Journal of Computational Mathematics, Vol.8, No.2, 1990, 118–127.

UPPER BOUNDS OF THE SPECTRAL RADII OF SOME ITERATIVE MATRICES*

Hu Jia-gan
(Institute of Applied Physics and Computational Mathematics, Beijing, China)

Abstract

In this paper, the concept of optimally scaled matrix and the estimate of $\|M^{-1}N\|_\infty$ in our previous paper are used to find the upper bounds of the spectral radii of the iterative matrices SOR, SSOR, AOR and SAOR. The sharpness of the upper bounds of the spectral radii of SOR and AOR is established. The proofs are very intuitive and may be considered as the geometrical interpretations of our theorems.

§1. Introduction

It is well-known that if the coefficient matrix A of a system of linear algebraic equations

$$Ax = f \tag{1}$$

is a nonsingular H-matrix and

$$0 \le \omega \le 2/[1 + S(|J|)], \tag{2}$$

then the spectral radius $S(L_\omega^A)$ of the SOR iterative matrix

$$L_\omega^A = (D - \omega L)^{-1}[(1-\omega)D + \omega U] \tag{3}$$

satisfies (see [1], for example)

$$S(L_\omega^A) \le |1-\omega| + \omega S(|J|) =: \delta \tag{4}$$

where $D = \mathrm{diag}(A), B = D - A, L$ and U are lower and upper triangular matrices of B respectively and

$$J = D^{-1}B \tag{5}$$

is the Jacobian iterative matrix of A.

In [2], it is proved that the upper bound in (4) is sharp, that is, given $v \in [0,1)$ and $\omega \in [0, 2/(1+v)]$, the equality

$$\sup_{A \in H_v} \{S(L_\omega^A)\} = |1-\omega| + \omega v$$

holds, where H_v is the set of all nonsingular H-matrices with

$$v = S(|J|), \tag{6}$$

which is obviously less than one, and L_ω^A is the SOR iterative matrix of A.

* Received July 1, 1987.

For the symmetric SOR (SSOR) iterative matrix

$$S_\omega^A = U_\omega^A L_\omega^A, \tag{7}$$

where

$$U_\omega^A = (D - \omega U)^{-1}[(1-\omega)D + \omega L], \tag{8}$$

we have from [3]

$$S(S_\omega^A) \le |1-\omega| + \omega S(|J|). \tag{9}$$

In this paper, we use the theorems abour $\|M^{-1}N\|_\infty$ and the optimally scaled matrix in [4] to derive the upper bound of L_ω^A and S_ω^A and then generalize our results to the AOR and SAOR matrices

$$L_{\tau,\omega}^A = (D - \tau L)^{-1}[(1-\omega)D + (\omega - \tau)L + \omega U], \tag{10}$$

and

$$S_{\tau,\omega}^A = U_{\tau,\omega}^A L_{\tau,\omega}^A, \tag{11}$$

where

$$U_{\tau,\omega}^A = (D - \tau U)^{-1}[(1-\omega)D + (\omega - \tau)U + \omega L]. \tag{12}$$

We obtain

$$S(S_\omega^A) \le (|1-\omega| + \omega v)^2, \tag{13}$$

$$S(L_{\tau,\omega}^A) \le |1-\omega| + \omega v \tag{14}$$

and

$$S(S_{\tau,\omega}^A) \le (|1-\omega| + \omega v)^2. \tag{15}$$

The upper bound in (13) is obviously better than in (9).

Further, we prove also the sharpness of the upper bounds of $S(L_\omega^A)$ and $S(L_{\tau,\omega}^A)$. Our method is very intuitive and may be considered as a geometrical interpretation of the upper bounds and their sharpness. Moreover, the matrices used here are more general than those in [3].

