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A COMPLETELY EXPONENTIALLY FITTED DIFFERENCE SCHEME FOR A SINGULAR PERTURBATION PROBLEM*

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Abstract

A completely exponentially fitted difference scheme is considered for the singular perturbation problem: $\varepsilon U'' + a(x)U' - b(x)U = f(x)$ for $0 < x < 1$, with $U(0)$, and $U(1)$ given, $\varepsilon \in (0,1]$ and $a(x) > \alpha > 0, b(x) \geq 0$. It is proven that the scheme is uniformly second-order accurate.

§1. Introduction

The singular perturbation problem

$$\begin{aligned} &L_\varepsilon U_\varepsilon(x) \equiv \varepsilon U'' + a(x)U' - b(x)U = f(x), \qquad 0 < x < 1 \\ &U(0) = \beta_0, \; U(1) = \beta_1, \end{aligned} \tag{1.1}$$

where ε is a parameter in $(0, 1]$, $a(x) > \alpha > 0, b(x) \geq 0, x \in [0,1]$, is one of those main problems computational mathematicians are trying to solve. In dealing with this problem, some mathematicians are most interested in those numerical methods and schemes which are valid for the small parameter ε. In 1969, Il' in [1] designed an exponentially fitted finite difference scheme in the case $b(x) \equiv 0$, and showed that its solution converges uniformly in ε, with order one, to the solution of (1.1). Kellogg and Tsan [2] (1978), Miller [3] (1979) and Emelyanov [4] (1978) independently extended this result to the case $b(x) \geq 0$. Wu Qiguang [5] (1985) studied a class of weighted exponentially fitted difference schemes and proved that these schemes are uniformly convergent with order one. We have reason to say that satisfactory results have been obtained from the researches on the schemes which converge uniformly in ε with order one. In recent years, some mathematicians began to study higher order schemes. Hegarty, Miller and O'Riordan [6], Berger, Solomon and Ciment [7] proved separately that the completely exponentially fitted difference scheme (also called exponential box scheme) is convergent uniformly in ε with order two in the case $b(x) = 0$, which was derived by El-Mistikawy and Werle [8] in 1978. However, they could only conjecture through their numerical experiments that the same is true in the case $b(x) \geq 0$. In this paper, the scheme will be written in the form of completely exponentially fitting factor by introducing a dominant algebraic quantity. We will prove theoretically the conjecture of Berger etal. and complete soundly El-Mistikawy and Werle's scheme.

For the sake of convenience, we assume $a(x), b(x)$ and $f(x)$ are smooth enough, and throughout this paper those positive constants which are independent of ε, h and x_j will be generically denoted by C.

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§2. Asymptotic Properties of the Solution

In order to give bounds for the discretisation error for the scheme which are uniform in ε and h, we need the asymptotic expansion of the solution of (1.1). We first consider the following problem

$$\begin{aligned} L_\varepsilon U_\varepsilon(x) &= f_\varepsilon(x), \qquad 0 < x < 1, \\ U_\varepsilon(0) &= \beta_0(\varepsilon),\ U_\varepsilon(1) = \beta_1(\varepsilon). \end{aligned} \tag{2.1}$$

where $|\beta_0(\varepsilon)| \le C, |\beta_1(\varepsilon)| \le C, |f_\varepsilon^{(i)}(x)| \le C\{1 + \varepsilon^{-i-1}\exp(-\alpha x/\varepsilon)\}, i \ge 0$.

Noticing the Uniform boundness of $\beta_0(\varepsilon)$ and $\beta_1(\varepsilon)$, we have

Lemma 2.1 (Kellogg and Tsan [3]). *Problem* (2.1) *has a unique solution satisfying*

$$|U_\varepsilon^{(i)}(x)| \le C\{1 + \varepsilon^{-i}\exp(-\alpha x/\varepsilon)\}, \quad i \ge 0. \tag{2.2}$$

Lemma 2.1 *leads to the following lemma immediatly.*

Lemma 2.2 *The solution of problem* (2.1) *can be written in the form*

$$U_\varepsilon(x) = \delta V_\varepsilon(x) + Z_\varepsilon(x) \tag{2.3}$$

where $|\delta| \le C$ *is a constant,* $V_\varepsilon(x) = \exp(-\alpha(0)x/\varepsilon)$, $\alpha_\varepsilon(x) = \sqrt{a(x)^2 + 4\varepsilon b(x)}$ *and* $|Z_\varepsilon^{(i)}(x)| \le C\{1 + \varepsilon^{-i+1}\exp(-\alpha x/\varepsilon)\}, i \ge 0$.

As for problem (1.1), *we have*

Lemma 2.3 (Smith [9]). *The Solution of Problem* (1.1) *can be written in the form*

$$U_\varepsilon(x) = A_\varepsilon(x) + C_0 B(x)\exp(-\frac{1}{\varepsilon}\int_0^x a(x)ds) + R_\varepsilon(X) \tag{2.4}$$

where $|C_0| \le C$ is a constant, $|A_\varepsilon^{(i)}(x)| \le C, i \ge 0, B(x) = \dfrac{1}{a(x)}\exp\Big(-\displaystyle\int_0^x \frac{b(x)}{a(x)}ds\Big)$ and $R_\varepsilon(x)$ satisfies

$$\begin{aligned} L_\varepsilon R_\varepsilon(x) &= F_\varepsilon(x), \qquad 0 < x < 1, \\ R_\varepsilon(0) &= 0, \quad R_\varepsilon(1) = \beta(\varepsilon), \end{aligned}$$

where

$$|\beta(\varepsilon)| \le C, |F_\varepsilon^{(i)}(x)| \le C\{1 + \varepsilon^{-i}\exp(-\alpha x/\varepsilon)\}, i \ge 0.$$

Summing up the above lemmas, we conclude

Theorem 2.4. *Problem* (1.1) *has a unique solution which can be expressed as*

$$U_\varepsilon(x) = A_\varepsilon(x) + C_0 G_\varepsilon(x) + \varepsilon R_\varepsilon(x)$$

where

$$G_\varepsilon(x) = W_\varepsilon(x)E_\varepsilon(x), \quad R_\varepsilon(x) = \delta V_\varepsilon(x) + Z_\varepsilon(x)$$

and $|C_0|, |\delta| \le C$ are constants, $|A_\varepsilon^{(i)}(x)| \le C, i \ge 0$,

$$W_\varepsilon(x) = \frac{1}{a(x)}\exp\Big[-\int_0^x \Big(\frac{b(s)}{a(s)} + \frac{a(s) - \alpha(s)}{\varepsilon}\Big)ds\Big], \quad E_\varepsilon(x) = \exp\Big(-\frac{1}{\varepsilon}\int_0^x \alpha(s)ds\Big),$$

$$V_\varepsilon(x) = \exp(-\alpha(0)x/\varepsilon), \quad |Z_\varepsilon^{(i)}(x)| \le C\{1 + \varepsilon^{-i+1}\exp(-\alpha x/\varepsilon), \quad i \ge 0.$$

It is clear that $|W_\varepsilon^{(i)}(x)| \le C, i \ge 0$. For the sake of convenience, the subscripts ε in the symbols of operators and functions will be omitted.

