EXTRAPOLATION FOR THE APPROXIMATIONS TO THE SOLUTION OF A BOUNDARY INTEGRAL EQUATION ON POLYGONAL DOMAINS

LIN QUN XIE RUI-FENG
(*Institute of Systems Science, Academia Sinica, Beijing, China*)

Abstract

In this paper, we consider a boundary integral equation of second kind rising from potential theory. The equation may be solved numerically by Galerkin's method using piecewise constant functions. Because of the singularities produced by the corners, we have to grade the mesh near the corner. In general, Chandler obtained the order 2 superconvergence of the iterated Galerkin solution in the uniform norm. It is proved in this paper that the Richardson extrapolation increases the accuracy from order 2 to order 4.

1. Introduction

Let $\Gamma \subset R^2$ be a simple closed polygon with corner points $x_0, x_1, x_2 \dots x_m = x_0$ For each $i, X_i \in (-1,1)$ is defined by requiring $(1-X_i)\pi$ to be the angle $x_{i-1}\widehat{x_i}x_{i+1}$. Let us consider the boundary integral equation of second kind

$$u_0(x) + Vu_0(x) = f(x), \quad x \in \Gamma \tag{1.1}$$

where

$$Vu_0(x) = \int_\Gamma k(x,y)u_0(y) \ dy$$

with

$$k(x,y) = \frac{1}{\pi}\frac{\partial}{\partial n_y} \ln |x-y|$$

Assume that Γ is parametrized by arc length s , with $s = s_i$ corresponding to the point x_i. We do not distinguish s from the point length s around Γ, and use the expressions such as $u(s), k(s,\sigma)$, etc. Let $s_{i+1/2} = (s_i + s_{i+1})/2, = s_{i-1/2} = (s_i + s_{i-1})/2, \Gamma_{2i} = [s_{i-1/2}, s_i]$ and $\Gamma_{2i+1} = [s_i, s_{i+1/2}]$. Each function u on Γ may be identified with the vector $(u_2, \dots, u_{2m+1}), u_k := u|_{\Gamma_k}$. Then (1.1) is equivalent to the $2m \times 2n$ system of equations

$$(I+T)u_0 = f \tag{1.2}$$

with $f = (f_2, \dots, f_{2m+1}), u_0 = (u_{0,2}, u_{0,3}, \dots\dots, u_{0,2m+1})$ and where the matrix operator T is defined by

$$T_{k,l}u_l(s) = \int_{\Gamma_l} k(s,\sigma)u_l(\sigma)\,d\sigma, \quad s \in \Gamma_k,$$

$$(Tu)_k = \sum_{l=2}^{2m+1} T_{k,l}u_l.$$

When

$$\begin{aligned}&\{k,l\} = \{2i, 2i+1\}, s \in \Gamma_k, \sigma \in \Gamma_l\\ &k(s,\sigma) = (\sin X_i\pi/\pi)\cdot(s-s_i)/[(s-s_i)^2 + (s_i-\sigma)^2\\ &+2(s-s_i)(s_i-\sigma)\cos X_i\pi].\end{aligned} \tag{1.3}$$

When

$$\begin{aligned}&k = l\ ,\ s \in \Gamma_k\ ,\ \sigma \in \Gamma_k,\\ &k(s,\sigma) = 0.\end{aligned}$$

T may be separated by writing $T = R + K$, where

$$\begin{aligned}R_{k,l} &= T_{k,l}, \qquad \{k,l\} = \{2i, 2i+1\} \quad \text{for some } i\\ &= 0, \qquad \text{otherwise}\end{aligned}$$

and the kernels of the components of K are smooth. For each i let R_i denote the 2×2 system of operators :

$$R_i = \begin{bmatrix} 0 & R_{2i,2i+1} \\ R_{2i+1,2i} & 0 \end{bmatrix}.$$

then $R = \text{diag}[R_1, R_2, \dots, R_m]$.

For any $\alpha_i > 0$ and integer $k \geq 0$, define the norm

$$\|u\|_{k,\alpha_i} = \max_{m \leq k} \sup\left\{\left|[s-s_i]^{m-\alpha_i}D^m u(s)\right| :\ s\ \in \Gamma_{2i} \cup \Gamma_{2i+1}\backslash\{s_i\}\right\}$$

with $[s]^\beta = |s|^{\max(\beta,0)}$, and the space

$$C^k_{\alpha_i}(\Gamma_{2i} \cup \Gamma_{2i+1}) = \left\{u | u \in C^k(\Gamma_{2i} \cup \Gamma_{2i+1}\backslash\{s_i\}), \|u\|_{k,\alpha_i} < \infty\right\}.$$

