THE ALGEBRAIC PERTURBATION METHOD FOR GENERALIZED INVERSES*

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1. Introduction

Algebraic perturbation methods were first proposed for the solution of nonsingular linear systems by R. E. Lynch and T. J. Aird [2]. Since then, the algebraic perturbation methods for generalized inverses have been discussed by many scholars [3]–[6]. In [4], a singular square matrix was perturbed algebraically to obtain a nonsingular matrix, resulting in the algebraic perturbation method for the Moore-Penrose generalized inverse. In [5], some results on the relations between nonsingular perturbations and generalized inverses of $m\times n$ matrices were obtained, which generalized the results in [4]. For the Drazin generalized inverse, the author has derived an algebraic perturbation method in [6].

In this paper, we will discuss the algebraic perturbation method for generalized inverses with prescribed range and null space, which generalizes the results in [5] and [6].

We remark that the algebraic perturbation methods for generalized inverses are quite useful. The applications can be found in [5] and [8].

In this paper, we use the same terms and notations as in [1].

2. Main Results

First, we will give two lemmas.

Lemma 1. *Let $A\in C_r^{n\times n}$, and let L and K be subspaces of C^n of dimension $s\le r$ and $n-s$ respectively. $AL\oplus K=C^n, B$ and $C^*\in C_{n-s}^{n\times(n-s)}$ are matrices whose columns form bases for K and $L^\perp$ respectively. Then*

$$\begin{bmatrix} T & B \\ C & 0 \end{bmatrix}$$

is nonsingular, and

$$\begin{bmatrix} T & B \\ C & 0 \end{bmatrix}^{-1} = \begin{bmatrix} A^{(2)}_{L,K} & P_{(A^*K^\perp)^\perp,L}C^+ \\ B^+P_{K,AL} & -I_{n-s} \end{bmatrix}$$

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where $T = A + BC - AP_{(A^*K^\perp)^\perp,L}$.

Proof. It is easy to show that

$$AL \oplus K = C^n \Longleftrightarrow (A^*K^\perp)^\perp \oplus L = C^n \qquad \text{(see [7])}$$

so that $P_{K,AL}, P_{(A^*K^\perp)^\perp,L}$ and $A^{(2)}_{L,K}$ exist.

From $L = N(C)$, it follows that

$$CA^{(2)}_{L,K} = 0, \quad CP_{L,(A^*K^\perp)^\perp} = 0 \tag{1}$$

and

$$\begin{aligned} TP_{(A^*K^\perp)^\perp,L}C^+ - B &= (A + BC - AP_{(A^*K^\perp)^\perp,L})P_{(A^*K^\perp)^\perp,L}C^+ - B \\ &= BCP_{(A^*K^\perp)^\perp,L}C^+ - B \\ &= BCC^+ - B = 0 \end{aligned} \tag{2}$$

and

$$CP_{(A^*K^\perp)^\perp,L}C^+ = CC^+ = I_{n-s}. \tag{3}$$

Finally, obviously $BB^+ = P_{R(B)} = P_K$, and $BB^+P_{K,AL} = P_{K,AL}$ so that

$$\begin{aligned} TA^{(2)}_{L,K} + BB^+P_{K,AL} &= (A + BC - AP_{(A^*K^\perp)^\perp,L})A^{(2)}_{L,K} + P_{K,AL} \\ &= AA^{(2)}_{L,K} + P_{K,AL} \\ &= P_{AL,K} + P_{K,AL} = I_n. \end{aligned} \tag{4}$$

Since $R(AA^{(2)}_{L,K}) = AL$ and $N(AA^{(2)}_{L,K}) = K$. From (1)-(4), we have

$$\begin{bmatrix} T & B \\ C & 0 \end{bmatrix} \cdot \begin{bmatrix} A^{(2)}_{L,K} & P_{(A^*K^\perp)^\perp,L}C^+ \\ B^+P_{K,AL} & -I_{n-s} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_{n-s} \end{bmatrix}$$

which is the required result.

