ASYMPTOTIC EXPANSIONS OF THE CUBIC SPLINE COLLOCATION SOLUTION FOR SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS*

HUANG YOU-QIAN (黄友谦) HAN GUO-QIANG (韩国强)
(Department of Computer Science, Zhongshan University, Guangzhou, China)

Abstract

In this paper, we consider the following problem

$$\begin{cases} u''(x)=f(x, u(x), u'(x)), & a\leqslant x\leqslant b, \\ u(a)=u(b)=0, \end{cases}$$

and obtain the following theorem.

Theorem. *Suppose that $s(x)$ is a unique collocation solution of the above equation. The solution $u(x)$ of the above equation exists uniquely and $u(x)\in C^{r+2}[a, b]$, $f(x, y, z)\in C^r([a, b]\times(-\infty, \infty)\times(-\infty,\infty))$, $l=\left[\frac{r-1}{2}\right]$, $\frac{\partial f}{\partial y}\geqslant 0$. Then*

$$s(x_i)=u(x_i)+\sum_{j=1}^{l} h^{2j}e_j(x_i)+O(h^r), \quad i=0, 1, \cdots, N,$$

where $e_j(x)$ are solutions of some linear ordinary differential equations.

We consider boundary-value problems of the first kind for second-order differential equations

$$\begin{cases} u''(x)=f(x, u(x), u'(x)), & a\leqslant x\leqslant b, \\ u(a)=u(b)=0, \end{cases} \tag{1}$$

where $u(x)$ is assumed to exist uniquely and $u(x)\in C^{r+2}[a, b]$.

$$f(x, y, z)\in C^r([a, b]\times(-\infty, \infty)\times(-\infty, \infty)), \text{ and } \frac{\partial f(x, y, z)}{\partial y}\geqslant 0.$$

Let Δ be an equidistant partition of $[a, b]$

$$\Delta: \ a=x_0<x_1<\cdots<x_N=b$$

and let $SP(3, \Delta)$ be a cubic spline space.

Definition. *If there is a unique $s(x)\in SP(3, \Delta)$ such that*

$$s(a)=s(b)=0,$$

$$s''(x_i)=f(x_i, s(x_i), s'(x_i)), \quad i=0, 1, \cdots, N,$$

then $s(x)$ is called the cubic spline collocation solution of boundary value problems of (1).

In the following, we always assume that collocation solutions exist uniquely.

Let
$$s(x)=\sum_{i=-1}^{N+1}\alpha_i B_i(x),$$

where $B_i(x)$ are B-Splines of a cubic spline. By using the values of $B_i(x_j)$, $B'_i(x_j)$,

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$B_i''(x_j)$ we have a system of nonlinear equations.

$$
\begin{cases}
\alpha_{-1}+4\alpha_0+\alpha_1=0, \\
\dfrac{\alpha_{i+1}-2\alpha_i+\alpha_{i-1}}{h^2}=f\left(x_i,\ \dfrac{\alpha_{i+1}+4\alpha_i+\alpha_{i-1}}{6},\ \dfrac{\alpha_{i+1}-\alpha_{i-1}}{2h}\right), \quad i=0,\ 1,\ \cdots,\ N, \\
\alpha_{N-1}+4\alpha_N+\alpha_{N+1}=0.
\end{cases}
$$

Cancelling α_{-1}, α_{N+1} we obtain

$$
\begin{cases}
-6\alpha_0=h^2 f\left(x_0,\ u_0,\ \dfrac{\alpha_1+2\alpha_0}{h}\right), \\
\dfrac{\alpha_{i+1}-2\alpha_i+\alpha_{i-1}}{h^2}=f\left(x_i,\ \dfrac{\alpha_{i+1}+4\alpha_i+\alpha_{i-1}}{6},\ \dfrac{\alpha_{i+1}-\alpha_{i-1}}{2h}\right), \quad i=1,\ 2,\ \cdots,\ N-1, \\
-6\alpha_N=h^2 f\left(x_N,\ u_N,\ -\dfrac{\alpha_{N-1}+2\alpha_N}{h}\right).
\end{cases} \tag{2}
$$

Solving this system by Newton's iteration we can obtain the collocation solution $s(x)$.