For any vector $\alpha = (\alpha_1, \alpha_2, \dots\dots, \alpha_m), \alpha_i > 0$, define the Banach space

$$C^k_\alpha(\Gamma) = \left\{u | u \in C^k(s_{i-1}, s_i), \forall_i = 1, 2, \dots m, \|u\|_{k.\alpha} < \infty\right\}$$

with the norm

$$\|u\|_{k,\alpha} = \max_i\left\{\|u|_{\Gamma_{2i}\cup\Gamma_{2i+1}}\|_{k,\alpha_i}\right\}.$$

The notation $\|\cdot\|_{p,\Omega}$ denotes the norm of the $L^p(\Omega)$ space. If L is a linear operator, $\|L\|_p$ denotes the norm of the appropriate L^p space. Then

$$\|R_i\|_\infty \le |X_i| < 1;$$
$$\|(I+R)^{-1}\|_\infty < \infty.$$

so $I+T$ is a Fredholm operator and

$$\left\|(I+T)^{-1}\right\|_\infty < \infty$$

Define $\bar{\alpha}$ by $\bar{\alpha}_i = 1/(1+|X_i|), i = 1,2,\dots,m$. It is known that for all $\alpha < \bar{\alpha}, f \in C^k_\alpha(\Gamma)$ implies $u_0 \in C^k_\alpha(\Gamma)$ and consequently

$$\left\|(I+T)^{-1}\right\|_{C^1_\alpha \to C^1_\alpha} < +\infty$$

We assume without further mention that $f \in C^4_\alpha(\Gamma)$ for all $\alpha < \bar{\alpha}$.

To solve (1.2) numerically, define the graded mesh $\{\sigma_{k,j}\}$ by

$$\sigma_{2i,j} = s_i - \left(\frac{n-j}{n}\right)^{q_i}(s_i - s_{i-1/2}), \qquad 0 \le j \le n;$$

$$\sigma_{2i+1,j} = s_i + \left(\frac{j}{n}\right)^{q_i}(s_{i+1/2} - s_i), \qquad 0 \le j \le n.$$

Set $e_{k,j} = [\sigma_{k,j}, \sigma_{k,j+1}]$ for each pair of k,j. Assign E to be the set of all elements $e_{k,j}$. Let $h = 1/n$ and S_h be the space of all piecewise constant functions with the break-points at $\{\sigma_{k,j}\}$. P_h denotes the orthogonal projection $L^2(\Gamma)$ onto S_h. Then for any $e \in E$

$$P_h u|_e = \frac{1}{\text{mes } e}\int_e u ds.$$

It is easy to check $\|P_h\|_\infty = 1$. So $(I+P_hT)^{-1}$ and $(I+TP_h)^{-1}$ is bounded because

$$\max\left\{\|RP_h\|_\infty, \|P_hR\|_\infty\right\} \le |X_i| < 1.$$

Assume u_h is the Galerkin solution to (1.2):

$$(I+P_hT)u_h = P_hf.$$

The iterated solution is defined by

$$u_h^* = f - Tu_h$$

Then the Chandler theorem [1] leads to

$$\left\|u_h^* - u_0\right\|_\infty \le ch^2.$$

Now, we divide each element $e \in E$ into two segments with the same length. Assume that $u_{h/2}$ is the associated Galerkin solution and $u^*_{h/2}$ is the iterated solution. We shall prove

$$\left\|(4u^*_{h/2} - u^*_h)/3 - u_o\right\|_\infty \leq ch^4.$$

2. Uniform estimate for extrapolation

To prove our main result, we need some properties for the projection operator P_h. In the following deduction, the generic constant c will always be independent of h. First it is easy to show

Lemma 1. Suppose $1 \leq p \leq \infty$ and $0 < \alpha < 1$.

(i) For any $e \in E$, if $u \in L^p(e)$ and $Du \in L^p(e)$, then

$$\|u - P_h u\|_{p;e} \leq ch_e \|Du\|_{\Gamma.e}; \tag{2.1}$$

(ii) If $e \in \{e_{2i,n}, e_{2i+1,0}\}$ for some i, and $u \in C^1_{\alpha_i}(\Gamma_{2i} \cup \Gamma_{2i+1})$, then

$$\|u - P_h u\|_\infty \leq ch_e^\alpha \|u\|_{C^1_\alpha}, \tag{2.2}$$

where $h_e = \text{diam} \quad e$.