Lemma 2. *Let* $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ *be a partitioned matrix which is nonsingular, and let the submatrix* A_{22} *also be nonsingular. Then*

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1}_{11,2} & -A^{-1}_{11,2}A_{12}A^{-1}_{22} \\ -A^{-1}_{22}A_{21}A^{-1}_{11,2} & A^{-1}_{22} + A^{-1}_{22}A_{21}A^{-1}_{11,2}A_{12}A^{-1}_{22} \end{bmatrix}$$

where $A_{11,2} = A_{11} - A_{12}A^{-1}_{22}A_{21}$.

Theorem 1. *Let* $A \in C^{m\times n}_r$. *L is a subspace of* C^n *of dimension* $s \le r$, *and K is a subspace of* C^m *of dimension* $m - s$. *Suppose* $AL \oplus K = C^m$, *and* $B \in C^{m\times(m-s)}_{m-s}, C^* \in C^{n\times(n-s)}_{n-s}$ *are matrices whose columns form bases for K and* $L^\perp$ *respectively. If* $m = n$, *let* $T = A + BC - AP_{(A^*K^\perp)^\perp,L}$. *If* $m > n$, *let* $B = [B_1 \vdots B_2]$ *where* $B_1 \in C^{m\times(n-s)}_{n-s}$, *and*

$M = [A + B_1C - AP_{(A^*K^\perp)^\perp,L} \vdots B_2]$. *If* $m < n$, *let* $C = \begin{bmatrix} C_1 \\ \cdots \\ C_2 \end{bmatrix}$ *where* $C_1 \in C_{m-s}^{(m-s)\times n}$ *and*

$N = \begin{bmatrix} A + BC_1 - P_{K,AL}A \\ \cdots \quad \cdots \\ C_2 \end{bmatrix}$. *Then*

(1) *when* $m = n, T$ *is nonsingular, and*

$$A_{L,K}^{(2)} = T^{-1} - P_{(A^*K^\perp)^\perp,L}C^+B^+P_{K,AL};$$

(2) *when* $m > n, M$ *is nonsingular. Let* $M^{-1} = \begin{bmatrix} M_1 \\ \cdots \\ M_2 \end{bmatrix}$ *where* $M_1 \in C_n^{n\times m}$; *then*

$$A_{L,K}^{(2)} = M_1 - P_{(A^*K^\perp)^\perp,L}C^+B_1^+(B_1B_1^+ + B_2B_2^+)^+P_{K,AL};$$

(3) *when* $m < n, N$ *is nonsingular. Let* $N^{-1} = [N_1 \vdots N_2]$ *where* $N_1 \in C_m^{n\times m}$; *then*

$$A_{L,K}^{(2)} = N_1 - P_{(A^*K^\perp)^\perp,L}(C_1^+C_1 + C_2^+C_2)^+C_1^+B^+P_{K,AL}.$$

Proof. (1) From Lemma 1, the matrix

$$\begin{bmatrix} T & B \\ C & 0 \end{bmatrix}$$

is nonsingular, and

$$\begin{bmatrix} T & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} A_{L,K}^{(2)} & P_{(A^*K^\perp)^\perp,L}C^+ \\ B^+P_{K,AL} & -I_{n-s} \end{bmatrix}^{-1}.$$

By using Lemma 2, we have

$$\begin{bmatrix} T & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} (A_{L,K}^{(2)} + P_{(A^*K^\perp)^\perp,L}C^+B^+P_{K,AL})^{-1} & * & * \\ * & * & * \end{bmatrix}$$

so that

$$T = (A_{L,K}^{(2)} + P_{(A^*K^\perp)^\perp,L}C^+B^+P_{K,AL})^{-1}$$

which is nonsingular, and therefore

$$A_{L,K}^{(2)} = T^{-1} - P_{(A^*K^\perp)^\perp,L}C^+B^+P_{K,AL}.$$

(2) Let $\tilde{A} = [A \vdots 0]C^{m\times m}$ where $0 \in C^{m\times(m-n)}$,

$$\tilde{C} = \begin{bmatrix} C & 0 \\ 0 & I_{m-n} \end{bmatrix} \in C_{m-s}^{(m-s)\times m},$$