§2. The Upper Bounds of the Spectral Radii of $L_\omega^A, S_\omega^A, L_{\tau,\omega}^A$ and $S_{\tau,\omega}^A$

First, for completeness, we present the theorems in [4], which will be used here, as our lemmas:

Lemma 1. *If $M = (m_{ij})$ and $N = (n_{ij})$ are $n \times n$ matrices and*

$$|m_{ii}| > \sum_{j \ne i} |m_{ij}|, \quad i = 1, 2, \cdots, n, \tag{16}$$

then

$$\|M^{-1}N\|_\infty \le \max_i \Big[\sum_j |n_{ij}| / (|m_{ii}| - \sum_{j \ne i} |m_{ij}|\Big]. \tag{17}$$

Lemma 2. *If A is an irreducible matrix, $D = \mathrm{diag}(A)$ is nonsingular and $B = D - A$, then there is a positive diagonal matrix $Q = \mathrm{diag}(q_1, q_2, \cdots, q_n)$ such that the matrix*

$$\tilde{A} = (\tilde{a}_{ij}) = AQ \tag{18}$$

satisfies

$$\sum_{j\neq i} |\tilde{a}_{ij}|/|\tilde{a}_{ii}| = S(|J|), \quad i = 1,2,\cdots,n. \tag{19}$$

We call this matrix $\tilde{A}$ the optimally scaled matrix of A.

Now, we prove our theorems.

Theorem 1. *If A is a nonsingular H-matrix and ω lies in the interval (2), then we have the estimates (4), (13), (14) and (15).*

Proof. First we assume that $A = (a_{ij})$ is irreducible and

$$0 < \omega < 2/[1 + S(|J|)]. \tag{20}$$

From Lemma 2, there is an optimally scaled matrix $\tilde{A} = (a_{ij})$ satisfying (18) and (19). But A is a nonsingular H-matrix; thus inequality (16) holds and $\tilde{A}$ is a daigonally dominant matrix, i.e.

$$|\tilde{a}_{ii}| > \sum_{j\neq i} |\tilde{a}_{ij}|, \quad i = 1,2,\cdots,n. \tag{21}$$

Now, from (20) we have

$$\begin{aligned}
|\tilde{a}_{ii}| - \omega\sum_{j<i}|\tilde{a}_{ij}| &\geq |\tilde{a}_{ii}| - \left\{2/[1+S(|J|)]\right\}\sum_{j<i}|\tilde{a}_{ij}| \\
&= \left\{|\tilde{a}_{ii}| + \sum_{j\neq i}|\tilde{a}_{ij}| - 2\sum_{j<i}|\tilde{a}_{ij}|\right\}\Big/\left\{1+S(|J|)\right\} \\
&= \left\{|\tilde{a}_{ii}| + \sum_{j>i}|\tilde{a}_{ij}| - \sum_{j<i}|\tilde{a}_{ij}|\right\}\Big/\left\{1+S(|J|)\right\} \\
&> 2\sum_{j>i}|\tilde{a}_{ij}|/[1+S(|J|)] \geq 0.
\end{aligned}$$

Hence, we can apply lemma 1 to the SOR iterative matrix $L_\omega^{\tilde{A}}$ of the optimally scaled matrix A and obtain

$$\begin{aligned}
S(L_\omega^A) &= S(L_\omega^{\tilde{A}}) \leq \|L_\omega^{\tilde{A}}\|_\infty \\
&\leq \max_i\left\{(|1-\omega|\,|\tilde{a}_{ii}| + \omega\sum_{j>i}|\tilde{a}_{ij}|)/(|\tilde{a}_{ii}| - \omega\sum_{j<i}|\tilde{a}_{ij}|)\right\} \\
&= \max_i\left\{[|1-\omega| + \omega\sum_{j>i}|\tilde{a}_{ij}|/|\tilde{a}_{ii}|]/[1-\omega\sum_{j<i}|\tilde{a}_{ij}|/|\tilde{a}_{ii}|]\right\} \\
&= \max_i\left\{[|1-\omega| + \omega S(|J|) - \omega\sum_{j<i}|\tilde{a}_{ij}|/|\tilde{a}_{ii}|]/[1-\omega\sum_{j<i}|\tilde{a}_{ij}|/|\tilde{a}_{ii}|]\right\} \\
&< \max_i\left\{|1-\omega| + \omega S(|J|)\right\} = |1-\omega| + \omega S(|J|).
\end{aligned}$$