An application of the argument used in [2,Section 2] gives

Lemma 2. Let $e \in E$. Assume $u \in C^4(e)$ and $g \in C^3(e)$. Then

$$\int_e (u - P_h u) g d\sigma = \frac{1}{12} h_e^2 \int_e DuDg d\sigma + R \tag{2.3}$$

where $h_e = \text{diam} \quad e$ and

$$|R| \leq ch_e^4 \sum_{i=0}^{3} \|D^{4-i} u\|_\infty D^i g\|_1. \tag{2.4}$$

With above lemmas, we come to prove the following results :

Lemma 3. Suppose $0 < \alpha < \bar{\alpha}, \beta_i = \alpha_i - 2/q_i$ for each i and $u \in C^1_\alpha(\Gamma)$. If $q_i > 2/\alpha_i$, then

$$\text{(i)} \quad \|T(I - P_h)u\|_\infty \leq ch^2 \|u\|_{c^1_\alpha}, \tag{2.5}$$

$$\text{(ii)} \quad \|T(I - P_h)u\|_{C^1_\beta} \leq ch^2 \|u\|_{C^1_\alpha} \tag{2.6}$$

Proof. To prove the lemma, it suffices to consider R_i instead of T. For $s \in \Gamma_{2i}$,

$$R_i(u - P_h u)(s) = \int_{\Gamma_{2i+1}} k(s,\sigma)(u - P_h u)(\sigma) d\sigma.$$

Denote $e_{2i+1,j}$ by e_j. By a change of variables, suppose $\Gamma_{2i+1} = [0,a]$ and $\Gamma_{2i} = [0,b]$ for some $a, b > 0$, and define $R_s(\sigma)$ by

$$R_s(\sigma) = \frac{\sin X\pi}{\pi} \frac{s}{s^2 + \sigma^2 + 2s\sigma \cos X\pi} \quad \forall \quad \sigma \in \Gamma_{2i+1}, s \in \Gamma_{2i}.$$

Then

$$R_i(u - P_h u)(s) = \int_0^a R_s(\sigma)(u - P_h u)(\sigma) d\sigma$$

and

$$\left| D_s^k D_\sigma^l R_s(\sigma) \right| \leq c(s^2 + \sigma^2)^{-(k+l+1)/2}$$

for any integer $k, 1 \geq 0$, with the constant c independent of s or σ.

Let $\sigma_j = a(\frac{j}{n})^q$ and $h_j = \sigma_{j+1} - \sigma_j$. Then $e_j = [\sigma_j, \sigma_{j+1}]$ and $h_j \leq ch\sigma_j^{1-1/q_i}$. For $j \geq 1$,

$$\begin{aligned}
\left| \int_{e_j} R_s(\sigma)(u - P_h u)(\sigma) d\sigma \right| &= \left| \int_{e_j} (R_s - P_h R_s)(\sigma)(u - P_h u)(\sigma) d\sigma \right| \\
&\leq \|R_s - P_h R_s\|_{1,e_j} \|u - P_h u\|_{\infty,e_j} \leq ch_j^2 \|DR_s\|_{1,e_j} \|Du\|_{\infty,e_j} \\
&\leq ch^2 \sigma_j^{2-2/q_i} \sigma_j^{\alpha_i - 1} \int_{e_j} (s^2 + \sigma^2)^{-1} d\sigma \|u\|_{C_\alpha^1} \\
&\leq ch^2 \int_{e_j} (s^2 + \sigma^2)^{\alpha_i/2 - 1/2 - 1/q} d\sigma \|u\|_{C_\alpha^1}.
\end{aligned}$$

$$\left| \sum_{j=1}^n \int_{e_j} R_s(\sigma)(u - P_h u)(\sigma) d\sigma \right| \leq ch^2 \int_0^a (s^2 + \sigma^2)^{\alpha_i/2 - 1/2 - 1/q_i} d\sigma \leq ch^2,$$

provided $q_i > 2/\alpha_i$.

For bounded $\int_{e_0} R_s(\sigma)(u - P_h u)(\sigma) d\sigma$ we use (2.2):

$$\left| \int_{e_0} R_s(\sigma)(u - P_h u)(\sigma) d\sigma \right| \leq c\|u - P_h u\|_{\infty,e_0} \leq ch_0^{\alpha_i} \|u\|_{C_\alpha^1} \leq ch^{\alpha_i q_i} \|u\|_{C_\alpha^1} \leq ch^2 \|u\| C_\alpha^1$$

which completes the proof of part (i).

