A REMARK ON MINIMAL SUPPORT FOR BIVARIATE SPLINES*

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§1. Introduction

This paper is concerned with the minimality of the support for bivariate splines, which was first investigated by de Boor and Höllig in [1]. Now let us introduce the problem briefly.

Let $\pi^{\rho}_{k,\Delta}$ be the space of bivariate pp (piecewise polynomial) functions in C^{ρ}, of degree $\leq k$, on the mesh Δ obtained from a uniformly unit square mesh by drawing all upward sloping diagonals. It is well-known that whenever $\rho > [\frac{2k-2}{3}]$, there is no element in space $\pi^{\rho}_{k,\Delta}$ with compact support and this kind of spaces has bad approximating properties. So we always suppose $\rho \leq [\frac{2k-2}{3}]$. In this case there exist compact support elements in space $\pi^{\rho}_{k,\Delta}$. Naturally, the problem of searching for the minimal support elements in this kind of spaces is raised. The very useful cases are $\rho = [\frac{2k-2}{3}]$. For different values of k mod 3, we obtain three different kinds of spaces $\pi^{\rho}_{k,\Delta}$. They will be denoted respectively by $S_{o,\mu} = \pi^{2\mu-3}_{3\mu-3,\Delta}$, $S_{1,\mu} = \pi^{2\mu-2}_{3\mu-2,\Delta}$ and $S_{2,\mu} = \pi^{2\mu-2}_{3\mu-1,\Delta}$, where μ is a positive integer.

In order to investigate $S_{o,\mu}$ and $S_{1,\mu}$, de Boor and Höllig introduced the following concepts.

A function f is said to have minimal support in space S if $f \in S$ and the only $g \in S$ having support strictly inside supp f is $g = 0$. And a function f is said to have unique minimal support in S if $f \in S$ and any $g \in S$ having support in supp f is a multiple of f (see[1]).

They determined all minimal support elements in $S_{0,\mu}$ and $S_{1,\mu}$. But when searching for minimal support elements in $S_{2,\mu}$, de Boor and Höllig noticed that if stuck to the definition of minimal support elements given earlier, one could not find a basis for $S_{2,\mu}$ consisting only of minimal support elements. As an example, they showed that the translates of four functions, whose supports are drawn in Figure 1, provide a suitable basis for $S_{2,1}$, but the notable fact is that the fourth element does not have minimal support. Without any definition or further illustration they still called them four "minimal support" elements and asserted at the end of their paper that for odd μ, the four "minimal support" elements in

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$S_{2,\mu}$ can be obtained from the preceding case by convolution with $M_{1,1,1}$ (for the definition of $M_{1,1,1}$ refer to [1]).

Since their "minimal support" in $S_{2,\mu}$ has an uncertain meaning, we cannot draw any clear conclusions from their discussion. We have to redefine the concept of minimal support for this case in order that we might discuss the problem precisely. That is what we want to do in this paper, which is arranged as follows: In Section 2, we calculate the dimension of the subspace of $S_{2,\mu}$, consisting of all the elements having their supports in a certain closed convex domain. In Section 3, the minimal support system in $S_{2,\mu}$ is defined and such a system in $S_{2,3}$ is given as well. We also show that the system consisting of the four "minimal support" elements in $S_{2,3}$, which were obtained and termed by de Boor and Höllig, is not a minimal support system.

§2. A local dimension theorem

Let $\Xi = (\xi_i)_1^n$ be a sequence in R^S. The truncated power (or cone spline) $C_\Xi(x)$ is defined as the following distribution on R^S

$$C_\Xi : \phi \to \int_{R_+^s} \phi\Big(\sum_{i=1}^n t(i)\xi_i\Big)dt,$$

where $\phi(x)$ is any test function on R^s and $R_+^s\{x \in R^s; x(i) \geq 0, i = \overline{1,s}\}$.

And the box spline $M_\Xi(x)$ is defined as follows:

$$\int_{R^S} \phi(x)M_\Xi(x)dx = \int_{[0,1]^n} \phi\Big(\sum_{i=1}^n t(i)\xi_i\Big)dt, \quad \phi \in C_0^\infty(R^s).$$

It is well-known that (see[2]) for $W \subset \Xi$

$$D_W C_\Xi = C_{\Xi\backslash W}.$$

In this paper, we only consider the case of R^2 and Ξ will be taken as $\Xi = (d_1 : r, d_2 : s, d_3 : t)$, where $d_1 = e_1 = (1,0), d_2 = e_2 = (0,1)$ and $d_3 = e_1 + e_2 = (1,1)$. For simplicity, we shall write $C_{r,s,t}, M_{r,s,t}$ instead of C_Ξ, M_Ξ respectively.

Lemma 2.1[3]. *Let $k = r + s + t - 2$; then*

$$C_{r,s,t}(x) = \begin{cases} p(x), & x \in V_1 = \{x; x \in R_+^2, x(1) - x(2) \leq 0\}; \\ q(x), & x \in V_2 = \{x; x \in R_+^2, x(1) - x(2) > 0\}, \end{cases}$$

where

$$p(x) = \frac{\binom{s+t-2}{t-1}}{k!} \sum_{j=0}^{s-1} (-1)^{s-1-j} \frac{\binom{s-1}{j}\binom{k}{j}}{\binom{s+t-2}{j}} x(1)^{k-j} x(2)^j,$$

$$q(x) = \frac{\binom{r+t-2}{t-1}}{k!} \sum_{j=0}^{r-1} (-1)^{r-1-j} \frac{\binom{r-1}{j}\binom{k}{j}}{\binom{r+t-2}{j}} x(1)^j x(2)^{k-j}.$$

By the way, we point out that formula (2.1) in [1], i.e. $C_{\Xi}(x) = \sum_{v \in Z_+^2} M_{\Xi}(x-v)$, is false, since for large enough $x(i), i = 1, 2$, $\sum_{v \in Z_+^2} M_{\Xi}(x-v) = 1$ while $C_{\Xi}(x)$ cannot be equal to a constant on any open set in R_+^2.

Now let $s(\leq k, \nu)$ denote the space of all pp functions, of degree $\leq k$, with supports in R_+^2, the possible singularities of which only appear on the three rays

$$R_+ d_i = \{t_1 d_i; t_1 \geq 0\}, \quad i = \overline{1,3},$$

where ν is such a triple index that all partial derivatives of order $\leq \nu(i)$ are required to be continuous across $R_+ d_i, i = \overline{1,3}$.

Lemma 2.2.[1]

$$\dim s(\leq k, \nu) = \sum_{l \leq k} \Big(\sum_{i=1}^{3} (l - \nu(i))_+ - l - 1\Big)_+.$$

Let $Q_v = [v(1), v(1)+1] \times [v(2), v(2)+1], v \in Z^2$. Let $\Omega = \text{conv } \{0, jd_1, jd_1 + d_3, d_2\}$, i.e., $\Omega = \bigcup_{n=0}^{j} Q_{(n,0)} \setminus \theta$ with θ the triangle conv $\{jd_1, jd_1 + d_3, (j+1)d_1\}$. Let $X = \{f|_\Omega; f \in S_{2,\mu}, \text{ supp } f \subseteq \{x(2) \geq 1\} \cup \Omega\}$. For convenience, we let $C_0 = C_{\mu+1,\mu,\mu}, C_1 = C_{\mu,\mu,\mu+1}$, $C_2 = C_{\mu-1,\mu+1,\mu+1}$, and $C_3 = C_{\mu,\mu,\mu}$.