$$\tilde{L} = N(\tilde{C}).$$

Hence

$$\tilde{L}=\left\{\begin{pmatrix}x\\0\end{pmatrix}:\quad x\in L \text{ and } 0\in C^{m-n}\right\},$$

$$\tilde{A}^*K^{\perp}=\left\{\begin{pmatrix}x\\0\end{pmatrix}:\quad x\in A^*K^{\perp} \text{ and } 0\in C^{m-n}\right\},$$

$$\tilde{A}\cdot\tilde{L}=AL \text{ and } \tilde{A}\cdot\tilde{L}\oplus K=C^m.$$

Since

$$\begin{aligned}P_{\tilde{L}}&=P_{N(\tilde{C})}=I_m-P_{R(\tilde{C}^+)}=I_m-\tilde{C}^+\tilde{C}\\&=\begin{bmatrix}I_n-C^+C&0\\0&0\end{bmatrix}=\begin{bmatrix}P_L&0\\0&0\end{bmatrix}.\end{aligned}$$

Similarly,

$$P_{\tilde{A}^*K^{\perp}}=\begin{bmatrix}P_{A^*K^{\perp}}&0\\0&0\end{bmatrix}$$

so that

$$P_{(\tilde{A}^*K^{\perp})^{\perp}}=I_m-P_{\tilde{A}^*K^{\perp}}=\begin{bmatrix}P_{(A^*K^{\perp})^{\perp}}&0\\0&I_{m-n}\end{bmatrix}.$$

By using Lemma 1 of [5],

$$\begin{aligned}P_{(\tilde{A}^*K^{\perp})^{\perp},\tilde{L}}&=P_{(\tilde{A}^*K^{\perp})^{\perp}}(P_{(\tilde{A}^*K^{\perp})^{\perp}}+P_{\tilde{L}})^{-1}\\&=\begin{bmatrix}P_{(A^*K^{\perp})^{\perp}}&0\\0&I_{m-n}\end{bmatrix}\left(\begin{bmatrix}P_{(A^*K^{\perp})^{\perp}}&0\\0&I_{m-n}\end{bmatrix}+\begin{bmatrix}P_L&0\\0&0\end{bmatrix}\right)^{-1}\\&=\begin{bmatrix}P_{(A^*K^{\perp})^{\perp}}(P_{(A^*K^{\perp})^{\perp}}+P_L)^{-1}&0\\0&I_{m-n}\end{bmatrix}\\&=\begin{bmatrix}P_{(A^*K^{\perp})^{\perp},L},&0\\0&I_{m-n}\end{bmatrix}.\end{aligned}$$

Thus

$$\tilde{A}+B\tilde{C}-\tilde{A}P_{(\tilde{A}^*K^{\perp})^{\perp},\tilde{L}}=[A+B_1C-AP_{(A^*K^{\perp})^{\perp}}\vdots B_2]=M.$$

By using Theorem 1 (1), we have that M is nonsingular, and

$$\tilde{A}^{(2)}_{\tilde{L},K}=M^{-1}-P_{(\tilde{A}^*K^{\perp})^{\perp},\tilde{L}}\tilde{C}^+B^+P_{K,AL}.$$