§3. The Difference Scheme and Its Basic Properties

The scheme was derived by El-Mistikawy and Werle by a piecewise constant coefficient approximate differential equation. Now we describe the process briefly. Let $[0, 1]$ be divided into N uniformly spaced mesh intervals, with mesh spacing $h = \frac{1}{N}$ and with mesh points $x_j = jh$, $j = 0, 1, \cdots, N$. Let $a_j = a(x_j), b_j = b(x_j), f_j = f(x_j)$. Consider the problem

$$\begin{aligned} &L_h U_h(x) \equiv \varepsilon U_h'' + a_h U_h' - b_h U_h = f_h, \qquad 0 < x < 1, \\ &U_h(0) = \beta_0, \quad U_h(1) = \beta_1 \end{aligned} \tag{3.1}$$

where $a_h(x) = (a_{j-1} + a_j)/2, x \in [x_{j-1}, x_j]$ and $b_h(x), f_h(x)$ are similarly defined.

Problem (3.1) has a unique generalized solution $U_h(x)$ in $C'[0, 1]$. Let $U_j = U_h(x_j)$ for $0 \le j \le N$. On each small interval $[x_{j-1}, x_j]$, solving the constant coefficient ordinary differential equation

$$\begin{aligned} &L_h U_h(x) = f_h, \qquad x_{j-1} < x < x_j, \\ &U_h(x_{j-1}) = U_{j-1}, \quad U_h(x_j) = U_j. \end{aligned} \tag{3.2}$$

We can express $U_h(x)$ by U_{j-1} and U_j. The contiunity of the first derivative of $U_h(x)$ will lead to the difference scheme which determines $\{U_j\}_{j=0}^N$

$$\begin{aligned} &\varepsilon h^{-2}(r_j^- U_{j-1} + r_j^c U_j + r_j^+ U_{j+1}) = q_j^- f_{j-1} + q_j^c f_j + q_j^+ f_{j+1}, \quad 1 \le j \le N-1, \\ &U_0 = \beta_0, \qquad U_N = \beta_1. \end{aligned} \tag{3.3}$$

This scheme is called completely exponentially fitted difference scheme and written in the simple form

$$\begin{aligned} &R^h U_j = Q^h f_j, \qquad 1 \le j \le N-1, \\ &U_0 = \beta_0, \qquad U_N = \beta_1, \end{aligned} \tag{3.4}$$

where

$$R^h V_j = \varepsilon h^{-2}(r_j^- V_{j-1} + r_j^c V_j + r_j^+ V_{j+1}), \quad Q^h V_j = q_j^- V_{j-1} + q_j^c V_j + q_j^+ V_{j+1}$$

and

$$\begin{aligned} &r_j^- = \exp(n_{1j})/g(n_{1j} - k_{1j}), \quad r_j^+ = \exp(-k_{2j})/g(n_{2j} - k_{2j}), \\ &r_{1j} = -n_{1j} - 1/g(n_{1j} - k_{1j}), \quad r_{2j} = k_{2j} - 1/g(n_{2j} - k_{2j}), \quad r_j^c = r_{1j} + r_{2j}, \\ &q_j^- = [g(n_{1j}) - \exp(n_{1j}) g(-k_{1j})] p(n_{1j} - k_{1j}), \\ &q_j^+ = [g(-k_{2j}) - \exp(-k_{2j}) g(n_{2j})] p(n_{2j} - k_{2j}), \quad q_j^c = q_j^- + q_j^+, \end{aligned} \tag{3.5}$$

where

$$g(m) = [\exp(m) - 1]/w, \quad g(0) = 1; \quad p(m) = 1/2[1 - \exp(m)]; \quad k_{1j}/h$$

denotes the nonnegative root of the characteristic equation of (3.2) and n_{1j}/h denotes the negative root; k_{2j} and n_{2j} can be defined similarly,

$$k_{1j} = -h\frac{a_j^- - \alpha_j^-}{2\varepsilon}, \quad n_{1j} = -h\frac{a_j^- + \alpha_j^-}{2\varepsilon}, \quad k_{2j} = -h\frac{a_j^+ - \alpha_j^+}{2\varepsilon}, \quad n_{2j} = -h\frac{a_j^+ + \alpha_j^+}{2\varepsilon},$$

$$\alpha_j^\mp = \sqrt{(a_j^\mp)^2 + 4\varepsilon b_j^\mp}, \quad a_j^\mp = \frac{a_{j\mp 1} + a_j}{2}, \quad b_j^\mp = \frac{b_{j\mp 1} + b_j}{2}.$$

The key to the research on the scheme (3.3) lies in choosing dominant algebraic quantity. Let $\rho(x) = h\alpha(x)/\varepsilon$, and (3.5) can be simplified as

$$r_j^- = r_{*j}^- \exp(k_{1j}), \quad r_j^+ = r_{*j}^+ \exp(-k_{2j}),$$
$$r_{1j} = -k_{1j} - r_{*j}^-, \quad r_{2j} = k_{2j} - r_{*j}^+, \quad r_j^c = r_{1j} + r_{2j},$$
$$q_j^- = \frac{\exp(k_{1j}) - 1}{k_{1j}} \cdot \frac{r_{*j}^-}{2n_{1j}} - \frac{1}{2n_{1j}}, \quad q_j^+ = \frac{\exp(-k_{2j}) - 1}{k_{2j}} \cdot \frac{r_{*j}^+}{2n_{2j}} + \frac{1}{2n_{2j}}, \quad q_j^c = q_j^+ + q_j^+,$$

where

$$r_{*j}^- = \frac{\rho_j^- \exp(-\rho_j^-)}{1 - \exp(-\rho_j^-)}, \quad r_{*j}^+ = \frac{\rho_j^+}{1 - \exp(-\rho_j^+)}, \quad \rho_j^\mp = h\alpha_j^\mp/\varepsilon.$$

The subscripts j in r_{*j}^-, q_j^-, etc. will be omitted in the following when there is no confusion.

The scheme (3.3) is determined by r^-, r^+, r^c, q^-, q^+ and q^c and these quantities can be expressed by r_*^- and r_*^+. Hence, it will be convenient to deal with scheme (3.3) by means of the properties of r_*^- and r_*^+.