Theorem 2.1.

$$\dim X = (j + 1 - \mu)_+ + (j + 2 - \mu)_+ + (2j + 1 - \mu)_+$$

and any $f \in X$ can be written as

$$f = \sum_{v=0}^{j} \sum_{i=0}^{3} a_{vi} C_i(\cdot - (v, 0)) \tag{2.1}$$

if and only if $(a_{vi})_{v=0,i=0}^{j\ \ 3}$ satisfy

$$\begin{cases} \sum_{v=0}^{j} [a_{v0}(\cdot - v)^{\mu} + \mu a_{v3}(\cdot - v)^{\mu-1} \equiv 0; \\ \sum_{v=0}^{j} a_{v1}(\cdot - v)^{\mu-1} \equiv 0; \\ \sum_{v=0}^{j} a_{v2}(\cdot - v)^{\mu-2} \equiv 0. \end{cases} \tag{2.2}$$

Proof. Since $k = 3\mu - 1, \rho = 2\mu - 2$, we conclude from Lemma 2.2 that

$$\dim s(\leq k, (\rho, \rho, \rho)) = 4.$$

Hence $s(\leq k(\rho,\rho,\rho))$ is spanned by the four independent cone splines C_0, C_1, C_2, C_3 and

$$X = \left\{ f \in \text{ span } (C_i(\cdot - (v,0))|_\Omega)_{v=0,i=0}^{j\ \ 3}; f|_\theta = 0 \right\}.$$

Thus for any $f \in X$, we have

$$f = \sum_{v=0}^{j} \sum_{i=0}^{3} a_{vi} C_i(\cdot - (v,0)) \text{ and } f|_\theta = 0.$$

Firstly, we shall show that $f|_\theta = 0$ implies (2.2). For this sake, let

$$\sum_{v=0}^{j} \sum_{i=0}^{3} a_{vi} C_i(x - (v,0)) = 0, \quad x \in \theta. \tag{2.3}$$

Using Lemma 2.1, we have

$$\sum_{v=0}^{j} \Bigg[a_{v0} \frac{\binom{2\mu-1}{\mu-1}}{(3\mu-1)!} \sum_{i=0}^{\mu} (-1)^{\mu-i} \frac{\binom{\mu}{i}\binom{3\mu-1}{i}}{\binom{2\mu-1}{i}} (x(1)-v)^i x(2)^{3\mu-1-i}$$

$$+ a_{v1} \frac{\binom{2\mu-1}{\mu}}{(3\mu-1)!} \sum_{i=0}^{\mu-1} (-1)^{\mu-1-i} \frac{\binom{\mu-1}{i}\binom{3\mu-1}{i}}{\binom{2\mu-1}{i}} (x(1)-v)^i x(2)^{3\mu-1-i}$$

$$+ a_{v2} \frac{\binom{2\mu-2}{\mu}}{(3\mu-1)!} \sum_{i=0}^{\mu-2} (-1)^{\mu-2-i} \frac{\binom{\mu-2}{i}\binom{3\mu-1}{i}}{\binom{2\mu-2}{i}} (x(1)-v)^i x(2)^{3\mu-1-i}$$

$$+ a_{v3} \frac{\binom{2\mu-2}{\mu-1}}{(3\mu-2)!} \sum_{i=0}^{\mu-1} (-1)^{\mu-1-i} \frac{\binom{\mu-1}{i}\binom{3\mu-2}{i}}{\binom{2\mu-2}{i}} (x(1)-v)^i x(2)^{3\mu-2-i} = 0, \quad x \in \theta. \tag{2.4}$$

Divide (2.4) by $x(2)^{2\mu-1+i}, i = \overline{0,2}$, and let $x(2)$ tend to sero. Then we obtain

$$\left\{\begin{array}{l}
\displaystyle\sum_{v=0}^{j} a_{v0}\frac{\binom{2\mu-1}{\mu-1}}{(3\mu-1)!}\cdot\frac{\binom{3\mu-1}{\mu}}{\binom{2\mu-1}{\mu}}(x(1)-v)^{\mu}+a_{v3}\frac{\binom{2\mu-2}{\mu-1}}{(3\mu-2)!}\cdot\frac{\binom{3\mu-2}{\mu-1}}{\binom{2\mu-2}{\mu-1}}(x(1)-v)^{\mu-1}\equiv 0;\\[2ex]
\displaystyle\sum_{v=0}^{j} -a_{v0}\frac{\binom{2\mu-1}{\mu-1}}{(3\mu-1)!}\cdot\frac{\binom{\mu}{\mu-1}\binom{3\mu-1}{\mu-1}}{\binom{2\mu-1}{\mu-1}}(x(1)-v)^{\mu-1}+a_{v1}\frac{\binom{2\mu-1}{\mu}}{(3\mu-1)!}\cdot\frac{\binom{3\mu-1}{\mu-1}}{\binom{2\mu-1}{\mu-1}}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times(x(1)-v)^{\mu-1}\\
\displaystyle -a_{v3}\frac{\binom{2\mu-1}{\mu-1}}{(3\mu-1)!}\cdot\frac{\binom{\mu-1}{\mu-2}\binom{3\mu-2}{\mu-2}}{\binom{2\mu-2}{\mu-2}}(x(1)-v)^{\mu-2}\equiv 0;\\[2ex]
\displaystyle\sum_{v=0}^{j} a_{v0}\frac{\binom{2\mu-1}{\mu-1}}{(3\mu-1)!}\cdot\frac{\binom{\mu}{\mu-2}\binom{3\mu-1}{\mu-2}}{\binom{2\mu-1}{\mu-2}}(x(1)-v)^{\mu-2}-a_{v1}\frac{\binom{2\mu-1}{\mu}}{(3\mu-1)!}\cdot\frac{\binom{\mu-1}{\mu-2}\binom{3\mu-1}{\mu-2}}{\binom{2\mu-1}{\mu-2}}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times(x(1)-v)^{\mu-2}\\
\displaystyle +a_{v2}\frac{\binom{2\mu-2}{\mu}}{(3\mu-1)!}\cdot\frac{\binom{3\mu-1}{\mu-2}}{\binom{2\mu-2}{\mu-2}}(x(1)-v)^{\mu-2}+a_{v3}\frac{\binom{2\mu-2}{\mu-1}}{(3\mu-2)!}\cdot\frac{\binom{\mu-1}{\mu-3}\binom{3\mu-2}{\mu-3}}{\binom{2\mu-2}{\mu-3}}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times(x(1)-v)^{\mu-3}\equiv 0,\quad x\in\theta
\end{array}\right.\tag{2.5}$$

It is easy to verify that they are equivalent to (2.2).