We turn to prove part (ii).

$$D(R_i(I - P_h)u)(s) = \int_0^a D_s R_s(u - P_h u) d\sigma$$

For $j \geq 1$,

$$\left|\int_{e_j} D_s R_s(u - P_h u)d\sigma\right| = \left|\int_{e_j} (D_s R_s - P_h D_s R_s)(\sigma)(u - P_h u)(\sigma)d\sigma\right|$$

$$\leq ch_j^2 \|D_\sigma D_s R_s\|_{1,e_j} \|Du\|_{\infty,e_j} \leq ch^2 \sigma_j^{\alpha_i+1-2/q_i} \int_{e_j} (s^2+\sigma^2)^{-3/2} d\sigma \|u\|_{C_\alpha^1}$$

$$\leq ch^2 \int_{e_j} (s^2+\sigma^2)^{\alpha_i/2-1-1/q_i} d\sigma \|u\|_{C_\alpha^1}.$$

$$\left|\sum_{j=1}^{n-1} \int_{e_j} D_s R_s(u - P_h u)d\sigma\right| \leq ch^2 \int_0^a (s^2+\sigma^2)^{\alpha_i/2-1-1/q_i} d\sigma \|u\|_{C_\alpha^1}$$

$$\leq ch^2 s^{\alpha_i - 2/q_i - 1} \|u\|_{C_\alpha^1} = ch^2 s^{\beta_i - 1} \|u\|_{C_\alpha^1}.$$

Observe that $\partial_s R_s(\sigma) = -\partial_\sigma R_\sigma(s)$,

$$\left|\int_{e_0} \partial_s R_s(\sigma)(u - P_h u)(\sigma)d\sigma\right| = \left|\int_{e_0} \partial_\sigma R_\sigma(s)(u - P_h u)(\sigma)d\sigma\right|$$

$$= \left|\int_{e_0} R_\sigma(s)u' d\sigma - R_{h_0}(s)(u - P_h u)(h_0)\right|.$$

Next

$$\left|\int_{e_0} R_\sigma(s)u' d\sigma\right| \leq c\int_0^{h_0} (s^2+\sigma^2)^{-1/2} \sigma^{\alpha_i - 1} d\sigma \|u\|_{C_\alpha^1}$$

$$\leq cs^{\beta_i-1} \int_0^{h_0} (s^2+\sigma^2)^{-\beta_i/2} \sigma^{\alpha_i-1} d\sigma \|u\|_{C_\alpha^1} \leq cs^{\beta_i-1} \int_0^{h_0} \sigma^{\alpha_i-\beta_i-1} d\sigma \|u\|_{C_\alpha^1}$$

$$\leq cs^{\beta_i-1} h_0^{\alpha_i-\beta_i} \|u\|_{C_\alpha^1} = cs^{\beta_i-1} h_0^{2/q_i} \|u\|_{C_\alpha^1} \leq cs^{\beta_i-1} h^2 \|u\|_{C_\alpha^1},$$

and

$$|R_{h_0}(s)(u - P_h u)(h_0)| \leq c(s^2+h_0^2)^{-1/2} h_0^{\alpha_i} \|u\|_{C_\alpha^1}$$

$$\leq cs^{\beta_i-1}(s^2+h_o^2)^{-\beta_i/2} h_0^{\alpha_i} \|u\|_{C_\alpha^1} \leq cs^{\beta_i-1} h_0^{\alpha_i-\beta_i} \|u\|_{C_\alpha^1}$$

$$\leq cs^{\beta_i-1} h^2 \|u\|_{C_\alpha^1}.$$

Therefore

$$\left\|T(I - P_h)u\right\|_{C_\beta^1} \leq ch^2 \|u\|_{C_\alpha^1}.$$

Lemma 4. Assume $u \in C_\alpha^4$. If $q_i > 4/\alpha_i$, then

$$T(u - P_h u)(s) = \sum_{e \in E_1} \frac{1}{12} h_e^2 \int_e D_\sigma k(s.\sigma)u(\sigma)d\sigma + R \tag{2.7}$$

where $E_1 = E \backslash E_0$ with $E_0 = \bigcup_i \{e_{2i.n}, e_{2i+1,0}\}$ and $|R| \leq ch^4 \|u\|_{C_\alpha^4}$.

Proof. We continue to use the notation introduced in Lemma 3. It suffices to prove the expansion (2.7) with T replaced by R_i. For $s \in \Gamma_{2i}$,

$$R_i(u - P_h u)(s) = \int_0^a R_s(\sigma)(u - P_h u)(\sigma) d\sigma.$$

Using Lemma 2, for $j \geq 1$ we have

$$\int_{e_j} R_s(\sigma)(u - P_h u)(\sigma) d\sigma = \frac{1}{12} h_j^2 \int_{e_j} D_\sigma R_s(\sigma) Du(\sigma) d\sigma + R_j$$