The first equality holds, since L_ω^A is similar to $L_\omega^{\tilde{A}}$ and the last inequality holds, if and only if

$$|1-\omega| + \omega S(|J|) < 1,$$

which is obviously true under the condition of (20). Thus, we have proved the estimate (4) when A is an irreducible H-matrix. Similarly, for the iterative matrix U_ω^A of (8) we have

$$S(U_\omega^A) = S(U_\omega^{\tilde{A}}) \le \|U_\omega^{\tilde{A}}\|_\infty \le |1-\omega| + \omega S(|J|),$$

where $U_\omega^{\tilde{A}}$ is the iterative matrix (8) corresponding to $\tilde{A}$. Now, for the SSOR matrix S_ω^A of (7) we have at once

$$S(S_\omega^A) = S(S_\omega^{\tilde{A}}) \le \|S_\omega^{\tilde{A}}\|_\infty \le \|L_\infty^{\tilde{A}}\| \, \|U_\omega^{\tilde{A}}\|_\infty \le \{|1-\omega| + \omega S(|J|)\}^2,$$

which proves the estimate (13).

For the iterative matrix of AOR in (10), we proceed similarly and obtain

$$\begin{aligned} S(L_{\tau,\omega}^A) &= S(L_{\tau,\omega}^{\tilde{A}}) \le \|L_{\tau,\omega}^{\tilde{A}}\|_\infty \\ &\le \max_i \Big\{ \big[|1-\omega||\tilde{a}_{ii}| + (\omega-\tau)\sum_{j<i}|\tilde{a}_{ij}| + \omega\sum_{j>i}|\tilde{a}_{ij}|\big] / \big[|\tilde{a}_{ii}| - \tau\sum_{j<i}|\tilde{a}_{ij}|\big] \Big\} \\ &= \max_i \Big\{ \big[|1-\omega| + \omega S(|J|) - \tau\sum_{j<i}|\tilde{a}_{ij}|/|\tilde{a}_{ii}|\big] / \big[1 - \tau\sum_{j<i}|\tilde{a}_{ij}|/|\tilde{a}_{ii}|\big] \Big\} \\ &\le |1-\omega| + \omega S(|J|). \end{aligned}$$

For the SAOR matrix $S_{\tau,\omega}^A$ of (11) we obtain

$$S(S_{\tau,\omega}^A) = S(S_{\tau,\omega}^{\tilde{A}}) \le \|S_{\tau,\omega}^{\tilde{A}}\|_\infty \le \|L_{\tau,\omega}^{\tilde{A}}\|_\infty \|U_{\tau,\omega}^{\tilde{A}}\|_\infty \le \{|1-\omega| + \omega S(|J|)\}^2,$$

where $S_{\tau,\omega}^{\tilde{A}}$, $L_{\tau,\omega}^{\tilde{A}}$ and $U_{\tau,\omega}^{\tilde{A}}$ are the matrices (11), (10) and (12) corresponding to the matrix $\tilde{A}$ respectively.

Up to now, we have proved the estimates (4), (13), (14) and (15) under the condition that A is an irreducible H-matrix. Now, we assume that A is a reducible H-matrix. As usual, let $D = \operatorname{diag}(A)$ and $B = D - A$. We construct an irreducible matrix A_ε by replacing some zero elements of B by a small positive number ε. Let $D_\varepsilon = \operatorname{diag}(A_\varepsilon)$ and $B_\varepsilon = D_\varepsilon - A_\varepsilon$. Obviously,

$$S(|D_\varepsilon|^{-1}|B_\varepsilon|) \to S(|D|^{-1}|B|), \quad \text{as } \varepsilon \to 0.$$

Since A is a nonsingular H-matrix, we have

$$S(|D|^{-1}|B|) < 1.$$

Thus we can find a small positive number ε_0 such that

$$S(|D_\varepsilon|^{-1}|B_\varepsilon|) < 1, \quad \forall\, \varepsilon \in (0, \varepsilon_0),$$

which means that A_ε is an irreducible nonsingular H-matrix for arbitrary $\varepsilon \in (0, \varepsilon_0)$. Hence, for the AOR iterative matrix $L_{\tau,\omega}$ as an example, we have

$$S(L_{\tau,\omega}^{A_\varepsilon}) \le |1-\omega| + \omega S(|J^{A_\varepsilon}|) \tag{22}$$