Secondly, we shall show that (2.1) together with (2.2) implies

$$f\in X.$$

From (2.2), we have

$$\left\{\begin{array}{l}
\displaystyle\sum_{v=0}^{j}\left[a_{v0}(x(1)-v)^{i}+ia_{v3}(x(1)-v)^{i-1}\right]\equiv 0,\quad i=\overline{0,\mu},\\
\displaystyle\sum_{v=0}^{j} a_{v1}(x(1)-v)^{i}\equiv 0,\quad i=\overline{o,\mu-1},\\
\displaystyle\sum_{v=0}^{j} a_{v2}(x(1)-v)^{i}\equiv 0,\quad i=\overline{0,\mu-2}.
\end{array}\right.\tag{2.6}$$

On the other hand, whenever $x\in\theta$, (2.1) becomes

$$f(x)=\sum_{i=1}^{\mu}(-1)^{\mu-i}\frac{(2\mu-1-i)!x(2)^{3\mu-1-i}}{(\mu-i)!i!(3\mu-1-i)!(\mu-1)!}\sum_{v=0}^{j}\left[a_{vo}(x(1)-v)^{i}+ia_{v3}(x(1)-v)^{i-1}\right]$$

$$+\frac{\binom{2\mu-1}{\mu-1}}{(3\mu-1)!}x(2)^{3\mu-1}\sum_{v=0}^{j}a_{v0}+\sum_{i=0}^{\mu-1}(-1)^{\mu-1-i}\frac{(2\mu-1-i)!x(2)^{3\mu-1-i}}{(\mu-1-i)!i!(3\mu-1-i)!\mu!}$$

$$\times\sum_{v=0}^{j}a_{v1}(x(1)-v)^i+\sum_{i=0}^{\mu-2}(-1)^{\mu-2-i}\frac{(2\mu-2-i)!x(2)^{3\mu-1-i}}{(\mu-2-i)!i!(3\mu-1-i)!\mu!}\sum_{v=0}^{j}a_{v2}(x(1)-v)^i. \tag{2.7}$$

By (2.6), we conclude that for any $x\in\theta$, $f(x)=0$, i.e., $f\in X$.

To calculate the dimension of X, we consider the following matrix

$$A_n=\begin{pmatrix} 1 & 1 & 1 & \cdots & 1\\ 1 & 2 & 2^2 & \cdots & 2^{2n-1}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 1 & n & n^2 & \cdots & n^{2n-1}\\ 2\cdot 1 & 3\cdot 1 & 4\cdot 1^2 & \cdots & (2n+1)\cdot 1^{2n-1}\\ 2\cdot 1 & 3\cdot 2 & 4\cdot 2^2 & \cdots & (2n+1)\cdot 2^{2n-1}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 2\cdot 1 & 3\cdot n & 4\cdot n^2 & \cdots & (2n+1)\cdot n^{2n-1}\end{pmatrix}.$$

Now let

$$A_n^*=\begin{pmatrix} 1 & 1 & 1 & \cdots & 1\\ 1 & 2 & 2^2 & \cdots & 2^{2n-1}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 1 & n & n^2 & \cdots & n^{2n-1}\\ 0 & 1 & 2 & \cdots & 2n-1\\ 0 & 1 & 2\cdot 2 & \cdots & (2n-1)2^{2n-2}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 1 & 2\cdot n & \cdots & (2n-1)\cdot n^{2n-2}\end{pmatrix}.$$

It is easy to see that $\det(A_n)=n!\cdot\det(A_n^*)$. Thus A_n is nonsingular, for A_n^* is only a Hermite interpolation matrix and certainly nonsingular. This shows

$$\dim X=(j+1-\mu)_+ + (j+2-\mu)_+ + (2j+1-\mu)_+.$$

Similarly we have

Theorem 2.2. *Let $\Omega'=\operatorname{conv}\{0,jd_2,jd_2+d_3,d_1\}$, and let $X'=\{f|_{\Omega'};f\in S_{2,\mu},$ $\operatorname{supp} f\subseteq\{x(1)\geq 1\}\cup\Omega'\}$. Then*

$$\dim X'=(j+1-\mu)_+ + (j+2-\mu)_+ + (2j+1-\mu)_+.$$

Furthermore, any $f \in X'$ can be written as

$$f = \sum_{v=0}^{j} \sum_{i=0}^{3} a_{vi} C_i(\cdot - (0, v)),$$

iff $(a_{vi})_{v=0,i=0}^{j\ \ 3}$ *satisfy*

$$\begin{cases} \sum_{v=0}^{j} a_{v2}(\cdot - v)^{\mu} + \mu a_{v3}(\cdot - v)^{\mu-1} \equiv 0; \\ \sum_{v=0}^{j} (a_{v0} + a_{v1} - a_{v2})(\cdot - v)^{\mu-1} \equiv 0; \\ \sum_{v=0}^{j} (a_{v1} - a_{v2})(\cdot - v)^{\mu-2} \equiv 0. \end{cases}$$

Now let $C_i^1(x) = C_i(x(2), x(2) - x(1))$, $C_i^2(x) = C_i(x(1) - x(2), x(1)), i = \overline{0,3}$.

Theorem 2.3. *Let* $\Omega_1 =$ conv $\{0, jd_3, jd_3 + d_2, -d_1\}$, *and* $X_1 = \{f|_{\Omega_1}; f \in S_{2,\mu}$, supp $f \subseteq \{x(2) \geq x(1) + 1\} \cup \Omega_1\}$. *Then*

$$\dim X_1 = (j + 1 - \mu)_+ + (j + 2 - \mu)_+ + (2j + 1 - \mu)_+.$$

Furthermore, any $f = \sum_{v=0}^{j} \sum_{i=0}^{3} a_{vi} C_i^1(\cdot - (v, v)) \in X_1$ *iff* $(a_{vi})_{v=0,i=0}^{j,\ \ 3}$ *satisfy*

$$\begin{cases} \sum_{v=0}^{j} (a_{v1} + a_{v2})(\cdot - v)^{\mu} + \mu a_{v3}(\cdot - v)^{\mu-1} \equiv 0; \\ \sum_{v=0}^{j} (a_{v0} - a_{v2})(\cdot - v)^{\mu-1} \equiv 0; \\ \sum_{v=0}^{j} a_{v0}(\cdot - v)^{\mu-2} \equiv 0. \end{cases}$$

Theorem 2.4. *Let* $\Omega_2 =$ conv $\{0, jd_3, jd_3 + d_1, -d_2\}$, *and* $X_2 = \{f|_{\Omega_2}; f \in S_{2,\mu}$, *supp* $f \subseteq \{x(1) \geq x(2) + 1\} \cup \Omega_2\}$. *Then*

$$\dim X_2 = (j + 1 - \mu)_+ + (j + 2 - \mu)_+ + (2j + 1 - \mu)_+.$$

And any $f=\sum_{v=0}^{j}\sum_{i=0}^{3}a_{vi}C_i^2(\cdot-(v,v))\in X_2$ *iff* $(a_{vi})_{v=0,i=0}^{j\ \ 3}$ *satisfy*

$$\begin{cases}\sum_{v=0}^{j}(a_{v1}+a_{v2})(\cdot-v)^{\mu}+\mu a_{v3}(\cdot-v)^{\mu-1}\equiv 0;\\ \sum_{v=0}^{j}(a_{v0}+a_{v2})(\cdot-v)^{\mu-1}\equiv 0;\\ \sum_{v=0}^{j}a_{v0}(\cdot-v)^{\mu-2}\equiv 0.\end{cases}$$

§3. The system of minimal supports in $S_{2,\mu}$

Let $S=\Pi_{k,\Delta}^{\rho}$. For a set $\Omega\subset R^2, S(\Omega)$ denotes the subspace of S which consists of functions having supports in Ω.

As we know, in $S_{2,\mu}$ the elements with minimal supports, which are defined by de Boor and Höllig in [1], fail to form a basis for $S_{2,\mu}(\Omega)$. So we have to introduce a more general definition about "minimal support" in S (here $\rho\le[\frac{2k-2}{3}]$).