Definition 3.1.

$$r_*^-(\rho) = \frac{\rho \exp(-\rho)}{1 - \exp(-\rho)}, \quad r_*^-(0) = 1; \quad r_*^+(\rho) = \frac{\rho}{1 - \exp(-\rho)}, \quad r_*^+(0) = 1,$$
$$q_*^-(\rho) = \frac{1 - r_*^-(\rho)}{2\rho}, \quad q_*^-(0) = 1/4; \quad q_*^+(\rho) = \frac{r_*^+(\rho) - 1}{2\rho}, \quad q_*^+(0) = 1/4.$$

Remark 3.1. The functions in Definition 3.1 are C^∞ on R' and for $\rho \in R'$; it is true that

$$r_*^-(\rho), \quad r_*^+(\rho), q_*^-(\rho), \quad q_*^+(\rho) > 0,$$
$$D_\rho r_*^-(\rho) < 0, \quad D_\rho r_*^+(\rho) > 0, \quad D_\rho q_*^-(\rho) < 0, \quad D_\rho q_*^+(\rho) > 0.$$

Remark 3.2. We have the following power series expansions about $\rho = 0$,

$$r_*^-(\rho) = 1 - \rho/2 + \rho^2/12 + 0(\rho^4), \quad r_*^+(\rho) = 1 + \rho/2 + \rho^2/12 + 0(\rho^4),$$
$$q_*^-(\rho) = 1/4 - \rho/24 + 0(\rho^3), \quad q_*^+(\rho) = 1/4 + \rho/24 + 0(\rho^3).$$

Remark 3.3. For $\rho \geq P$,

$$|D_\rho^i r_*^-(\rho)| \leq C\rho \exp(-\rho) \leq C \exp(-\rho/2), \quad 0 \leq i \leq 2,$$
$$|r_*^+(\rho)| \leq C\rho, \quad |D_\rho r_*^+(\rho)| \leq C,$$
$$|q_*^-(\rho)| \leq C\rho^{-1}, \quad |D_\rho q_*^-(\rho)| \leq C\rho^{-2},$$
$$|q_*^+(\rho)| \leq C,$$

where P is a constant independent of h and ε.

Remark 3.4.

$$|k_i| \leq Ch, \quad i = 1, 2,$$
$$\alpha h\varepsilon^{-1} \leq |n_i|, \quad \rho^\mp \leq Ch\varepsilon^{-1}, \quad i = 1, 2.$$

Definition 3.2.

$$r^-(\rho^-) = r_*^-(\rho^-)\exp(k_1), \quad r^-(\rho^+) = r_*^-(\rho^+)\exp(k_2),$$

$$r^+(\rho^-) = r_*^+(\rho^-)\exp(-k_1), \quad r^+(\rho^+) = r_*^+(\rho^+)\exp(-k_2),$$

$$q^-(\rho^-) = \frac{\exp(k_1)-1}{k_1}\cdot\frac{r_*^-(\rho^-)}{2n_1} - \frac{1}{2n_1}, \quad q^-(\rho^-) = q_*^-(\rho^-), \quad \text{for } b^- = 0,$$

$$q^-(\rho^+) = \frac{\exp(k_2)-1}{k_2}\cdot\frac{r_*^-(\rho^+)}{2n_2} - \frac{1}{2n_2}, \quad q^-(\rho^+) = q_*^-(\rho^+), \quad \text{for } b^+ = 0,$$

$$q^+(\rho^-) = \frac{\exp(-k_1)-1}{k_1}\cdot\frac{r_*^+(\rho^-)}{2n_1} + \frac{1}{2n_1}, \quad q^+(\rho^-) = q_*^+(\rho^-), \quad \text{for } b^- = 0,$$

$$q^+(\rho^+) = \frac{\exp(-k_2)-1}{k_2}\cdot\frac{r_*^+(\rho^+)}{2n_2} + \frac{1}{2n_2}, \quad q^+(\rho^+) = q_*^+(\rho^+), \quad \text{for } b^+ = 0.$$

Remark 3.5. The algebraic quantites in Definition 3.2 have the following estimates:

$$|r^-(\rho^\mp)| \le C\exp(-\alpha h/\varepsilon), \quad |r^+(\rho^\mp)| \le Ch\frac{1}{\min(h,\varepsilon)},$$

$$|q^-(\rho^\mp)| \le C\varepsilon\frac{1}{\max(h,\varepsilon)}, \quad |q^+(\rho^\mp)| \le C,$$

$$|r^-(\rho^+) - r^-(\rho^-)| \le C\varepsilon^{-1}h^2\exp(-\alpha h/\varepsilon),$$

$$|q^-(\rho^+) - q^-(\rho^-)| \le Ch^2\frac{1}{\max(h,\varepsilon)}.$$

Remark 3.6. $r^- > 0, r^+ > 0, -r_1 \ge r^-. - r_2 \ge r^+, q^- > 0, q^+ > 0.$

§4. Analysis of the Truncation Error

The truncation error for the scheme (3.3) can be written in the form

$$\tau_j(U) = R^h(U(x_j) - U_j) = R^hU(x_j) - Q^h(LU(x_j)) = \sum_{i=0}^{n} T^iU^{(i)}(x_j) + R_n^*(U) \tag{4.1}$$

where

$$T^0 = \varepsilon h^{-2}(r^- + r^c - r^+) + (b_{j-1}q^- + b_jq^c + b_{j+1}q^+),$$

$$T^1 = \varepsilon h^{-1}(-r^- + r^+) - (a_{j-1}q^- + a_jq^c + a_{j+1}q^+) + (-b_{j-1}q^- + b_{j+1}q^+),$$

$$T^2 = \frac{\varepsilon}{2}(r^- + r^+) - \varepsilon(q^- + q^c + q^+) - h(-a_{j-1}q^- + a_{j+1}q^+) + \frac{h^2}{2}(b_{j-1}q^- + b_{j+1}q^+),$$

$$T^i = \frac{\varepsilon}{h^2}\frac{h^i}{i!}\Big[(-1)^ir^- + r^+\Big] - \varepsilon\frac{h^{i-2}}{(i-2)!}\Big[(-1)^iq^- + q^+\Big]$$

$$-\frac{h^{i-1}}{(i-1)!}\Big[(-1)^{i-1}a_{j-1}q^- + a_{j+1}q^+\Big] + \frac{h^i}{i!}\Big[(-1)^ib_{j-1}q^- + b_{j+1}q^+\Big], 3 \le i \le n,$$

$$
\begin{aligned}
R_n^*(U) &= \varepsilon h^{-2}[r^- R_n(x_j, x_{j-1}, U) + r^+ R_n(x_j, x_{j+1}, U)] \\
&\quad -\varepsilon[q^- R_{n-2}(x_j, x_{j-1}, U'') + q^+ R_{n-2}(x_j, x_{j+1}, U'')] \\
&\quad -[a_{j-1} q^- R_{n-1}(x_j, x_{j-1}, U') + a_{j+1} q^+ R_{n+1}(x_j, x_{j+1}, U')] \\
&\quad +[b_{j-1} q^- R_n(x_j, x_{j-1}, U) + b_{j+1} q^+ R_n(x_j, x_{j+1}, U)]
\end{aligned} \tag{4.2}
$$

$$
\begin{aligned}
R_n(a, b, g) &= g(b) - \sum_{i=0}^{n} g^{(i)}(a) \frac{(b-a)^i}{i!} \\
&= g^{(n+1)}(\xi) \frac{(b-a)^{n+1}}{(n+1)!} = \frac{1}{n!} \int_a^b (b-s)^n g^{(n+1)}(s) ds
\end{aligned} \tag{4.3}
$$

Here ξ is a point between the points a and b.