From a theorem in [1] (p. 210, Theorem 6), it follows that

$$B^+=[B_1\vdots B_2]^+=\begin{bmatrix}B_1^+(B_1B_1^++B_2B_2^+)^+\\B_2^+(B_1B_1^++B_2B_2^+)^+\end{bmatrix}$$

so that

$$\begin{aligned}
\tilde{A}^{(2)}_{\tilde{L},K} &= \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} - \begin{bmatrix} P_{(A^*K^\perp)^\perp,L} & 0 \\ 0 & I_{m-n} \end{bmatrix} \\
&\quad \times \begin{bmatrix} C^+ & 0 \\ 0 & I_{m-n} \end{bmatrix} \cdot \begin{bmatrix} B_1^+(B_1B_1^+ + B_2B_2^+)^+ \\ B_2^+(B_1B_1^+ + B_2B_2^+)^+ \end{bmatrix} \cdot P_{K,AL} \\
&= \begin{bmatrix} M_1 - P_{(A^*K^\perp)^\perp,L}C^+B_1^+(B_1B_1^+ + B_2B_2^+)^+P_{K,AL} \\ M_2 - B_2^+(B_1B_1^+ + B_2B_2^+)^+P_{K,AL} \end{bmatrix}.
\end{aligned}$$

Let $\tilde{A}^{(2)}_{\tilde{L},K} = \begin{bmatrix} X \\ Y \end{bmatrix}$ where $X \in C^{n\times m}$ and $Y \in C^{(m-n)\times m}$. Since $\begin{bmatrix} X \\ Y \end{bmatrix}[A \vdots 0]\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix}$ so that $XAX = X$, so $N(X) = N(AX) = N([A \vdots 0]\begin{bmatrix} X \\ Y \end{bmatrix}) = N(\tilde{A} \cdot \tilde{A}^{(2)}_{\tilde{L},K}) = N(\tilde{A}^{(2)}_{\tilde{L},K}) = K$. From $\tilde{L} = R(\tilde{A}^{(2)}_{\tilde{L},K}) = \left\{\begin{bmatrix} X \\ Y \end{bmatrix} x : x \in C^m\right\}$, we have $R(X) = L$. Hence $X = A^{(2)}_{L,K}$, i.e. $A^{(2)}_{L,K} = M_1 - P_{(A^*K^\perp)^\perp,L}C^+B_1^+(B_1B_1^+ + B_2B_2^+)^+P_{K,AL}$.

(3) Consider $A^* \in C_r^{n\times m}$. It is observed that (see [7])

$$A \cdot L \oplus K = C^m \Longleftrightarrow A^* \cdot K^\perp \oplus L^\perp = C^n.$$

From $R(B) = K$ and $N(C) = L$, it follows that $R(C^*) = L^\perp$ and $N(B^*) = K^\perp$. By using Theorem 1 (2), we have that

$$[A^* + C_1^*B^* - A^*P_{(AL)^\perp,K^\perp} \vdots C_2^*] = N^*$$

is nonsingular, and

$$A^{*(2)}_{K,L} = N_1^* - P_{(AL)^\perp,K^\perp}B^{*+}C_1^{*+}(C_1^*C_1^{*+} + C_2^*C_2^{*+})^+P_{L^\perp,A^*K^\perp}.$$

Notice that $(A^{(2)}_{L,K})^* = A^{*(2)}_{K^\perp,L^\perp}$. Therefore, N is nonsingular, and

$$A^{(2)}_{L,K} = N_1 - P_{(A^*K^\perp)^\perp,L}(C_1^+C_1 + C_2^+C_2)^+C_1^+B^+P_{K,AL}.$$

Under the conditions of Theorem 1, if the dimension of subspace $L = r$, then $\mathrm{rank}(A^{(2)}_{L,K})$ =dimension of $L = r$ and so $A^{(2)}_{L,K} = A^{(1,2)}_{L,K}$. From [1] (p. 62, Corollary 9), it follows that

$$AL \oplus K = C^m, \dim(L) = r \Longleftrightarrow L \oplus N(A) = C^n, \quad K \oplus R(A) = C^m.$$

It is easy to show that

$$(A^*K^\perp)^\perp = N(A) \text{ and } AL = R(A).$$

Therefore

$$P_{(A^*K^\perp)^\perp,L} = P_{N(A),L},$$

$$P_{K,AL} = P_{K,R(A)},$$

$$A \cdot P_{(A^*K^\perp)^\perp,L} = 0.$$

The following is a direct consequence from the discussion above and Theorem 1.

