

$$\gamma - t_{i+1} \ge M(\gamma - t_{i-1}). \tag{4.6}$$

From (4.2), (4.4), (3.3), (3.4) and (4.6) it follows, that

$$\bar{h}_i|r_i^0| \le M\mu[1 + (\gamma - t_{i+1})^{-4} \exp(-M/(\gamma - t_{i-1}))]/n^2 \le M\mu/n^2. \tag{4.7}$$

3° Finally, if $\gamma - 3/n < t_{i-1} < \alpha$, we use (4.3), (4.4), (3.3) and (3.4) to get

$$\begin{aligned} \bar{h}_i|r_i^0| &\le M\mu[1 + \varepsilon^{-1} y_\varepsilon(x_{i-1})]/n \le M[\mu + \varepsilon^p \exp(-M/(\gamma - t_{i-1}))]/n \\ &\le M[\mu + \varepsilon^p \exp(-m_0 n)]/n. \end{aligned} \tag{4.8}$$

Note that there are no more than three points t_i which satisfy case 3°. Having this in mind, from (4.5), (4.7) and (4.8) we get (4.1). In fact, we have obtained somewhat better estimate — the quantity

$$[\varepsilon/n + \exp(-m_0 n)]/n$$

comes from the component r_i''.

Theorem 4.2. *Let the conditions of Theorem 3.1 hold. Then we have*

$$\|w_\varepsilon^h - u_\varepsilon^h\|_\infty \le \begin{cases} M[n^{-1} + \varepsilon^p + \varepsilon^{-1} \exp(-m_0 n)], & 0 < p < 1, \\ M[n^{-1} + \varepsilon^{-1} \exp(-m_0 n)], & p \ge 1 \end{cases}$$

with the same m_0 as in Theorem 4.1.

Proof. Since

$$h_i \ge M\varepsilon/n, \quad i = 1(1)n,$$

we get

$$\|w_h\|_\infty \le Mn\varepsilon^{-1}\|w_h\|_1^h$$

and the assertion follows from Theorem 4.1.

§5. Numerical Results

Let us consider the following test example:

$$-\varepsilon^2 u'' + \mu u u' + u - s(x) = 0, \quad x \in I, \quad u(0) = U_0, \quad u(1) = U_1,$$

where $s(x)$ and U_0, U_1 are determined so that the solution reads

$$u_\varepsilon(x) = -\exp(-x/\varepsilon) + \exp((x-1)/\varepsilon).$$

We take $b^* = 1$. let

$$E_\infty = \|w_\varepsilon^h - u_\varepsilon^h\|_\infty, \quad E_1 = \|w_\varepsilon^h - u_\varepsilon^h\|_1^h.$$

We shall be interested in the numerical order of convergence, both in $\|\cdot\|_\infty$ and $\|\cdot\|_1^h$:

$$\mathrm{Ord}_\infty(n) = (\ln 2)^{-1} \ln(E_\infty(n)/E_\infty(2n)),$$

$$\mathrm{Ord}_1(n) = (\ln 2)^{-1} \ln(E_1(n)/E_1(2n)),$$

where $E_\infty(n), E_1(n)$ mean the errors E_∞, E_1, respectively, on the mesh with n subintervals.

We use the function λ with the parameters:

$$\alpha = 0.25, \quad \beta = 1.$$

This gives the following percentage of the mesh points in the layers, represented by the intervals $[0, \varepsilon]$ and $[1 - \varepsilon, 1]$: about 35% for $\varepsilon = 1.-2$, 30% for $\varepsilon = 1.-3$ and 25% for $\varepsilon = 1.-4$ (as usual, the notation $a.-k$ means $a10^k$).

Table 1. Values of Ord_1 (40)

p	ε		
	1.−2	1.−3	1.−4
1	2.02	1.98	2.01
1/2	1.76	2.02	2.02
1/4	1.03	2.01	2.15
0	0.89	0.81	0.73

Table 2. Values of Ord_∞ (40)

p	ε		
	1.−2	1.−3	1.−4
1	1.99	1.94	2.06
1/2	1.88	1.93	2.07
1/4	1.34	1.93	2.12
0	0.52	0.67	1.12

The numerical results of Table 1 confirm the results of Theorem 4.1. Note that for a greater value of ε the order of convergence decreases from 2 to 1, as p does. Furthermore, the order does not decrease together with ε. The case $p = 0$ is beyond the theory presented in this paper.

A similar analysis holds for Table 2. In fact, the numerical results are even better than the theoretical ones, since the uniform pointwise convergence can not be observed from Theorem 4.2.

Finally, in Table 3 we give more detailed results for $p = 1$. As ε decreases, the error E_∞ increases slightly, while E_1 decreases.

Table 3. Values of E_∞ and E_1

n	ε			
	1.−2	1.−3	1.−4	
20	3.08−2	4.70−2	8.79−2	E_∞
	1.99−3	6.24−4	7.54−5	E_1
40	8.94−3	1.69−2	2.45−2	
	4.80−4	1.15−4	1.95−5	
80	2.25−3	4.39−3	5.89−3	
	1.18−4	2.91−5	4.84−6	

The nonlinear system (3.6) was solved by the Newton method. The iterations were carried out until the maximal pointwise difference between two successive iterations became less than 1.−6.

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