The first step in estimating the truncation error is to show the following

Lemma 4.1. *The algebraic quantities $T^i (i = 0, 1, 2, 3)$ from (4.1) satisfy $T^0 = 0, |T^i| \le Ch^2$, for $i = 1, 2, 3$.*

Proof. It is obvious that $T^0 \equiv 0$. From (3.5),

$$
\begin{aligned}
T^1 &= \varepsilon h^{-1}(-r^- + r^+) - 2(a^- q^- + a^+ q^+) + h(-b^- q^- + b^+ q^+) + O(h^2) \\
&= T_1^1 + T_2^1 + O(h^2)
\end{aligned}
$$

where

$$
T_1^1 = \varepsilon h^{-1}(r^-(\rho^+) + r^+(\rho^+)) - 2a^+(q^-(\rho^+) + q^+(\rho^+)) + hb^+(-q^-(\rho^+) + q^+(\rho^+)),
$$

$$
T_2^1 = \varepsilon h^{-1}(r^-(\rho^+) - r^-(\rho^-)) + 2(a^+ q^-(\rho^+) - a^- q^-(\rho^-)) + h(b^+ q^-(\rho^+) - b^- q^-(\rho^-)).
$$

Using Definition 3.2 and Remarks 3.2, 3.3 and 3.4, we can obtain

$$
\begin{aligned}
T_1^1 &= -(2\varepsilon h^{-1} K_2 + \frac{a^+ k_2}{n_2} + \frac{hb^+}{n_2})(\frac{\rho^+}{2} \mathrm{Coth} \frac{\rho^2}{2} - 1) + O(h^2) \\
&= \frac{\frac{1}{2} a^+(\alpha^+ - a^+) - \varepsilon b^+}{a^+ + \alpha^+} (\frac{\rho^+}{2} \mathrm{Coth} \frac{\rho^+}{2} - 1) + O(h^2)
\end{aligned}
$$

and so

$$
|T_1^1| \le Ch^2.
$$

Next, as

$$
\begin{aligned}
T_2^1 &= (\frac{\varepsilon}{h} + \frac{a^+}{n_2})(\rho^+ - \rho^-) D_\rho r_*^-(\rho^-) + \left[\frac{\varepsilon}{h}(k_2 - k_1) + \frac{1}{2}(\frac{a^+ k_2}{n_2} - \frac{a^- k_1}{n_1})\right] r_*^-(\rho^-) \\
&\quad + (\frac{a^+}{n_2} - \frac{a^-}{n_1})(r_*^-(\rho^-) - 1) + O(h^2),
\end{aligned}
$$

estimating T_2^1 for the two cases $0 < \rho \le P$ and $\rho \ge P$ respectively, we can obtain

$$
|T_2^1| \le Ch^2,
$$

hence

$$
|T^1| \le Ch^2.
$$

Similarly, we can prove

$$|T^2| \le Ch^2, \qquad |T^3| \le Ch^2.$$

The proof is completed by combining the above results.

We estimate the truncation error by using the "Method of Singularity Decomposition", Let $n = 3$ in (4.1). From Theorem 2.1, Lemma 4.1, and Remark 3.5, we can easily see

Lemma 4.2. *If $\{A_j\}$ is the solution of*

$$R^h A_j = Q^h(LA(x_j)), \qquad 1 \le j \le N-1$$
$$A_0 = A(0), \qquad A_N = A(1),$$

then

$$|\tau_j(A)| \le Ch^2 \qquad \text{for } 1 \le j \le N-1.$$

Lemma 4.3. *If $\{Z_j\}$ is the solution of*

$$R^h Z_j = Q^h(LZ(x_j)), \qquad 1 \le j \le N-1,$$
$$Z_0 = Z(0), \quad Z_N = z(1),$$

then for $1 \le j \le N-1$,

$$|\tau_j(Z)| \le Ch^2 + C\varepsilon^{-2}h^2\exp(-\alpha x_j/\varepsilon) \qquad \text{for } h \le \varepsilon,$$
$$|\tau_j(Z)| \le Ch^2 + C\varepsilon h^{-1}\exp(-\alpha x_{j-1}/\varepsilon) \qquad \text{for } h \ge \varepsilon.$$

Proof. From Theorem 2.1,

$$|Z^{(i)}(x)| \le \{1 + \varepsilon^{-i+1}\exp(-\alpha x/\varepsilon)\} \qquad \text{for } i \ge 0.$$

We first consider the case that $h \le \varepsilon$. Letting $n = 3$ in (4.1) , we have

$$\tau_j(Z) = \sum_{i=0}^{3} T^i Z^{(i)}(x_j) + R_3^*(Z).$$

Clearly,

$$\left|\sum_{i=0}^{3} T^i Z^{(i)}(x_j)\right| \le Ch^2 + C\varepsilon^{-2}h^2\exp(-\alpha x_j/\varepsilon) \quad \text{for} \quad 1 \le j \le N-1.$$

By using $|r^+| \le C, |q^+| \le C, |r^-| \le C, |q^-| \le C, \exp(-\alpha\xi/\varepsilon) \le C\exp(-\alpha x_j/\varepsilon)$. when $\xi \in (x_{j-1}, x_{j+1})$ and the differential form of (4.3), we can obtain

$$|\varepsilon h^{-2} r^+ R_3(x_j, x_{j+1}, Z)| \le Ch^2 + Ch^2\varepsilon^{-2}\exp(-\alpha x_j/\varepsilon) \qquad \text{for } 1 \le j \le N-1$$

and similar estimates for $|\varepsilon q^+ R_1(x_j, x_{j+1}, Z^N)|$, $|a_{j+1}q^+ R_2(x_j, x_{j+1}, Z^1)|$, $|b_{j+1}q^+ R_3(x_j, x_{j+1}, Z)|$, $|\varepsilon h^{-2} r^- R_3(x_j, x_{j-1}, Z)|$, $|a_{j-1}q^- R_2(x_j, x_{j-1}, Z^1)|$, $|\varepsilon q^- R_1(x_j, x_{j-1}, Z'')|$, $|b_{j-1}q^- R_3(x_j, x_{j-1}, Z)|$.