with

$$\begin{aligned} |R_j| &\leq c h_j^4 \sum_{k=0}^{3} \left\| D^{4-k} u \right\|_{\infty, e_j} \left\| D_\sigma^k R_s \right\|_{1.e_j} \\ &\leq c h^4 \sigma_j^{4-4/q_i} \|u\|_{C_\alpha^4} \sum_{k=0}^{3} \sigma_j^{\alpha_i + k - 4} \int_{e_j} (\sigma^2 + s^2)^{-(k+1)/2} d\sigma \\ &\leq c h^4 \sigma_j^{\alpha_i - 4/q_i} \int_{e_j} (\sigma^2 + s^2)^{-1/2} d\sigma \|u\|_{C_\alpha^4} \\ &\leq c h^4 \|u\|_{C_\alpha^4} \int_{e_j} (\sigma^2 + s^2)^{-1/2 + \alpha_i/2 - 2/q_i} d\sigma, \end{aligned}$$

and

$$\sum_{j=1}^{n-1} |R_j| \leq c h^4 \int_0^a (\sigma^2 + s^2)^{-1/2 + \alpha_i/2 - 2/q_i} d\sigma \leq c h^4 \|u\|_{C_\alpha^4}$$

provided $q_i > 4/\alpha_i$. Thus

$$\sum_{j=1}^{n-1} \int_{e_j} R_s(\sigma)(u - P_h u)(\sigma) d\sigma = \frac{1}{12} \sum_{j=1}^{n-1} h_j^2 \int_{e_j} D_\sigma R_s(\sigma) Du(\sigma) d\sigma + 0(h^4) \|u\|_{C_\alpha^4}.$$

Next we estimate the remaining part using (2.2) :

$$\left| \int_{e_0} R_s(\sigma)(u - P_h u)(\sigma) d\sigma \right| \leq c \|u - P_h u\|_{\infty . e_0} \leq c h_0^\alpha \|u\|_{C_\alpha^1} \leq c h^4 \|u\|_{C_\alpha^4}.$$

Collecting the above relations gives

$$\int_0^a R_s(\sigma)(u - P_h u)(\sigma) d\sigma = \frac{1}{12} \sum_{j=1}^{n-1} h_j^2 \int_{e_j} D_\sigma R_s(\sigma) Du(\sigma) d\sigma + O(h^4) \|u\|_{C_\alpha^4},$$

which completes the proof of (2.7).

Now we are able to prove our main result.

Theorem 1. Provide $q_i > 4/\bar{\alpha}_i$ for each i, then

$$\left\| (4u_{h/2}^* - u_h^*)/3 - u_0 \right\|_\infty \leq c h^4.$$

Proof. Because $u_h^* = f - Tu_h$ we deduce

$$u_h^* - u_0 = (I + TP_h)^{-1}T(I - P_h)u_0.$$

Let

$$\begin{aligned} W^h &= T(I - P_h)u_0, u_h^* - u_0 = (I + TP_h)^{-1}W^h \\ &= (I + T)^{-1}W^h + (I + TP_h)^{-1} \cdot T \cdot (I - P_h)(I + T)^{-1}W^h. \end{aligned}$$

Select $\alpha < \bar{\alpha}$ such that $q_i > 4/\alpha_i$. Let $\beta_i = \alpha_i - 2/q_i > 2/q_i$. Using Lemma 3, we have

$$\begin{aligned} &\left\|(I + TP_h)^{-1}T(I - P_h)(I + T)^{-1}W^h\right\|_\infty \le c\left\|T(I - P_h)(I + T)^{-1}W^h\right\|_\infty \\ &\le ch^2\left\|(I + T)^{-1}W^h\right\|_{C_\beta^1} \le ch^2\|W^h\|_{C_\beta^1} \le ch^4\|u_0\|_{C_\alpha^1}. \end{aligned}$$

Hence

$$u_h^* - u_0 = (I + T)^{-1}W^h + 0(h^4)\|u_0\|_{C_\alpha^1}.$$

With the same argument we have

$$u_{h/2}^* - u_0 = (I + T)^{-1}W^{h/2} + 0(h^4)\|u_0\|_{C_\alpha^1}$$

Form Lemma 4, we can derive

$$\left\|4W^{h/2} - W^h\right\|_\infty \le ch^4.$$

Collecting all above relations we obtain

$$\left\|(4u_{h/2}^* - u_h^*)/3 - u_0\right\|_\infty \le ch^4.$$

References

[1] G. A. Chandler, Superconvergent approximations to the solution of a boundary integral equation on polygonal domains, Research Report, The Australian National University, 1985.

[2] Lin Qun , Xie Rui-feng , Some advances in the study of error expansion for finite elements, *J. Comp. Math.*, 4,(1986), 368-382.