For simplicity let us assume

$$
l=\left[\frac{r-1}{2}\right] \quad \text{and} \quad 0!=\frac{1}{3}.
$$

For any function $d_j(x)\in C^{r+2-2j}[a,\ b]\ (j=0,\ 1,\ \cdots,\ l)$, if we take $\beta_i=\sum\limits_{j=0}^{l} h^{2j}d_j(x_i)$ $(i=0,\ 1,\ \cdots,\ N)$, then for $i=1,\ 2,\ \cdots,\ N-1$, from Taylor's formula we have

$$
\frac{\beta_{i+1}-2\beta_i+\beta_{i-1}}{h^2}=\sum_{j=0}^{l} h^{2j}\sum_{k=0}^{j}\frac{2\cdot d_{j-k}^{(2k+2)}(x_i)}{(2k+2)!}+O(h^r),
$$

$$
\frac{\beta_{i+1}+4\beta_i+\beta_{i-1}}{6}=\sum_{j=0}^{l} h^{2j}\sum_{k=0}^{j}\frac{d_{j-k}^{(2k)}(x_i)}{3\cdot(2k)!}+O(h^r),
$$

$$
\frac{\beta_{i+1}-\beta_{i-1}}{2h}=\sum_{j=0}^{l} h^{2j}\sum_{k=0}^{j}\frac{d_{j-k}^{(2k+1)}(x_i)}{(2k+1)!}+O(h^r),
$$

$$
\begin{aligned}
& f\left(x_i,\ \frac{\beta_{i+1}+4\beta_i+\beta_{i-1}}{6},\ \frac{\beta_{i+1}-\beta_{i-1}}{2h}\right) \\
&=f(x_i,\ d_0(x_i),\ d_0'(x_i))+\sum_{1<s+m<l}\frac{1}{s!\,m!}\,\frac{\partial^{s+m}}{\partial y^s\partial z^m}f(x_i,\ d_0(x_i),\ d_0'(x_i)\,) \\
&\quad\times\left(\sum_{j=1}^{l}h^{2j}\sum_{k=0}^{j}\frac{d_{j-k}^{(2k+1)}(x_i)}{(2k+1)!}+O(h^r)\right)^m \\
&\quad\times\left(\sum_{j=1}^{l}h^{2j}\sum_{k=0}^{j}\frac{d_{j-k}^{(2k)}(x_i)}{3\cdot(2k)!}+O(h^r)\right)^s+O(h^r) \\
&=f(x_i,\ d_0(x_i),\ d_0'(x_i))+\sum_{1<s+m<l}\frac{1}{s!\,m!}\,\frac{\partial^{s+m}}{\partial y^s\partial z^m}f(x_i,\ d_0(x_i),\ d_0'(x_i)) \\
&\quad\times\sum_{j=s+m}h^{2j}\left\{\sum_{\substack{t_1+\cdots+t_{s+m}=j \\ t_r\geq 1}}\prod_{n=1}^{s}\left(\sum_{k=0}^{t_n}\frac{d_{t_n-k}^{(2k)}(x_i)}{3\cdot(2k)!}\right)\prod_{n=s+1}^{s+m}\left(\sum_{k=0}^{t_n}\frac{d_{t_n-k}^{(2k+1)}(x_i)}{(2k+1)!}\right)\right\}+O(h^r) \\
&=f(x_i,\ d_0(x_i),\ d_0'(x_i))+\sum_{j=1}^{l}h^{2j}\sum_{1<s+m<j}\frac{1}{s!\,m!}\,\frac{\partial^{s+m}}{\partial y^s\partial z^m}f(x_i,\ d_0(x_i),\ d_0'(x_i)) \\
&\quad\times\left\{\sum_{\substack{t_1+\cdots+t_{s+m}=j \\ t_r\geq 1}}\prod_{n=1}^{s}\left(\sum_{k=0}^{t_n}\frac{d_{t_n-k}^{(2k)}(x_i)}{3\cdot(2k)!}\right)\cdot\prod_{n=s+1}^{s+m}\left(\sum_{k=0}^{t_n}\frac{d_{t_n-k}^{(2k+1)}(x_i)}{(2k+1)!}\right)\right\}+O(h^r),
\end{aligned} \tag{3}
$$

where $g^{(r)}(x_i)=\dfrac{d^r g(x_i)}{dx^r}$.