So

$$|R_3^*(Z)| \le Ch^2 + C\varepsilon^{-2}h^2\exp(-\alpha x_j/\varepsilon) \qquad \text{for } 1 \le j \le N-1.$$

In the case $h \geq \varepsilon$, we let $n = 2$ in (4.1) and obtain

$$\tau_j(Z) = \sum_{i=0}^{2} T^i Z^{(i)}(x_j) + R_2^*(Z).$$

In what follows, we will use this inequality repeatedly:

$$t^i \cdot \exp(-t) \leq C \exp(-t/2) \tag{4.4}$$

where $t > 0$ and i is any given positive integer. It is easily seen that

$$\Big|\sum_{i=0}^{2} T^i Z^{(i)}(x_j)\Big| \leq Ch^2 + C\varepsilon \exp(-\alpha x_j/\varepsilon) \qquad \text{for } 1 \leq j \leq N-1.$$

From $|r^+| \leq C\varepsilon^{-1}h, |q^+| \leq C, \exp(-\alpha\xi/\varepsilon) < \exp(-\alpha x_j/\varepsilon)$ with $\xi \in (x_j, x_{j+1})$, and the differential form of (4.3) and noticing (4.4), we have

$$|\varepsilon h^{-2} r^+ R_2(x_j, x_{j+1}, Z)| \leq Ch^2 + C\varepsilon h^{-1} \exp(-\alpha x_j/\varepsilon) \qquad \text{for } 1 \leq j \leq N-1$$

and similar estimates for $|\varepsilon q^+ R_0(x_j, x_{j+1}, Z'')|$, $|a_{j+1} q^+ R_1(x_j, x_{j+1}, Z^1)|$, and $|b_{j+1} q^+ R_2(x_j,$ $x_{j+1}, Z)|$.

Similarly, we can prove that

$$|\varepsilon h^{-2} R_2(x_j, x_{j-1}, Z)| \leq Ch^2 + C\varepsilon h^{-1} \exp(-\alpha x_{j-1}/\varepsilon) \qquad \text{for } 2 \leq j \leq N-1$$

and similar estimates for

$$|\varepsilon q^- R_0(x_j, x_{j+1}, Z'')|, \quad |a_{j-1} q^- R_1(x_j, x_{j+1}, Z')|,$$

and

$$|b_{j-1} q^- R_2(x_j, x_{j-1}, Z)|.$$

For $j = 1$, we use the integral form of (4.3). Then,

$$|\varepsilon h^{-2} \cdot r_1^- R_2(h, 0, Z)| = |\varepsilon h^{-2} r_1^- R_1(h, 0, Z) + \frac{h^2}{2} Z''(h)|$$

$$\leq C\varepsilon h^{-1} \int_0^h \Big[1 + \varepsilon^{-1} \exp(-\alpha x/\varepsilon)\Big] dx + C\varepsilon\Big[1 + \varepsilon^{-1} \exp(-\alpha h/\varepsilon)\Big] \leq C\varepsilon h^{-1}.$$

Similarly, we can obtain the estimates for $|\varepsilon q_1^- R_0(h, 0, Z'')|$, $|a_0 q^1 R_1(h, 0, Z')|$ and $|b_0 q_1^- R_2(h,$ $0, Z)|$. Hence,

$$|R_2^*(Z)| \leq Ch^2 + C\varepsilon h^{-1} \exp(-\alpha x_{j-1}/\varepsilon) \qquad \text{for } 1 \leq j \leq N-1$$

and the desired result follows.

Corollary 4.3. For $\{Z_j\}$ in Lemma 4.3,

$$|\tau_j(Z)| \leq C\varepsilon^{-1} h^2 \Big\{1 + \frac{1}{\max(h, \varepsilon)} \exp(-\alpha x_{j-1}/\varepsilon)\Big\} \qquad \text{for } | \leq j \leq N-1.$$

Lemma 4.4. *If $\{V_j\}$ is the solution of*

$$R^h V_j = Q^h(LV(x_j)), \qquad 1 \le j \le N-1,$$
$$V_0 = V(0), \quad V_N = V(1),$$

then, for $1 \le J \le N-1$, the following hold:

$$|\tau_J(V)| \le C\varepsilon^{-2}h^2 \exp(-\alpha x_j/\varepsilon) \qquad \text{when } h \le \varepsilon,$$
$$|\tau_j(V)| \le C\varepsilon h^{-1} \exp(-\alpha x_{j-1}/\varepsilon) \qquad \text{when } h \ge \varepsilon.$$

Proof. Instead of using the Taylor expansion of the truncation error we estimate directly

$$\tau_j(V) = R^h V(x_j) - Q^h(LV(x_j)).$$

Setting $\tau^r = R^k(V(x_j)), \tau^q = Q^h(LV(x_j))$, we have

$$\tau^r = \varepsilon h^{-2}[r^- \exp(\alpha(0)h/\varepsilon) + r_1 + r_2 + r^+ \exp(-\alpha(0)h/\varepsilon)]V(x_j),$$
$$\tau^q = [q^-\delta_{j-1}\exp(\alpha(0)h/\varepsilon) + (q^- + q^+)\delta_j + q^+\delta_{j+1}\exp(-\alpha(0)h/\varepsilon)]V(x_j)$$

where $\delta_j = \delta(x_j) = \alpha(0)(\alpha(0) - a(x_j))\varepsilon^{-1} - b(x_j)$.

We first suppose that $h \ge \varepsilon$. Let M denote those quantities which satisfy $|M| \le C\varepsilon h^{-1}\exp(-\alpha x_{j-1}/\varepsilon)$. We first estimate $|\tau^r|$. From Remark 3.3, it is clear that

$$\tau^r = \tau^r_* + M$$

where

$$\tau^r_* = \varepsilon h^{-2}[r^-_*(\rho^-)\exp(\alpha(0)h/\varepsilon) - r^-_*(\rho^-) - r^+_*(\rho^+) + r^+_*(\rho^+)\exp(-\alpha(0)h/\varepsilon)[V(x_j).$$

This can be written in the form

$$\tau^r_* = \varepsilon h^{-2} r^+_*(\rho^+)[\exp(\alpha(0)h/\varepsilon) - 1][\frac{r^-_*(\rho^-)}{r^+_*(\rho^+)} - \exp(-\alpha(0)h/\varepsilon]V(x_j).$$

We can prove that when $h \le C_1 (C_1 > 0$ is a constant independent of $\varepsilon)$,

$$|\tau^r_*| \le C\varepsilon h^{-1}\exp(-\alpha x_j/\varepsilon) \qquad \text{for } 1 \le j \le N-1,$$

so

$$|\tau^r| \le C\varepsilon h^{-1}\exp(-\alpha x_j/\varepsilon) \qquad \text{for } 1 \le j \le N-1.$$