Definition 3.1. *A finite collection* $\{f_i\}_1^m$ *of functions in* S*, whose supports are convex, is called a minimal basic system iff*

(a) *For arbitrary finite subsets* $V_i\subset Z^2, i=\overline{1,m}$,

$$\sum_{i=1}^{m}\sum_{v\in V_i}a_{i,v}f_i(\cdot-v)=0$$

implies

$$a_{i,v}=0, i=\overline{1,m}; v\in V_i.$$

(b) *For each function* $\phi\in S$ *having compact support, there exist subsets* $V_i(\subset Z^2)$ *and real numbers* $a_{i,v}, i=\overline{1,m}$; $v\in V_i$*, such that*

$$\phi=\sum_{i=1}^{m}\sum_{v\in V_i}a_{i,v}f_i(\cdot-v).$$

Furthermore, if supp ϕ *is convex,*

$$\bigcup_{i=1}^{m}\ \bigcup_{\substack{v\in V_i\\ a_{i,v}\neq 0}}\text{supp } f_i(\cdot-v)\subseteq\text{supp }\phi\ .$$

We shall see that the number m is completely determined by S.

For simplicity, for a vector $y\in R^2$ and a subset $\Omega\subset R^2$ we write $\Omega+y=\{x+y; x\in\Omega\}$.

Definition 3.2. *Two compact domains Ω and Ω' are said to be translatively coincidental iff there exists a vector $v \in Z^2$ such that*

$$\Omega' = \Omega + v.$$

It is evident that the definition gives an equivalence relation on the compact domains in R^2.

Definition 3.3. *Two minimal basic system $\{f_i\}_1^m$ and $\{g_i\}_1^m$ in S are said to be coincidental with respect to support iff there exists a rearrangement π of $(1, 2, \cdots, m)$ such that* supp $g_{\pi(i)}$ *and* supp $f_i, i = \overline{1,m}$, *is translatively coincidental.*

Also, the coincidental relation with respect to support is an equaivalence relation on the minimal basic system in S.

Definition 3.4. *The support collection* $\{$ supp $f_i\}_1^m$ *of a minimal basic system in S is called a system of minimal support in S.*

By Definition 3.1 and [1], we are aware that the only one minimal support element (box spline) in $S_{1,\mu}$ is a minimal basic system in $S_{1,\mu}$, and the two minimal support splines in $S_{0,\mu}$ form a minimal basic system in $S_{0,\mu}$.

Theorem 3.1. *The number of functions in a minimal basic system is completely determined by S. And the minimal basic system in S is unique in the sense of the equivalence relation.*

Proof. Assume that there exist two minimal basic systems in $S, \{f_i\}_1^m$ and $\{g_i\}_1^{m'}$. Define

$$F_k = \{f_i; 1 \le i \le m, \text{mes (supp } f_i) = m(k, f) = \min_{\substack{1 \le j \le m \\ j \in I_{k-1}}} \{\text{mes(supp } f_j\},\},$$

$$I_0 = \phi, I_k = \{i; f_i \in F_k\}, k = \overline{1, n_f};$$

$$G_k = \{g_j; 1 \le j \le m', \text{ mes (supp } g_j) = m(k.g)$$

$$= \min_{\substack{1 \le i \le m' \\ i \in J_{k-1}}} \{ \text{ mes (supp } g_i)\},\},$$

$$J_0 = \phi, J_k = \{j; g_j \in F_k\}, k = \overline{1, n_g}.$$

For any function f, let $F_k(f)(G_k(f))$ be the set of all functions in $F_k(G_k)$ which are translatively coincidental with f. For any $v \in Z^2, F_k(\cdot - v) = \{f_i(\cdot - v); f_i \in F_k\}, k = \overline{1, n_f}$. Then

$$\Big\{f_1, \cdots, f_m\Big\} = \bigcup_{k=1}^{n_f} F_k, F_k \cap F_{k'} = \phi; \quad \Big\{g_1, \cdots, g_{m'}\Big\} = \bigcup_{k=1}^{n_g} G_k, G_k \cap G_{k'} = \phi,$$

for $k \ne k'$. We shall show that $n_f = n_g$ and there exists a 1–1 map T from F_k onto G_k such that Supp $T(f)$ and supp f are translatively coincidental for all $f \in F_k, k = \overline{1, n_f}$.

When $k = 1$, by (b) of Definition 3.1 we know

$$m(1, f) = m(1, g).$$

For any $f \in F_1$, let $F_1(f) = \{f_{i_l}\}_1^{r_1}, G_1(f) = \{g_{j_l}\}_1^{r_1'}$. By (b) of Definition 3.1 we are aware that any function $f \in F_1(f)$ must be a linear combination of functions in $G_1(f)$, i.e., there exist $\{a_j\}_1^{r_1'} \subset R$ such that

$$f_{i_k} = \sum_{l=1}^{r_1'} a_l g_{j_l}(\cdot - v_l), k = \overline{1, r_1}, v_l \in Z^2.$$

From (a) of Definition 3.1 we conclude

$$r_1 \le r_1'.$$

By symmetry we have

$$r_1' \le r_1.$$

Thus

$$r_1 = r_1'.$$

This means that for any $f \in F_1$ (or G_1),

$$|F_1(f)| = |G_1(f)|.$$

It follows that there exists a $1-1$ map T_1 from F_1 onto G_1 such that $T_1(f)$ and f are translatively coincidental with respect to support for all $f \in F_1$.

Inductively assume that for all $k' \le k-1$ there exists a $1-1$ map $T_{k'}$ from $F_{k'}$ onto $G_{k'}$ such that $T_{k'}(f)$ and f are translatively coincidental with respect to support for all $f \in F_{k'}$. Consequently, $m(k', f) = m(k', g)$. Then for the case k, for any $f \in F_k$ let

$$F_k(f) = \{f_{i_l}^k\}_{l=1}^{r_k}, \quad G_k(f) = \{g_{j_l}^k\}_{l=1}^{r_k'}.$$

Again by (b) of Definition 3.1 we have $m(k, f) = m(k, g)$ and any $\bar{f} \in F_k(f)$ must be a linear combination of functions in $G_k(f) \bigcup \left(\bigcup_{i=1}^{k-1} \bigcup_{v \in V_i'} F_i(\cdot - v)\right)$, i.e., there exist real numbers $\{b_\lambda\}_{\lambda=1}^{r_k'}$ and $v_{\lambda,k} \in Z^2, \lambda = \overline{1, r_k'}$ such that

$$f_{i_l}^k = \sum_{\lambda=1}^{r_{k'}} b_\lambda g_{j_\lambda}^k(\cdot - v_{\lambda,k}) + \bar{f}_l . l = \overline{1, r_k},$$

where $\bar{f}_l \in$ span $\left\{\bigcup_{l=1}^{k-1} \bigcup_{v \in V_l'} F_i(\cdot - v)\right\}, V_l'$ is a subset of $Z^2, l = \overline{1, r_k}$. Similarly, we know from (a) of Definition 3.1 that

$$r_k \le r_k'.$$

By symmetry, we have

$$r_k' \le r_k.$$

Thus

$$r_k = r_k.$$

This means that for any $f \in F_k$ (or G_k),

$$|F_k(f)| = |G_k(f)|.$$

This shows that there exists a $1-1$ map T_k from F_k onto G_k such that $T_k(f)$ and f are translatively coincidental with respect to support for all $f \in F_k$.