Comparing the coefficients of $h^2, h^4, \cdots, h^{2l}$, we shall see that if we choose

$$d_0(x)=u(x)$$

and if for $j=1, 2, \cdots, l$, $d_j(x)$ satisfy differential equations

$$\begin{aligned}
&d_j''(x)-\frac{\partial f}{\partial z}(x, u(x), u'(x))d_j'(x)-\frac{\partial f}{\partial y}(x, u(x), u'(x))\cdot d_j(x)\\
&=-\sum_{k=1}^{j}\frac{2\cdot d_{j-k}^{(2k+2)}(x)}{(2k+2)!}+\sum_{2<s+m<j}\frac{1}{s!m!}\frac{\partial^{s+m}f}{\partial y^s\partial z^m}(x, u(x), u'(x))\\
&\quad\times\Big\{\sum_{\substack{t_1+\cdots+t_{s+m}=j\\ t_r\geq 1}}\prod_{n=1}^{s}\Big(\sum_{k=0}^{t_n}\frac{d_{t_n-k}^{(2k)}(x)}{3\cdot(2k)!}\Big)\prod_{n=s+1}^{s+m}\Big(\sum_{k=0}^{t_n}\frac{d_{t_n-k}^{(2k+1)}(x)}{(2k+1)!}\Big)\Big\}\\
&\quad+\frac{\partial f}{\partial z}(x, u(x), u'(x))\cdot\sum_{k=1}^{j}\frac{d_{j-k}^{(2k+1)}(x)}{(2k+1)!}+\frac{\partial f}{\partial y}(x, u(x), u'(x))\cdot\sum_{k=1}^{j}\frac{d_{j-k}^{(2k)}(x)}{3\cdot(2k)!},
\end{aligned}\tag{4}$$

then

$$\frac{\beta_{i+1}-2\beta_i+\beta_{i-1}}{h^2}-f\Big(x_i, \frac{\beta_{i+1}+4\beta_i+\beta_{i-1}}{6}, \frac{\beta_{i+1}-\beta_{i-1}}{2h}\Big)=O(h^r),$$
$$i=1, 2, \cdots, N-1. \tag{5}$$

At boundary points we have

$$\frac{2\beta_0+\beta_1}{h}=\sum_{j=1}^{l}h^{2j-1}\sum_{k=0}^{j}\frac{d_k^{(2j-2k)}(x_0)}{(2j-2k)!}+\sum_{j=0}^{l}h^{2j}\sum_{k=0}^{j}\frac{d_{j-k}^{(2k+1)}(x_0)}{(2k+1)!}+O(h^r).$$

So if we assume

$$\sum_{k=0}^{j}\frac{d_{j-k}^{(2k)}(x_0)}{(2k)!}=0$$

or

$$d_j(x_0)=-\frac{1}{3}\sum_{k=1}^{j}\frac{d_{j-k}^{(2k)}(x_0)}{(2k)!}, \quad j=1, 2, \cdots, l, \tag{6}$$

then

$$\frac{2\beta_0+\beta_1}{h}=\sum_{j=0}^{l}h^{2j}\sum_{k=0}^{j}\frac{d_{j-k}^{(2k+1)}(x_0)}{(2k+1)!}+O(h^r),$$

$$\begin{aligned}
f\Big(x_0, u_0, \frac{\beta_1+2\beta_0}{h}\Big)&=f\Big(x_0, \sum_{j=1}^{l}h^{2j}\sum_{k=0}^{j}\frac{d_{j-k}^{(2k)}(x_0)}{3\cdot(2k)!}, \sum_{j=0}^{l}h^{2j}\sum_{k=0}^{j}\frac{d_{j-k}^{(2k+1)}(x_0)}{(2k+1)!}+O(h^r)\Big)\\
&=f(x_0, u_0, u_0')+\sum_{j=1}^{l}h^{2j}\sum_{1<s+m<j}\frac{1}{s!m!}\frac{\partial^{s+m}}{\partial y^s\partial z^m}f(x_0, u_0, u_0')\\
&\quad\times\Big\{\sum_{\substack{t_1+\cdots+t_{s+m}=j\\ t_r\geq 1}}\prod_{n=1}^{s}\Big(\sum_{k=0}^{t_n}\frac{d_{t_n-k}^{(2k)}(x_0)}{3\cdot(2k)!}\Big)\\
&\quad\times\prod_{n=s+1}^{s+m}\Big(\sum_{k=0}^{t_n}\frac{d_{t_n-k}^{(2k+1)}(x_0)}{(2k+1)!}\Big)\Big\}+O(h^r).
\end{aligned}$$