We next estimate $|\tau^q|$. Since

$$\begin{aligned} |\delta(x_j)| &= |\alpha(0)[(\alpha(0) - a(0)) + (a(0) - a(x_j))]\varepsilon^{-1} - b(x_j)| \\ &\le C\varepsilon^{-1}x_j \qquad \text{for } 1 \le j \le N-1 \end{aligned}$$

we immediately see that

$$|(q^- + q^+)\delta_j V(x_j)| \le C\varepsilon h^{-1}\exp(-\alpha x_j/\varepsilon) \qquad \text{for } 1 \le j \le N-1,$$
$$|q^+\delta_{j+1}V(x_{j+1})| \le C\varepsilon h^{-1}\exp(-\alpha x_{j+1}/\varepsilon) \qquad \text{for } 1 \le j \le N-1,$$
$$|q^-\delta_{j-1}V(x_{j-1})| \le C\varepsilon h^{-1}\exp(-\alpha x_{j-1}/\varepsilon) \qquad \text{for } 2 \le j \le N-1.$$

For $j = 1$, we find

$$|q_1^-, \delta_0 V(0)| = |q_1^- [\alpha(0)(\alpha(0) - a(0))\varepsilon^{-1} - b(0)]| \le C\varepsilon h^{-1}.$$

Thus

$$|\tau^q| \le C\varepsilon h^{-1} \exp(-\alpha x_{j-1}/\varepsilon) \qquad \text{for } 1 \le j \le N-1$$

and then

$$|\tau_j(V)| \le |\tau^r| + |\tau^q| \le C\varepsilon h^{-1} \exp(-\alpha x_{j-1}/\varepsilon) \qquad \text{for } 1 \le j \le N-1.$$

To bound $|\tau_j(V)|$ in the case $h \le \varepsilon$, we let M denote those quantities which satisfy $|M| \le C\varepsilon^{-2}h^2 \exp(-\alpha x_j/\varepsilon)$. Then

$$\tau^r = \tau_0^r + \tau_1^r + \tau_2^r + \tau_3^r + M$$

where

$$\tau_0^r = \varepsilon h^{-2}\{r_*^-(\rho^-)[\exp(\alpha(0)h/\varepsilon) - 1] + r_*^+(\rho^+)[\exp(-\alpha(0)h/\varepsilon) - 1]\}V(x_j),$$
$$\tau_1^r = \varepsilon h^{-2}\{k_1[r_*^-(\rho^-)\exp(\alpha(0)h/\varepsilon) - 1] - k_2[r_*^+(\rho^+)\exp(-\alpha(0)h/\varepsilon) - 1]\}V(x_j),$$
$$\tau_2^r = \varepsilon h^{-2}[\frac{1}{2}k_1^2 r_*^-(\rho^-)\exp(\alpha(0)h/\varepsilon) + \frac{1}{2}k_2^2 r_*^+(\rho^+)\exp(-\alpha(0)h/\varepsilon)]V(x_j),$$
$$\tau_3^r = \varepsilon h^{-2}[\frac{1}{6}k_1^3 r_*^-(\rho^-)\exp(\alpha(0)h/\varepsilon) - \frac{1}{6}k_2^3 r_*^+(\rho^+)\exp(-\alpha(0)h/\varepsilon)]V(x_j).$$

It can be showed that

$$\tau_0^r = \alpha_0(\alpha_0 - \alpha_j)\varepsilon^{-1}V(x_j) + M,$$

$$\tau_1^r = \frac{1}{2}(2\alpha_0 - \alpha_j)(\alpha_j - a_j)\varepsilon^{-1}V(x_j) + M,$$

$$\tau_2^r = \frac{1}{4}(\alpha_j - a_j)^2\varepsilon^{-1}V(x_j) + M,$$

$$\tau_3^r = M.$$

Hence

$$\tau^r = \delta(x_j)V(x_j) + M.$$

Similarly,

$$\tau^q = \tau_0^q + \tau_1^q + \tau_j^q + M$$

where

$$\tau_0^q = -\left\{\frac{\rho^-}{n_1}q_*^-(\rho^-)[\delta_{j-1}\exp(\alpha(0)h/\varepsilon) + \delta_j] + \frac{\rho^+}{n_2}q_*^+(\rho^+)[\delta_{j+1}\exp(-\alpha(0)h/\varepsilon) + \delta_j]\right\}V(x_j),$$

$$\tau_1^q = \left\{\frac{k_1}{4n_1}r_*^-(\rho^-)[\delta_{j-1}\exp(\alpha(0)h/\varepsilon) + \delta_j] + \frac{k_2}{4n_2}r_*^+(\rho^+)[\delta_{j+1}\exp(-\alpha(0)h/\varepsilon) + \delta_j]\right\}V(x_j),$$

$$\tau_2^q = \left\{\frac{k_1^2}{12n_1}r_*^-(\rho^-)[\delta_{j-1}\exp(\alpha(0)h/\varepsilon) + \delta_j] - \frac{k_2^2}{12n_2}r_*^+(\rho^+)[\delta_{j+1}\exp(-\alpha(0)h/\varepsilon) + \delta_j]\right\}V(x_j).$$

We have

$$\tau_0^q = -\frac{\rho}{n}\delta(x_j)V(x_j) + M,$$

$$\tau_1^q = \frac{k}{n}\delta(x_j)V(x_j) + M,$$

$$\tau_2^q = M.$$

So

$$\tau^q = \delta(x_j)V(x_j) + M.$$

We conclude that

$$|\tau_j(V)| \le C\varepsilon^{-2}h^2 \exp(-\alpha x_j/\varepsilon), \qquad 1 \le j \le N-1.$$

The proof is then completed by combining the above results.

Corollary 4.4. For the $\{V_j\}$ in Lemma 4.4,

$$|\tau_j(V)| \le C\varepsilon^{-1}h^2 \frac{1}{\max(h,\varepsilon)} \exp(-\alpha x_{j-1}/\varepsilon), \qquad 1 \le j \le N-1.$$

Lemma 4.5. *If $\{G_j\}$ is the solution of*

$$R^h G_j = Q^h(LG(x_j)), \quad 1 \le j \le N-1,$$

$$G_0 = G(0), \quad G_N = G(1),$$

then

$$|\tau_j(G)| \le Ch^2 \frac{1}{\max(h,\varepsilon)} \exp(-\alpha x_{j-1}/\varepsilon) \qquad \textit{for } 1 \le j \le N-1.$$

Proof. We write $\tau_j(G) = \tau^r - \tau^q$ where $\tau^r = R^h(G(x_j)), \tau^q = Q^h(LG(x_j))$. Let M denote those quantities which satisfy $|M| \le Ch^2 \dfrac{1}{\max(h,\varepsilon)} \exp(-\alpha x_{j-1}/\varepsilon)$.