Finally we show that $n_f = n_g$. If, on the contrary, without loss of generality we suppose $n_f < n_g$, then all $g_j \in G_{n_g}$, by Definition 3.1, must be a linear combination of functions in $\bigcup_{i=1}^{n_f} \bigcup_{v \in V_i} F_i(\cdot - v)$, where V_i is a finite subset of $Z^2, i = \overline{1, n_f}$. But any function in $\bigcup_{i=1}^{n_f} \bigcup_{v \in V_i} F_i(\cdot - v)$ must be a linear combination of functions in $\bigcup_{i=1}^{n_f} \bigcup_{v \in V_i'} G_i(\cdot - v)$, where V_i' is a finite subset of $Z^2, i = \overline{1, n_f}$. Thus g_j would be a linear combination of functions in $\bigcup_{i=1}^{n_f} \bigcup_{v \in V_i'} G_i(\cdot - v)$. This contradicts (a) of Definition 3.1. So $n_f = n_g$ and consequently $m = m'$.

Define

$$T(f) = T_k(f), \quad \text{for } f \in F_k (k = \overline{1, n_f}).$$

Then T is such a $1-1$ map from $\{f_1, \cdots, f_m\}$ onto $\{g_1, \cdots, g_m\}$ that $T(f)$ and f are translatively coincidental with respect to support for all $f \in \{f_1, \cdots, f_m\}$. This means that there exists a rearrangement π of $(1, 2, \cdots, m)$ such that $g_{\pi(i)}$ and $f_i, i = \overline{1, m}$ are translatively coincidental with respect to support. This completes the proof.

Corollary 3.1. The system of minimal support in S is unique in the sense of the equivalence relation.

The main result of this section is

Theorem 3.2. *In $S_{2,3}$ there exist four functions $\{M_3^i\}_0^3$ which form a minimal basic system in $S_{2,3}$. And the collection $\{\text{supp } M_3^i\}_0^3$, which are illustrated in Figure 2, is a system of minimal support in $S_{2,3}$.*

To prove Theorem 3.2 we first establish some lemmas.

Lemma 3.1. *Let $P_3 = \text{conv}\{0, 2d_1, 2d_1 + 3d_3, 5d_3, 3d_3 + 2d_2, 2d_2\}$, then*

$$\dim S_{2,3}(P_3) = 3.$$

Proof. By Lemma 2.2, any $f \in S_{2,3}(P_3)$ can be written as

$$\begin{aligned} f = & \sum_{v=0}^{2}\sum_{i=0}^{3} a_{vi}C_i(\cdot-(v,0)) + \sum_{v=0}^{2}\sum_{i=0}^{3} b_{vi}C_i(\cdot-(v+1,1)) \\ & + \sum_{v=0}^{2}\sum_{i=0}^{3} c_{vi}C_i(\cdot-(v+2,2)) + \sum_{v=0}^{2}\sum_{i=0}^{3} d_{vi}C_i(\cdot-(v+3,3)) \\ & + \sum_{v=0}^{1}\sum_{i=0}^{3} e_{vi}C_i(\cdot-(v+4,4)) + \sum_{v=1}^{2}\sum_{i=0}^{3} a^{vi}C_i(\cdot-(0,v)) \\ & + \sum_{v=1}^{2}\sum_{i=0}^{3} b^{vi}C_i(\cdot-(1,v+1)) + \sum_{v=1}^{2}\sum_{i=0}^{3} c^{vi}C_i(\cdot-(2,v+2)) \\ & + \sum_{v=1}^{2}\sum_{i=0}^{3} d^{vi}C_i(\cdot-(3,v+3)) + \sum_{i=0}^{3} e^{1i}C_i(\cdot-(4,v+4)). \end{aligned}$$

Since supp $f \subseteq P_3$, by theorems 2.1–2.4 the coefficients $(a_{vi},b_{vi},c_{vi},d_{vi})_{v=0,i=0}^{2,\ \ 3}$, $(e_{vi})_{v=0,i=0}^{1,\ \ 3}$, $(a^{vi},b^{vi},c^{vi},d^{vi})_{v=1,i=0}^{2,\ \ 3}$ and $(e^{1i})_{i=0}^{3}$ must satisfy the following equations

$$\begin{cases} \sum_{v=0}^{2}\Big[a_{v0}(\cdot-v)^3+3a_{v3}(\cdot-v)^2\Big]\equiv 0; \\ \sum_{v=0}^{2} a_{v1}(\cdot-v)^2\equiv 0; \\ \sum_{v=0}^{2} a_{v2}(\cdot-v)\equiv 0, \end{cases}$$

$$\begin{cases} \sum_{v=-1}^{2}\Big[b_{v0}(\cdot-v)^3+3b_{v3}(\cdot-v)^2\Big]\equiv 0; \\ \sum_{v=-1}^{2} b_{v1}(\cdot-v)^2\equiv 0; \\ \sum_{v=-1}^{2} b_{v2}(\cdot-v)\equiv 0; \\ b_{-1,i}=a^{1,i}, \quad i=\overline{0,3}. \end{cases}$$

$$\begin{cases} \sum_{v=-2}^{2}\Big[c_{v0}(\cdot-v)^3+3c_{v3}(\cdot-v)^2\Big]\equiv 0; \\ \sum_{v=-2}^{2} c_{v1}(\cdot-v)^2\equiv 0; \\ \sum_{v=-2}^{2} c_{v2}(\cdot-v)\equiv 0; \\ c_{-1,i}=b^{1,i}, \quad c_{-2,i}=a^{2,i}, \quad i=\overline{0,3}, \end{cases}$$

$$\begin{cases} \sum_{v=-2}^{2}\Big[d_{v0}(\cdot-v)^3+3d_{v3}(\cdot-v)^2\Big]\equiv 0; \\ \sum_{v=-2}^{2} d_{v1}(\cdot-v)^2\equiv 0; \\ \sum_{v=-2}^{2} d_{v2}(\cdot-v)\equiv 0; \\ d_{-1,i}=c^{1,i}, \quad d_{-2,i}=b^{2,i}, \quad i=\overline{0,3}, \end{cases}$$

$$
\begin{cases}
(a_{21}-a_{22})(\cdot)^3+3a_{23}(\cdot)^2+(b_{21}-b_{22})(\cdot-1)^3+3b_{23}(\cdot-1)^2\\
+(c_{21}-c_{22})(\cdot-2)^3+3c_{23}(\cdot-2)^2+(d_{21}-d_{22})(\cdot-3)^3+3d_{23}(\cdot-3)^2\equiv 0;\\
(a_{20}-a_{22})(\cdot)^2+(b_{20}-b_{22})(\cdot-1)^2+(c_{20}-c_{22})(\cdot-2)^2+(d_{20}-d_{22})(\cdot-3)^2\equiv 0;\\
a_{20}(\cdot)+b_{20}(\cdot-1)+c_{20}(\cdot-2)+d_{20}(\cdot-3)\equiv 0,
\end{cases}
$$

$$
\begin{cases}
(a_{11}-a_{12})(\cdot)^3+3a_{13}(\cdot)^2+(b_{11}-b_{12})(\cdot-1)^3+3b_{13}(\cdot-1)^2\\
+(c_{11}-c_{12})(\cdot-2)^3+3c_{13}(\cdot-2)^2+(d_{11}-d_{12})(\cdot-3)^3+3d_{13}(\cdot-3)^2\\
+(e_{11}-e_{12})(\cdot-4)^3+3e_{13}(\cdot-4)^2\equiv 0;\\
(a_{10}-a_{12})(\cdot)^2+(b_{10}-b_{12})(\cdot-1)^2+(c_{10}-c_{12})(\cdot-2)^2\\
+(d_{10}-d_{12})(\cdot-3)^2+(e_{10}-e_{12})(\cdot-4)^2\equiv 0;\\
a_{10}(\cdot)+b_{10}(\cdot-1)+c_{10}(\cdot-2)+d_{10}(\cdot-3)+e_{10}(\cdot-4)\equiv 0,
\end{cases}
$$