In addition, $d_j(x)$ $(j=1, 2, \cdots, l)$ satisfy equations (4). So we have

$$6\beta_0+h^2f\left(x_0,\ u_0,\ \frac{2\beta_0+\beta_1}{h}\right)$$
$$=6\sum_{j=0}^{l}h^{2j}d_j(x_0)+h^2f\left(x_0,\ u_0,\ \frac{2\beta_0+\beta_1}{h}\right)$$
$$=-\sum_{j=1}^{l}h^{2j}\sum_{k=1}^{j}\frac{2d_{j-k}^{(2k)}(x_0)}{(2k)\,!}+h^2f\left(x_0,\ u_0,\ \frac{2\beta_0+\beta_1}{h}\right)=O(h^r).$$

Similarly, let

$$d_j(x_N)=-\frac{1}{3}\sum_{k=1}^{j}\frac{d_{j-k}^{(2k)}(x_N)}{(2k)\,!}. \tag{7}$$

Then

$$6\beta_N+h^2f\left(x_N,\ u_N,\ -\frac{2\beta_N+\beta_{N-1}}{h}\right)=O(h^r).$$

Lemma. *For equation*

$$\begin{cases}-(p(x)v'(x))'+q(x)v(x)=R(x), & x\in[a,\ b],\\ v(b)=v(a)=0\end{cases}$$

suppose $p(x)\geqslant c_1>0$, $q(x)\geqslant 0$, $x\in[a,\ b]$, $q(x)$, $R(x)\in C^r[a,\ b]$, $p(x)\in C^{r+1}[a,\ b]$. *Then there is a unique solution* $v(x)\in C^{r+2}[a,\ b]$.

Corollary. For $j=1,\ 2,\ \cdots,\ l$, there is a unique function $d_j(x)\in C^{r+2-2j}\ [a,\ b]$ and $d_j(x)$ satifies (4), (6) and (7).

Proof. The first step is to find the function $d_1(x)$. Equations (4) for this function have a simple form

$$-d_1''(x)+\frac{\partial f}{\partial z}(x,\ u(x),\ u'(x))d_1'(x)+\frac{\partial f}{\partial y}(x,\ u(x),\ u'(x))d_1(x)$$
$$=-\frac{1}{12}u^{(4)}(x)+\frac{1}{6}\ \frac{\partial f}{\partial z}(x,\ u(x),\ u'(x))\cdot u'''(x)$$
$$+\frac{1}{6}\ \frac{\partial f}{\partial y}(x,\ u(x),\ u'(x))\cdot u''(x).$$

The right-hand side of this equation contains only the function u and is $r-2$ times continuously differentiable. Since the equation is linear and the coefficients are $r-2$ times continuously differentiable, there is a solution satisfying the boundary conditions (6) and (7), which is unique and belongs to $C^r[a,\ b]$. Let us assume that the functions $d_0,\ \cdots,\ d_{j-1}$ (with $d_k\in C^{r+2-2k}[a,\ b]$) have been already determined. From (4) we see that the right-hand side of the equation for d_j does not contain functions d_k whose index k is greater than $j-1$, and is $r-2j$ times continuously differentiable on the interval $[a,\ b]$. The latter follows from a simple calculation on the smoothness of d_k and the number of the derivatives one may take. The coefficients of the linear equation obtained are functions of $C^{r-1}[a,\ b]$; therefore there is a unique solution satisfying the boundary conditions (6), (7) as well as equations (4). This solution will belong to $C^{r+2-2j}[a,\ b]$. Thus, all $l+1$ functions d_j have been found and satisfy the condition

$$d_j(x)\in C^{r+2-2j}[a,\ b],\quad j=0,\ 1,\ 2,\ \cdots,\ l.$$

Theorem 1. *If* $d_0(x)=u(x)$, $d_j(x)$ *satisfy equations* (4), (6) *and* (7) for $j=1$, 2, $\cdots$, l, *and*

$$\beta_i = \sum_{j=0}^{l} h^{2j} d_j(x_i), \quad i=0, 1, \cdots, N,$$

then

$$\begin{cases} 6\beta_0 + h^2 f\left(x_0, u_0, \dfrac{2\beta_0+\beta_1}{h}\right) = O(h^r), \\ \dfrac{\beta_{i+1}-2\beta_i+\beta_{i-1}}{h^2} - f\left(x_i, \dfrac{\beta_{i+1}+4\beta_i+\beta_{i-1}}{6}, \dfrac{\beta_{i+1}-\beta_{i-1}}{2h}\right) = O(h^r), \\ \qquad i=1, 2, \cdots, N-1, \\ 6\beta_N + f\left(x_N, u_N, -\dfrac{2\beta_N+\beta_{N-1}}{h}\right) = O(h^r). \end{cases} \tag{8}$$