In the case $h \ge \varepsilon$, we first estimate $|\tau^q|$.$G(x)$ is written in the form

$$G(X) = B(x) \exp(-\frac{1}{\varepsilon}\int_0^x a(s)ds).$$

Then

$$LG(x) = \varepsilon B''(x) \exp(-\frac{1}{\varepsilon}\int_0^x a(s)ds)$$

and

$$|LG(x_j)| \le C\varepsilon \exp(-\alpha x_j/\varepsilon) \qquad \text{for } 0 \le j \le N.$$

From Remark 3.5, we obtain immediately

$$\begin{aligned} |\tau^q| &\le |q^- LG(x_{j-1})| + |q^c LG(x_j)| + |q^+ LG(x_{j+1})| \\ &\le C\varepsilon \exp(-\alpha x_{j-1}/\varepsilon) \qquad \text{for } 1 \le j \le N-1. \end{aligned}$$

To estimate $|\tau^r|$, we write $G(x) = W(x)E(x)$ and $S(k,m) = \exp(-\frac{1}{\varepsilon}\int_{x_k}^{x_m} \alpha(s)ds)$ and $W_j = W(x_j)$. Then

$$\begin{aligned} \tau^r &= \varepsilon h^{-2}\{r^- W_{j-1} + (r_1 + r_2)W_j S(j-1,j) + r^+ W_{j+1} S(j-1,j+1)\} E(x_{j-1}) \\ &= \tau_0^r + \tau_1^r + M \end{aligned}$$

where

$$\begin{aligned}\tau_0^r = \quad & \varepsilon h^{-2}\{r_*^-(\rho^-)W_{j-1} - [r_*^-(\rho^-) + r_*^+(\rho^+)]W_j S(j-1,j) \\ & + r_*^+(\rho^+)W_{j+1}S(j-1,j+1)\}E(x_{j-1}), \\ \tau_1^r = \quad & \varepsilon h^{-2}\{k_1 r_*^-(\rho^-)W_{j-1} - (k_1 - k_2)W_j S(j-1,j) \\ & - k_2 r_*^+(\rho^+)W_{j+1}S(j-1,j+1)\}E(x_{j-1}).\end{aligned}$$

It can be showed that

$$\begin{aligned}\tau_0^r &= -\varepsilon h^{-2} kW\rho\exp(-\rho)E(x_{j-1}) + M, \\ \tau_1^r &= \varepsilon h^{-2} kW\rho\exp(-\rho)E(x_{j-1}) + M.\end{aligned}$$

So

$$|\tau^r| \le Ch\exp(-\alpha x_{j-1}/\varepsilon) \qquad \text{for } 1 \le j \le N-1.$$

In the case $h \le \varepsilon$, we also first estimate $|\tau^q|$. From $G(x) = W(x)E(x)$ we have $LG(x) = H(x)E(x)$, where

$$H(x) = (-\alpha'(x) + \frac{\alpha^2(x)}{\varepsilon} - \frac{a(x)\alpha(x)}{\varepsilon} - b(x))W(x) + (-2\alpha(x) + a(x))W'(X) + \varepsilon W''(x).$$

Clearly

$$|LG(x)| \le C\varepsilon, \quad |H^{(i)}(x)| \le C\varepsilon, \qquad i \ge 0.$$

We find

$$\tau^q = \tau_0^q + \tau_1^q + \tau_2^q + M$$

where

$$\begin{aligned}\tau_0^q &= -\Big\{\frac{\rho^-}{n_1}q_*^-(\rho^-)[H_{j-1}S(j,j-1) + H_j] + \frac{\rho^+}{n_2}q_*^+(\rho^+)[H_j + H_{j+1}S(j,j+1)]\Big\}E(x_j), \\ \tau_1^q &= \Big\{\frac{k_1}{4n_1}r_*^-(\rho^-)[H_{j-1}S(j,j-1) + H_j] + \frac{k_2}{4n_2}r_*^+(\rho^+)[H_j + H_{j+1}S(j,j+1)]\Big\}E(x_j), \\ \tau_2^q &= \Big\{\frac{k_1^2}{12n_1}r_*^-(\rho^-)[H_{j-1}S(j,j-1) + H_j] - \frac{k_2^2}{12n_2}r_*^+(\rho^+)[H_j + H_{j+1}S(j,j+1)]\Big\}E(x_j).\end{aligned}$$

It can be proven that

$$\begin{aligned}\tau_0^q &= -\frac{\rho}{n}H(x_j)E(x_j) + M, \\ \tau_1^q &= \frac{k}{n}H(x_j)E(x_j) + M, \qquad \tau_2^q = M.\end{aligned}$$

Hence

$$\tau^q = H(x_j)E(x_j) + M.$$

Finally, we estimate $|\tau^r|$. We see that

$$\tau^r = \tau_0^r + \tau_1^r + \tau_2^r + \tau_3^r + M$$

where

$$\tau_0^r = \varepsilon h^{-2}[r_*^-(\rho^-)W_{j-1}S(j,j-1) - (r_*^-(\rho^-) + r^+(\rho^+))W_j + r_*^+(\rho_*^+)W_{j+1}S(j,j+1)]E(x_j),$$

$$\tau_1^r = \varepsilon h^{-2}[k_1 r_*^-(\rho^-)W_{j-1}S(j,j-1) - (k_1 - k_2)W_j - r_*^+(\rho^+)k_2 W_{j+1}S(j,j+1)]E(x_j),$$

$$\tau_2^r = \varepsilon h^{-2}[\frac{1}{2}\cdot k_1^2 r_*^-(\rho^-)W_{j-1}S(j,j-1) + \frac{1}{2}k_2^2 r_*^+(\rho^+)W_{j+1}S(j,j-1)]E(x_j),$$

$$\tau_3^r = \varepsilon h^{-2}[\frac{1}{6}k_1^3 r_*^-(\rho^-)W_{j-1}S(j,j-1) - \frac{1}{6}k_2^3 r_*^+(\rho^+)W_{j+1}S(j,j+1)]E(x_j).$$

We can prove that

$$\tau_0^r = \varepsilon h^{-2}[(-\frac{\alpha'}{\alpha}hW_j - hW_j')\rho + h^2 W_j'']E(x_j) + M,$$

$$\tau_1^r = \varepsilon h^{-2}(kW_j\rho - 2hkW_j') + M,$$

$$\tau_2^r = \varepsilon h^{-2}k^2 W_j E(x_j) + M,$$

$$\tau_3^r = M.$$

So

$$\tau^r = H(x_j)E(x_j) + M$$

and we have

$$|\tau_j(G)| \le Ch^2\varepsilon^{-1}\exp(-\alpha x_j/\varepsilon) \qquad \text{for } 1 \le j \le N-1.$$

The proof is then completed by combining the above results.