$$
\begin{cases}
\sum\limits_{v=0}^{2}a^{v2}(\cdot-v)^3+3a^{v3}(\cdot-v)^2\equiv 0;\\
\sum\limits_{v=0}^{2}(a^{v0}+a^{v1}-a^{v2})(\cdot-v)^2\equiv 0;\\
\sum\limits_{v=0}^{2}(a^{v1}-a^{v2})(\cdot-v)\equiv 0;\\
a^{0,i}=a_{0,i},\quad i=\overline{0,3},
\end{cases}
\qquad
\begin{cases}
\sum\limits_{v=-1}^{2}b^{v2}(\cdot-v)^3+3b^{v3}(\cdot-v)^2\equiv 0;\\
\sum\limits_{v=-1}^{2}(b^{v0}+b^{v1}-b^{v2})(\cdot-v)^2\equiv 0;\\
\sum\limits_{v=-1}^{2}(b^{v1}-b^{v2})(\cdot-v)\equiv 0;\\
b^{-1,i}=a_{1,i},\quad b^{0,i}=b_{0,i},\quad i=\overline{0,3},
\end{cases}
$$

$$
\begin{cases}
\sum\limits_{v=-2}^{2}c^{v2}(\cdot-v)^3+3c^{v3}(\cdot-v)^2\equiv 0;\\
\sum\limits_{v=-2}^{2}(c^{v0}+c^{v1}-c^{v2})(\cdot-v)^2\equiv 0;\\
\sum\limits_{v=-2}^{2}(c^{v1}-c^{v2})(\cdot-v)\equiv 0;\\
c^{-2,i}=a_{2,i},\quad c^{-1,i}=b_{1,i},\quad c^{0,i}=c_{0,i},\quad i=\overline{0,3},
\end{cases}
$$

$$\begin{cases} \sum_{v=-2}^{2} d^{v2}(\cdot-v)^3+3d^{v3}(\cdot-v)^2\equiv 0; \\ \sum_{v=-2}^{2} (d^{v0}+d^{v1}-d^{v2})(\cdot-v)^2\equiv 0; \\ \sum_{v=-2}^{2} (d^{v1}-d^{v2})(\cdot-v)\equiv 0; \\ d^{-2,i}=b_{2,i}, \quad d^{-1,i}=c_{1,i}, \quad d^{0,i}=d_{0,i}, \quad i=\overline{0,3}, \end{cases}$$

$$\begin{cases} \sum_{v=-2}^{1} e^{v2}(\cdot-v)^3+3e^{v3}(\cdot-v)^2\equiv 0; \\ \sum_{v=-2}^{1} (e^{v0}+e^{v1}-e^{v2})(\cdot-v)^2\equiv 0; \\ \sum_{v=-2}^{1} (e^{v1}-e^{v2})(\cdot-v)\equiv 0; \\ e^{-2,i}=c_{2,i}, e^{-1,i}=d_{1,i}, e^{0,i}=e_{0,i}, \quad i=\overline{0,3}, \end{cases}$$

$$\begin{cases} (a^{21}-a^{22})(\cdot)^3+3a^{23}(\cdot)^2+(b^{21}-b^{22})(\cdot-1)^3+3b^{23}(\cdot-1)^2 \\ +(c^{21}-c^{22})(\cdot-2)^3+3c^{23}(\cdot-2)^2+(d^{21}-d^{22})(\cdot-3)^3+3d^{23}(\cdot-3)^2\equiv 0; \\ (a^{20}-a^{22})(\cdot)^2+(b^{20}-b^{22})(\cdot-1)^2+(c^{20}-c^{22})(\cdot-2)^2+(d^{20}-d^{22})(\cdot-3)^2\equiv 0; \\ a^{20}(\cdot)+b^{20}(\cdot-1)+c^{20}(\cdot-2)+d^{20}(\cdot-3)\equiv 0, \end{cases}$$

$$\begin{cases} (a^{11}-a^{12})(\cdot)^3+3a^{13}(\cdot)^2+(b^{11}-b^{12})(\cdot-1)^3+3b^{13}(\cdot-1)^2 \\ +(c^{11}-c^{12})(\cdot-2)^3+3c^{13}(\cdot-2)^2+(d^{11}-d^{12})(\cdot-3)^3+3d^{13}(\cdot-3)^2 \\ +(e^{11}-e^{12})(\cdot-4)^3+3e^{13}(\cdot-4)^2\equiv 0; \\ (a^{10}-a^{12})(\cdot)^2+(b^{10}-b^{12})(\cdot-1)^2+(c^{10}-c^{12})(\cdot-2)^2 \\ +(d^{10}-d^{12})(\cdot-3)^2+(e^{10}-e^{12})(\cdot-4)^2\equiv 0; \\ a^{10}(\cdot)+b^{10}(\cdot-1)+c^{10}(\cdot-2)+d^{10}(\cdot-3)+e^{10}(\cdot-4)\equiv 0. \end{cases}$$

After complicated computations, we find out that all equations above constitute 89 independent conditions. Thus

$$\dim S_{2,3}(P_3)=23\times 4-89=3.$$

Lemma 3.2. *Let* $T_1 = \text{conv }\{0,2d_1,2d_1+d_3,d_3\}\cup \text{conv }\{2d_2,2d_2+d_3,d_3,0\}$, $Y_1=\{f|_{T_1}; f\in S_{2,3}, \text{supp } f\subseteq\{x(i)\geq 1, i=1,2\}\cup T_1\}$. *Then*

$$\dim Y_1=2.$$

Proof. Since the method is analogous to that employed in the proof of Theorem 2.1, we only sketch the main points.

Let $\theta_1 = \text{conv}\,\{2d_1, 3d_1, 2d_1 + d_3\}, \theta_2 = \text{conv}\,\{2d_2, 3d_2, 2d_2 + d_3\}$. Then

$$Y_1 = \left\{ f \in \text{ span } \left((C_i(-(n,0))\Big|_{T_1})_{n=0,i=0}^{2\ \ 3}(C_i(-(0,m))\Big|_{T_1})_{m=1,i=0}^{2\ \ 3}\right) f\Big|_{\theta_i} = 0, i = 1,2. \right\}.$$

Thus for any $f \in Y_1$ we have

$$f = \sum_{n=0}^{2}\sum_{i=0}^{3} a_{ni}C_i(\cdot - (n,0)) + \sum_{m=1}^{2}\sum_{i=0}^{3} b_{mi}C_i(\cdot - (0,m)).$$

$f|_{\theta_1} = 0$ and $f|_{\theta_2} = 0$ imply

$$\begin{cases} \sum_{n=0}^{2} a_{n0}(\cdot - n)^3 + 3a_{n3}(\cdot - n)^2 \equiv 0; \\ \sum_{n=0}^{2} a_{n1}(\cdot - n)^2 \equiv 0; \\ \sum_{n=0}^{2} a_{n2}(\cdot - n) \equiv 0, \end{cases} \tag{3.1}$$

$$\begin{cases} (a_{01} - a_{02})(\cdot) + \sum_{m=1}^{2} (b_{m1} - b_{m2})(\cdot - m) \equiv 0; \\ (a_{00} + a_{01} - a_{02})(\cdot)^2 + \sum_{m=1}^{2} (b_{m0} + b_{m1} - b_{m2})(\cdot - m)^2 \equiv 0; \\ a_{02}(\cdot)^3 + 3a_{03}(\cdot)^2 + \sum_{m=1}^{2} b_{m2}(\cdot - m)^3 + 3b_{m3}(\cdot - m)^2 \equiv 0. \end{cases} \tag{3.2}$$

Equations (3.1) and (3.2) constitute eighteen independent conditions. Therefore

$$\dim Y_1 = 20 - 18 = 2.$$

The lemma is proven.