whenever

$$0 < \omega < 2/[1 + S(|J^{A_\varepsilon}|) =: \mu_\varepsilon, \tag{23}$$

where $L^{A_\varepsilon}_{r,\omega}$ and J^{A_ε} are the AOR and Jacobian iterative matrices corresponding to the matrix A_ε respectively. Now, for arbitrary ω in the interval

$$0 < \omega < 2/[1 + S(|J|)] =: \mu,$$

we can find $\varepsilon_1 \in (0, \varepsilon_0)$ such that

$$0 < \omega < 2/[1 + S(|J^{A_\varepsilon}|)], \quad \forall\, \varepsilon \in (0, \varepsilon_1)$$

sine

$$\mu_\varepsilon \longrightarrow \mu, \quad \text{as} \quad \varepsilon \to 0.$$

Thus for arbitrary $\varepsilon \in (o, \varepsilon_1)$, inequality (22) holds. Putting $\varepsilon \to 0$, we obtain (14) at once. Hence (14) is proved when A is a reducible nonsingular H-matrix. The estimates (4), (13) and (15) can be proved similarly.

Finally, from the continuity of the eigenvalues with respect to the elements of the matrix considered, it is obvious that our conclusion also holds if we replace inequality (20) by (2). This completes our proof.

Note that the proof of this theorem may be considered as another proof of the convergence of SOR, SSOR, AOR and SAOR iterative matrices.

§3. The Sharpness of the Upper Bounds of $S(L^A_\omega)$

In [2], in order to prove the sharpness of $A(L^A_\omega)$, the authors use the matrices:

$$A = I - B \quad \text{and} \quad B = v \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{bmatrix}, \quad 0 < v < 1. \tag{24}$$

In this paper, we use instead of (24), matrices of order $p = q + r$:

$$A = I - B \quad \text{and} \quad B = v \begin{bmatrix} O_{q\times r} & I_q \\ I_r & O_{r\times q} \end{bmatrix}, \quad 0 < v < 1, (q, r) = 1, \tag{25}$$

where $O_{q\times r}$ and $O_{r\times q}$ are the null-matrices of order $q * r$ and $r * q$ respectively. Obviously, (25) is more general than (24). Moreover, our proof is very intuitive and interesting, and it may be considered as a geometrical interpretation of the upper bounds and their sharpness. Besides, this method is also applied in this paper to prove the sharpness of the upper bound of $S(L_{r,\omega})$ (see §4).

Now, we consider the matrices (25), where r is fixed and p changes to infinity. There are two cases:

1. $0 < \omega \le 1$.

Since the matrix A in (25) is consistently ordered, it follows from [5] that

$$(\lambda + \omega - 1)^p = v^p \omega^p \lambda^r, \tag{26}$$

i.e.

$$(\lambda + \omega - 1)/\omega = V\lambda^{r/p}, \tag{27}$$

where λ is the eigenvalue of $L_\omega^A, p = q + r$ and $\lambda^{r/p}$ is one of the roots of $x^p = \lambda^r$. Let

$$y = (\lambda + \omega - 1)/\omega =: d(\lambda) \quad \text{and} \quad y = v\lambda^{r/p} =: c(\lambda). \tag{28}$$

From Fig. 1 it may be seen that the two curves in (28) have a point of intersection in the right half $\lambda - y$ plane. Thus, there is a positive λ_0 which satisfies (27) and hence (26). This λ_0 is therefore an eigenvalue of L_ω^A. Moreover, since the slope of the line $y = d(\lambda)$ is $1/\omega = PR/FR = 1/FR$, we have $FR = \omega$. Besides, from Fig. 1 we see that

$$QR = v, \quad PQ = 1 - v.$$

Hence,

$$SQ = \omega(1 - v), \quad TS = 1 - \omega(1 - v) = |1 - \omega| + \omega v, \quad ED = \lambda_0 < TS$$

and ED increases as p does. On the other hand, if λ_∞ is the limit of λ as p tends to infinity, from (27) we have

$$(\lambda_\infty + \omega - 1)/\omega = v, \quad \text{i.e.} \quad \lambda_\infty = |1 - \omega| + \omega v.$$

But

$$S(L_\omega^A) \leq |1 - \omega| + \omega v.$$

These prove the sharpness of $S(L_\omega^A)$ when $0 < \omega \leq 1$.