Putting together Lemmas 2.2, 4.3 and 4.4, we obtain the bound of the truncation error for scheme (3.3). For $1 \le j \le N-1$,

$$|\tau_j(U)| \le Ch^2 + C\varepsilon^{-2}h^2\exp(-\alpha x_j/\varepsilon) \qquad \text{when } h \le \varepsilon,$$

$$|\tau_j(U)| \le Ch^2 + C\varepsilon h^{-1}\exp(-\alpha x_{j-1}/\varepsilon) \qquad \text{when } h \ge \varepsilon.$$

From Theorem 2.1, Lemma 4.2, Corollaries 4.3, 4.4 and Lemma 4.5, we conclude

$$|\tau_j(U)| \le Ch^2\Big\{1 + \frac{1}{\max(h,\varepsilon)}\exp(-\alpha x_{j-1}/\varepsilon)] \qquad \text{for } 1 \le j \le N-1.$$

§5. Principal Theorems

We first establish the Maximum Pronciple and the Comparison Theorem.

Lemma 5.1. *Suppose $\{V_j\}$ is a set of Values at the grid points x_j satisfying $R^hV_j \ge 0$ for $1 \le j \le N-1, V_0 \le 0, V_N \le 0$. Then $V_j \le 0$ for $0 \le j \le N$.*

Proof. From Remark 3.6, $r^- > 0, r^+ > 0, -r^c = -r_1 - r_2 \ge r^- + r^+$, the matrix of coefficients is an M matrix and so the result follows [10].

From Lemma 5.1, we immediately see

Lemma 5.2. *Suppose $\{v_j\}$ and $\{V_j\}$ are two sets of values at the grid points x_j satisfying $|R^hv_j| \le R^hV_j$ for $1 \le j \le N-1, |v_0| \le -V_0, |v_N| \le -V_n$. Then $|v_j| \le V_j$ for $0 \le j \le N$.*

For the rest of this paper, it will be convenient to take h as bounded above by some "small" constant (independent of ε). This is permissible by uniform boundedness of the solutions $U(x)$ and $U_h(x)$.

Lemma 5.3. *Let $\varphi_j = -2 + x_j$. Then , when $h \le C_1$, $R^h\varphi_j \ge C$ for $1 \le j \le N-1$.*

Proof. Clearly, $R^h\varphi_j - Q^h(L\varphi(x_j)) = O(h^2)$ and $Q^h(L\varphi(x_j)) \ge q^- a_{j-1} + q^c a_j + q^+ a_{j+1} \ge \alpha q^+$. From Remark 3.1, when $h \le C_1$, we have

$$q^+ \ge \frac{1}{4}\frac{a^+}{a^+ + \alpha^+} \ge C > 0.$$

So

$$R^h\varphi_j \ge C.$$

This completes the proof.

Lemma 5.4. *Let $\psi_j = -\exp(-\beta x_j/\varepsilon)$. Then, when $h \le C_1, 0 < \beta < \alpha$, we have*

$$R^h\psi_j \ge C\frac{1}{\max(h,\varepsilon)}\exp(-\beta x_j/\varepsilon) \qquad \textit{for } 1 \le j \le N-1.$$

Proof. Set $\mu = \exp(-\beta h/\varepsilon)$. Using $r^c \le -(r^- + r^+)$ and $\psi_j \le 0$, we have

$$\begin{aligned} R^h\psi_j &\ge \varepsilon h^{-2}(r^-\psi_{j-1} - (r^- + r^+)\psi_j + r^+\psi_{j+1}) \\ &= \varepsilon h^{-2}\exp(-\beta x_{j-1}/\varepsilon) r^+(1-\mu)(\mu - \frac{r^-}{r^+}). \end{aligned}$$

The result is obtained by estimating the individual factors in the above expression for the three cases (a) $\rho \le \rho_0$, (b) $\rho \ge P_0$, and (c) $\rho_0 \le \rho \le P_0$ (for appropriately chosen ρ_0 and P_0).

For case (a) and for ρ_0 sufficiently small, $r^+(\rho^+) \ge C$, $1-\mu \ge Ch/\varepsilon$, $\mu - \frac{r^-(\rho^-)}{r^+(\rho^+)} \ge Ch/\varepsilon$ and then $R^h\psi_j \ge C\varepsilon^{-1}\exp(-\beta x_j/\varepsilon)$. For case (b) and for P_0 sufficiently large, $r^+(\rho^+) \ge Ch/\varepsilon$, $1-\mu \ge C$, $\mu - \frac{r^-(\rho^-)}{r^+(\rho^+)} \ge C\exp(-\beta h/\varepsilon)$, and so $R^h\psi_j \ge Ch^{-1}\exp(-\beta x_j/\varepsilon)$. For case (c) ($\rho_0$ and P_0 are now fixed) and for h sufficiently small, $r^+(p^+) \ge C$, $1-\mu \ge C$, $\mu - \frac{r^-(\rho^-)}{r^+(\rho^+)} \ge C$, hence $R^h\psi_j \ge C\frac{1}{\max(h,\varepsilon)}\exp(-\beta x_j/\varepsilon)$.

The proof is completed.

Corollary 5.4. Let $\chi_j = \psi_j \exp(\beta h/\varepsilon)$. Then

$$R^h\chi_j \ge C\frac{1}{\max(h,\varepsilon)}\exp(-\beta x_{j-1}/\varepsilon), \qquad \text{for } 1 \le j \le N-1.$$

From the estimates of the truncation error at the end of the last section and Lemmas 5.2, 5.3, 5.4 and Corollary 5.4 in this section, the main results can be deduced.

Theorem 5.5. *Let $\{U_j\}$ be the approximation to the solution $U(x)$ of (1.1) obtained using (3.3). Then there are positive constants β and C, independent of ε, h and x_j, such that for $0 \le j \le N$*

$$\begin{aligned} |U(x_j) - U_j| &\le Ch^2 + C\varepsilon^{-1}h^2\exp(-\beta x_j/\varepsilon), && \textit{when } h \le \varepsilon, \\ |U(x_j) - U_j| &\le Ch^2 + C\varepsilon\exp(-\beta x_{j-1}/\varepsilon), && \textit{when } h \ge \varepsilon. \end{aligned}$$

Theorem 5.6. *Let $\{U_j\}$ be the approximation to the solution $U(x)$ of (1.1) obtained using (3.3). Then there is a positive constant C, independent of ε, h and x_j, such that*

$$|U(x_j) - U_j| \le Ch^2 \qquad \text{for } 0 \le j \le N.$$

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