The solution to equations (3.1) and (3.2) is useful in the following. For convenience, we write out the solution as follows:

$$\begin{cases} a_{00} = a_{02}, a_{01} = 0, a_{02} = a_{02}, a_{03} = a_{03}; \\ a_{10} = 4a_{02} + 12a_{03}, a_{11} = 0, a_{12} = -2a_{02}, a_{13} = -4a_{02} - 8a_{03}; \\ a_{20} = -5a_{02} - 12a_{03}, a_{21} = 0, a_{22} = a_{02}, a_{23} = -2a_{02} - 5a_{03}; \\ b_{10} = -2a_{02}, b_{11} = 6a_{02} + 12a_{03}, b_{12} = 4a_{02} + 12a_{03}, b_{13} = -4a_{02} - 8a_{03}; \\ b_{20} = a_{02}, b_{21} = -6a_{02} - 12a_{03}, b_{22} = -5a_{02} - 12a_{03}, b_{23} = -2a_{02} - 5a_{03}. \end{cases} \tag{3.3}$$

Lemma 3.3. *Let* $P_2 = \text{conv}\ \{0, 2d_1, 2d_1 + 2d_3, 4d_3, 2d_2 + 2d_3, 2d_2\}$. *Then*

$$\dim S_{2,3}(P_2) = 1.$$

Proof. From [1] we know that there exists a spline $s \in S_{2,2}$ whose support coincides with the support of $M_{1,1,1}$. Thus $S * M_{1,1,1} \in S_{2,3}$ and supp $(s * M_{1,1,1}) = P_2$. This implies

$$\dim S_{2,3}(P_2) \geq 1.$$

On the other hand, from Lemma 3.1 we have

$$\dim S_{2,3}(P_2) \leq 1$$

for, if , on the contrary, $\dim S_{2,3}(P_2) \geq 2$, then we would have

$$\dim S_{2,3}(P_3) \geq 4.$$

This leads to a contradiction. Lemma 3.3 is proven.

Let $A_0 = \text{conv}\ \{0, d_1, d_3, d_2\}$. A direct calculation shows that there exists a function $M_3^3 \in S_{2,3}(P_2)$ such that

$$M_3^3\Big|_{A_0} = C_3(\cdot) - 2C_0(\cdot) - 2C_2(\cdot).$$

By lemma 3.1 and (3.3) we can find $s^0 \in S_{2,3}(P_3)$ such that

$$S^0|_{A_0} = C_3(\cdot).$$

Let $P_3^1 = \text{conv}\ \{0, 2d_1, 2d_1+2d_3, 4d_3+d_2, 2d_3+3d_2, 3d_2\}$, and $P_3^2 = \text{conv}\ \{0, 2d_2, 2d_2+2d_3, 4d_3+d_1, 2d_3+3d_1, 3d_1\}$.

By symmetry there exist two splines s^1 and s^2 in $S_{2,3}(P_3^1)$ and $S_{2,3}(P_3^2)$, respectively, such that

$$s^1|_{A_0} = -12C_1(\cdot) - 12C_2(\cdot) + 5C_3(\cdot),$$
$$s^2|_{A_0} = -12C_0(\cdot) + 5C_3(\cdot).$$

Thus s^0, s^1, s^2 and M_3^3 are linearly independent over A_0.

We can choose $M_3^0 \in S_{2,3}(P_3)$ such that

$$M_3^0(x) = a_0 M_3^0(5 - x(1), 5 - x(2)), \quad x \in A_0, a_0 \neq 0.$$

Indeed, if $s^0(x) \not\equiv a\ s^0(5 - x(1), 5 - x(2))$, $x \in A_0$, for all $a \in R$, since $s^0(x)|_{A_0}$, $s^0(5 - x(1), 5 - x(2))|_{A_0}$ and $M_3^3(x)|_{A_0}$ are linearly dependent, there exist $\{c_i\}_1^3 \subset R \setminus \{0\}$ such that

$$c_1 s^0(x) + c_2 s^0(5 - x(1), 5 - x(2)) + c_3 M_3^3(x) \equiv 0, \quad x \in A_0.$$

Let

$$M_3^0(x) = s^0(x) + (c_3/c_1) M_3^3(x) \in S_{2,3}(P_3).$$

Then

$$M_3^0(x) = -(c_2/c_1)M_3^0(5 - x(1), 5 - x(2)), \quad x \in A_0.$$

Similarly, $M_3^1(\in S_{2,3}(P_3^1))$ and $M_3^1(\in S_{2,3}(P_3^2))$ can be chosen such that

$$M_3^1(x) = a_1 M_3^1(2 - x(1), 5 - x(2)),$$

$$M_3^2(x) = a_2 M_3^2(5 - x(1), 2 - x(2)), x \in A_0, a_1 \cdot a_2 \neq 0.$$

In addition, $\{M_3^i\}_0^3$ are linearly independent over A_0.

Lemma 3.4. *Let $f \in S_{2,3}$,* supp$f$ $\subseteq R_+^2 \cap [0,m]^2$, *and* supp$f$ *lie between the rays* $2d_1 + R_+ d_3$ *and* $2d_2 + R_+ d_3$. *Then*

$$f = \sum_{i=0}^{m-5} b_i^0 M_3^0(\cdot - (i,i)) + \sum_{i=0}^{m-4} b_i^3 M_3^3(\cdot - (i,i)).$$

The lemma is easily obtained from Lemmas 3.2 and 3.3.

Once having Lemmas 3.1–3.4, we can utilize the same technique as employed in [1] to obtain the following

Lemma 3.5. *For any function ϕ in $S_{2,3}$ with a compact support there exists a unique linear combination of functions in* $\bigcup_{i=0}^{3} \bigcup_{v \in V_i} M_3^i(\cdot - v)$, *i.e.,*

$$\phi = \sum_{i=0}^{3} \sum_{v \in V_i} a_{i,v} M_3^i(\cdot - v)$$

where V_i is a finite subset of $Z^2, i = \overline{0,3}$. Furthermore, if supp ϕ *is convex,*

$$\bigcup_{i=0}^{3} \bigcup_{\substack{v \in V \\ a_{i,v} \neq 0}} \text{supp } M_3^i(\cdot - v) \subseteq \text{supp } \phi.$$

Now Theorem 3.2 easily follows from the linear independence of $\{M_3^i\}_0^3$ and Lemma 3.5.

Let $\{N_3^i\}_0^3$ be the four splines obtained from the four "minimal support" splines in $S_{2,2}$ by convolution with $M_{1,1,1}$. Their supports are shown in Figure 3. By the uniqueness of the minimal basic system, we can readily know $\{N_3^i\}_0^3$ fails to be a minimal basic system in $S_{2,3}$.