2. $1 < \omega < 2/\{1 + S(|J|)\}$.

Without loss of generality, we assume both p ang q are odd numbers. Obviously, this does not affect the proof of the sharpness. Putting

$$\eta = -\lambda$$

in (26), we obtain

$$(\eta + 1 - \omega)^p = v^p \omega^p \eta. \tag{29}$$

From Fig. 2 the two curves

$$y = d(\eta) := (\eta + 1 - \omega)/\omega \quad \text{and} \quad y = c(\eta) := v\eta^{r/p} \tag{30}$$

have a point of intersection (η_0, y_0) in the right half $\eta - y$ plane and $\lambda_0 = -\eta_0$ is an eigenvalue of L_ω^A. Just as in the previous case, it can easily be found that

$$EG = |1 - \omega| + \omega S(|J|)$$

and $|\lambda_0| = \eta_0 = OH$ increases to EG as p increases to infinity. This completes the proof of the sharpness of $S(L_\omega^A)$.

Notice that if we keep q fixed, then the second equation of (28) becomes

$$y = v\lambda^{(p-q)/p} = v\lambda/\lambda^{q/p} := c(\lambda),$$

which decreases to $v\lambda$ as p increases to infinity (see Fig. 1). Since the point of intersection of the two lines

$$y = (\lambda + \omega - 1)/\omega \quad \text{and} \quad y = v\lambda$$

is $((1-\omega)/(1-\omega v), \{(1-\omega)v/(1-\omega v))$, we can only obtain

$$\sup_{A\in H_\nu} \{S(L_\omega^A)\} \geq (1-\omega)/(1-\omega v).$$

On the other hand, (30) becomes

$$y = v\eta^{(p-q)/p} = v\eta/\eta^{q/p},$$

which decreases to $v\eta$ as p increases to infinity (see Fig. 2). The point of intersection of

$$y = (\eta + 1 - \omega)/\omega \quad \text{and} \quad y = v\eta$$

is $((\omega-1)/(1-\omega v), (\omega-1)v/(1-\omega v))$. Hence,

$$\sup_{A\in H_\nu} S(L_\omega^A) \geq (\omega-1)/(1-\omega v).$$

In both cases, we obtain

$$|\omega - 1|/(1-\omega v) \leq \sup S(L_\omega^A) \leq |1-\omega| + \omega v$$

(Notice that $\omega v < 2v/(1+v) < 1$). The left-hand side is zero when $\omega = 1$. This is really the case; in fact, for arbitrary $A \in H_\nu (0 < v < 1), S(L_1) = v^{p/q} \to 0$ as $p \to \infty$.

§4. The Sharpness of the Upper Bound of $S(L_{\tau,\omega}^A)$

Now let us consider $S(L_{\tau,\omega}^A)$. Since SOR is the special case of AOR with $\tau = \omega$, we have, of course,

$$\sup_{A\in H_\nu, 0<\tau\leq\omega} S(L_{\tau,\omega}^A) = |1-\omega| + \omega\nu.$$

But here by the sharpness of the upper bound of $S(L_{\tau,\omega}^A)$ we mean

$$\sup_{A\in H_\nu} S(L_{\tau,\omega}^A) = |1-\omega| + \omega v.$$

That is, for arbitrary $v \in (0,1)$ and arbitrary ω, τ satisfying the inequality

$$0 \leq \tau \leq \omega \leq 2/(1+v),$$

there is a matrix $A \in H_\nu$ such that $\delta - S(L_{\tau,\omega}^A)$ is as small as we please. Now, from (26) it may be proved the eigenvalues λ of $L_{\tau,\omega}^A$ satisfy

$$(\lambda + \omega - 1)^p = v^p \omega^q (\tau\lambda + \omega - \tau)^r \tag{31}$$

or

$$[(\lambda + \omega - 1)/\omega] = v(\tau/\omega)^{r/p}[\lambda + (\omega/\tau) - 1]^{r/p}. \tag{32}$$

First, we consider the case $0 < \omega \leq 1$. let

$$y = (\lambda + \omega - 1)/\omega =: d(\lambda) \tag{33}$$

and

$$y = v(\tau/\omega)^{r/p}[\lambda + (\omega/\tau) - 1]^{r/p} =: c(\lambda). \tag{34}$$

Here for convenience, we still use $c(\lambda)$ to denote the function in (34). Note that

$$d(1) = 1 \quad \text{and} \quad c(1) = v < 1.$$

From Fig. 3 we see that the situation is almost the same as that of §3 and the sharpness of the upper bound of $S(L^A_{\tau,\omega})$ may be proved in the same manner as that of $S(L^A_\omega)$.