Corollary 1 [5]. Let $A \in C_r^{m\times n}, B \in C_{m-r}^{m\times(m-r)}$, and $C \in C_{n-r}^{(n-r)\times n}$ such that

$$R(A) \oplus R(B) = C^m \text{ and } N(C) \oplus N(A) = C^n.$$

If $m = n$, let $\tilde{T} = A + BC$. If $m > n$, let $B = [B_1 \ \vdots \ B_2]$ where $B_1 \in C_{n-r}^{m\times(n-r)}$, and $\tilde{M} = [A + B_1C \ \vdots \ B_2]$. If $n > m$, let $C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ where $C_1 \in C_{m-r}^{(m-r)\times n}$, and $\tilde{N} = \begin{bmatrix} A + BC_1 \\ C_2 \end{bmatrix}$. Then

(1) when $m = n, \tilde{T}$ is nonsingular, and

$$A^{(1,2)}_{N(C),R(B)} = \tilde{T}^{-1} - P_{N(A),N(C)}C^+B^+P_{R(B),R(A)};$$

(2) when $m > n, \tilde{M}$ is nonsingular. Let $\tilde{M}^{-1} = \begin{bmatrix} \tilde{M}_1 \\ \tilde{M}_2 \end{bmatrix}$ where $\tilde{M}_1 \in C_n^{m\times m}$; then

$$A^{(1,2)}_{N(C),R(B)} = \tilde{M}_1 - P_{N(A),N(C)}C^+B_1^+(B_1B_1^+ + B_2B_2^+)^+P_{R(B),R(A)};$$

(3) when $m < n, \tilde{N}$ is nonsingular, and let $\tilde{N}^{-1} = [\tilde{N}_1 \vdots \tilde{N}_2]$ where $\tilde{N}_1 \in C_m^{n\times m}$. Then

$$A^{(1,2)}_{N(C),R(B)} = \tilde{N}_1 - P_{N(A),N(C)}(C_1^+C_1 + C_2^+C_2)^+C_1^+B^+P_{R(B),R(A)}.$$

Since the weighted Moore-Penrose generalized inverse $A^+_{(M,N)} = A^{(1,2)}_{N^{-1}R(A^*),M^{-1}N(A^*)}$ ([1], p.127), especially $A^+ = A^{(1,2)}_{R(A^*),N(A^*)}$, the algebraic perturbation method for $A^+_{(M,N)}$ and A^+ can be derived from Corollary 1. For details see [5].

Corollary 2 [6]. Let $A \in C_r^{n\times n}, k = \text{index}(A), s = \text{rank}(A^k)$, and $B, C^* \in C_{n-s}^{n\times(n-s)}$ be matrices whose columns form bases for $N(A^k)$ and $R(A^k)^\perp$ respectively. Let $T = A + BC - AB(CB)^{-1}C$. Then T is nonsingular, and $A^D = T^{-1} - B(CB)^{-2}C$.

Proof. It is observed that

$$A^D = A^{(2)}_{R(A^k),N(A^k)} = A^{(2)}_{N(C),R(B)}$$

and

$$A \cdot N(C) \oplus R(B) = C^m (\text{ from } R(A^k) \oplus N(A^k) = C^n).$$

Finally, notice that

$$\begin{aligned} (A^*R(B)^\perp)^\perp &= (A^*N(A^k)^\perp)^\perp &&= (R(A^*)^{k+1})^\perp \\ &= (R(A^*)^k)^\perp &&= (N(A^k)^\perp)^\perp \\ &= N(A^k) &&= R(B) \end{aligned}$$

and

$$AN(C) = AR(A^k) = R(A^k) = N(C)$$

together with ([6])

$$P_{R(B),N(C)} = B(CB)^{-1}C.$$

So T is nonsingular, and $A^D = T^{-1} - B(CB)^{-2}C$ from Theorem 1.

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