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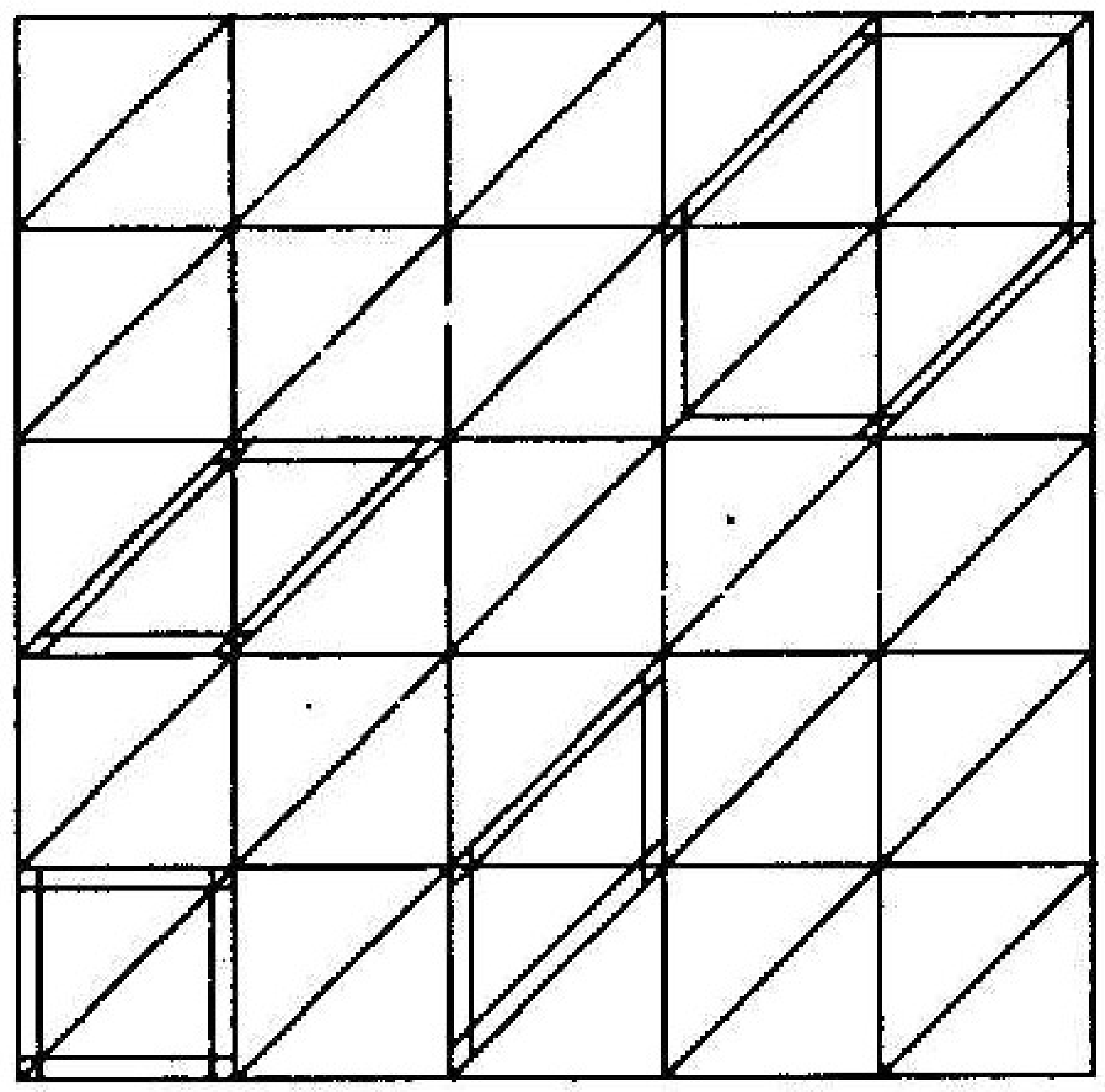
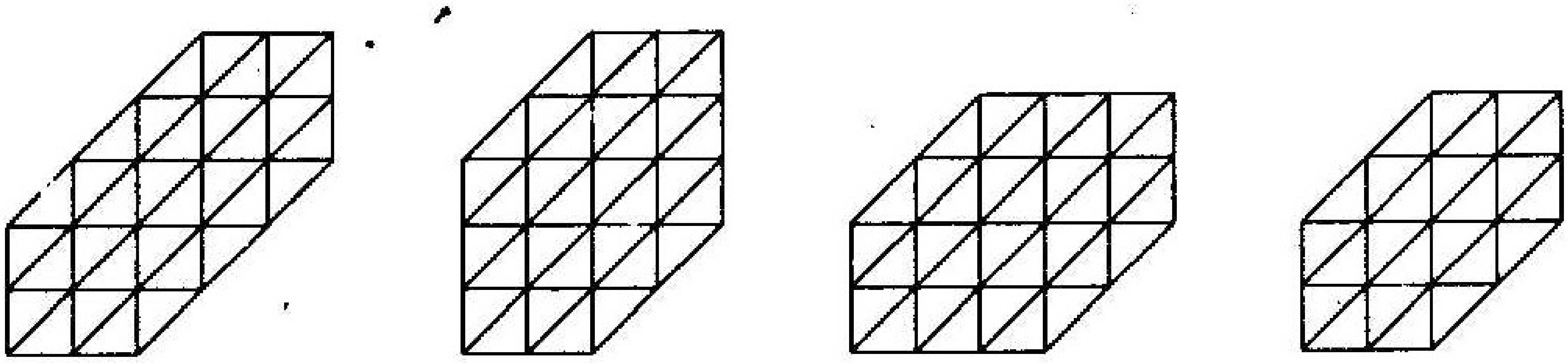
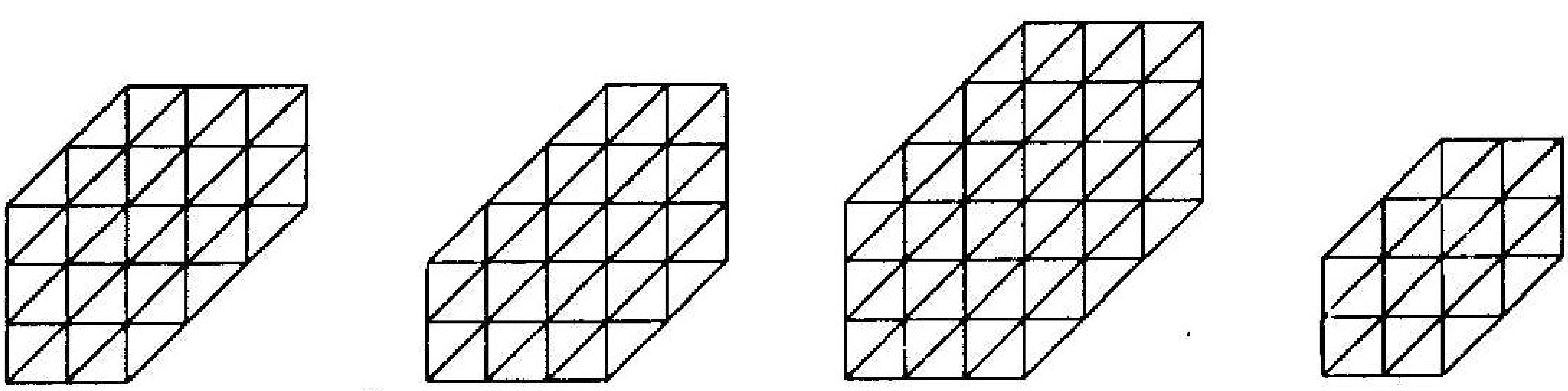


Figure 1



supp M_3^0 supp M_3^1 supp M_3^2 supp M_3^3

Figure 2



supp N_3^0 supp N_3^1 supp N_3^2 supp N_3^3

Figure 3