Now we consider the case $1 < \omega < 2/(1 + S(|J|))$. There are two cases again:

1. $1/\tau < 1 + v$.

Assuming p and r are odd numbers and putting $\eta = -\lambda$, we have from (30) the equation

$$(\eta - \omega + 1)^p = v^p \omega^q (\tau\eta - \omega + \tau)^r. \tag{35}$$

The two curves

$$y = (\eta - \omega + 1)/\omega \quad \text{and} \quad y = v(\tau/\omega)^{r/p}[\eta - (\omega/\tau) + 1]^{r/p} \tag{36}$$

intersect at (η_0, y_0) in the right-half (η, y) plane if p is sufficiently large, since $1/\tau < 1 + v$ implies

$$(\omega/\tau) - 1 < \omega - 1 + \omega v$$

and $(\omega/\tau) - 1$ is the vertex of the second curve of (36) (see Fig. 4). Proceeding as in the second part of §3, we have the sharpness of the upper bound of $S(L^A_{\tau,\omega})$ at once.

2. $1/\tau \geq 1 + v$.

In this case, from Fig. 6 we see that there is no point of intersection in the neighbourhood of G, even if p is sufficiently large. Thus $L^A_{\tau,\omega}$ has no real eigenvalue which tends to δ in absolute value as p increases to infinity. But there is another point of intersection. If $(\omega - 1)/\omega \geq v$, from Fig. 5 it may be seen that the two curves have a point of intersection (η_0, y_0) with $\eta_0 > 0$. But η_0 deceases to $\omega - 1 - \omega v$ as p increases to infinity, so we can only obtain the result

$$\omega - 1 - \omega v \leq \sup_{A \in H_v} \{S(L^A_{\tau,\omega})\} \leq \omega - 1 + \omega v.$$

If $(\omega - 1)/\omega < v$, from Fig. 6 we see that the two curves intersect at (η_0, y_0) with $\eta_0 < 0$ and $|\eta_0|$ increases to $\omega v - (\omega - 1)$, as p increases to infinity. Thus we have

$$\omega v - (\omega - 1) \leq \sup_{A \in H_v} \{S(L^A_{\tau,\omega})\} \leq \omega v + (\omega - 1). \tag{37}$$

Note that we can also use the curves $y = d(\lambda)$ and $y = c(\lambda)$ in the $\lambda - y$ plane to obtain the same result (37) (see Fig. 7).

Finally, we notice that due to the continuity of the eigenvalues of a matrix with respect to its elements, the sharpness of the upper bounds of $S(L^A_\omega)$ and $S(L^A_{\tau,\omega})$ still hlods if we replace the condition (20) by the condition (2). Hence, we have

Theorem 2. *If A is a nonsingular H-matrix, then for arbitrary $v \in [0, 1)$ and arbitrary $\omega \in [0, 2/(1 + v)]$, the following equality holds:*

$$\sup_{A \in H_v} \{S(L^A_\omega)\} = |1 - \omega| + \omega v.$$

Theorem 3. *Let A be a nonsingular H-matrix and $v \in [0, 1)$; then*

$$\sup_{A \in H_v} \{S(L^A_{\tau,\omega})\} \begin{cases} = |1 - \omega| + \omega v, & \text{when } 0 < \omega \leq 1, \\ = |1 - \omega| + \omega v, & \text{when } 1 < \omega \leq 2/(1 + v), \quad 1/\tau \leq 1 + v, \\ \geq |(\omega - 1) - \omega v|, & \text{when } 1 < \omega \leq 2/(1 + v), \quad 1/\tau > 1 + v. \end{cases}$$