Theorem 2. *Suppose that* $s(x) = \sum_{i=-1}^{N+1} \alpha_i B_i(x)$ *is a unique collocation solution of problem* (1), $u(x) \in C^{r+2}[a, b]$,

$$f(x, y, z) \in C^r([a, b] \times (-\infty, \infty) \times (-\infty, \infty)) \text{ and } \frac{\partial f(x, y, z)}{\partial y} \geqslant 0.$$

Then α_i *can be expanded as*

$$\alpha_i = u(x_i) + \sum_{j=1}^{l} h^{2j} d_j(x_i) + O(h^r), \quad i=0, 1, \cdots, N,$$

where $d_j(x) \in C^{r+2-2j}[a,b]$ ($j=0, 1, 2, \cdots, l$) *are the solutions of equations* (4), (6) *and* (7).

Proof. For $i=0, 1, \cdots, N$, let

$$\gamma_i = \alpha_i - \beta_i.$$

Subtracting (2) from (8) and then applying the mean value formula we obtain

$$\begin{cases} 6\gamma_0 + h\dfrac{\partial f}{\partial z}(x_0, u_0, \varphi_0)(2\gamma_0+\gamma_1) = O(h^r), \\ \dfrac{\gamma_{i+1}-2\gamma_i+\gamma_{i-1}}{h^2} - \dfrac{\partial f}{\partial z}(x_i, \theta_i, \varphi_i)\dfrac{\gamma_{i+1}-\gamma_{i-1}}{2h} - \dfrac{\partial f}{\partial y}(x_i, \theta_i, \varphi_i)\dfrac{\gamma_{i+1}+4\gamma_i+\gamma_{i-1}}{6} \\ \quad = O(h^r), \quad i=1, 2, \cdots, N-1, \\ 6\gamma_N - h\dfrac{\partial f}{\partial z}(x_N, u_N, \varphi_N)(2\gamma_N+\gamma_{N-1}) = O(h^r). \end{cases}$$

By using the maximum principle for difference we have

$$|\gamma_i| \leqslant ch^r.$$

So

$$\alpha_i = u(x_i) + \sum_{j=1}^{l} h^{2j} d_j(x_i) + O(h^r), \quad i=0, 1, \cdots, N.$$

The proof of the theorem is thus completed.

Remark 1. When $j=1$, $d_1(x)$ satisfies the equation

$$\begin{cases} d_1''(x)-\dfrac{\partial f}{\partial z}(x,\ u(x),\ u'(x))d_1'(x)-\dfrac{\partial f}{\partial y}(x,\ u(x),\ u'(x))d_1(x) \\ \quad =-\dfrac{1}{12}u^{(4)}(x)+\dfrac{1}{6}\dfrac{\partial f}{\partial z}(x,\ u(x),\ u'(x))\cdot u'''(x) \\ \qquad +\dfrac{1}{6}\dfrac{\partial f}{\partial y}(x,\ u(x),\ u'(x))\cdot u''(x), \\ d_1(x_0)=-\dfrac{1}{6}u''(x_0),\quad d_1(x_N)=-\dfrac{1}{6}u''(x_N). \end{cases}$$

When $j=2$, $d_2(x)$ satisfies the equation

$$\begin{cases} d_2''(x)-\dfrac{\partial f}{\partial z}(x,\ u(x),\ u'(x))d_2'(x)-\dfrac{\partial f}{\partial y}(x,\ u(x),\ u'(x))d_2(x) \\ \quad =-\dfrac{1}{12}d_1^{(4)}(x)-\dfrac{1}{360}u^{(6)}(x)+\dfrac{1}{6}\dfrac{\partial f}{\partial z}(x,\ u(x),\ u'(x))\cdot d_1^{(3)}(x) \\ \qquad +\dfrac{1}{120}\dfrac{\partial f}{\partial z}(x,\ u(x),\ u'(x))\cdot u^{(5)}(x)+\dfrac{1}{6}\dfrac{\partial f}{\partial y}(x,\ u(x),\ u'(x))d_1''(x) \\ \qquad +\dfrac{1}{72}\dfrac{\partial f}{\partial y}(x,\ u(x),\ u'(x))\cdot u^{(4)}(x) \\ \qquad +\dfrac{1}{2}\dfrac{\partial^2 f}{\partial y^2}(x,\ u(x),\ u'(x))\left(d_1(x)+\dfrac{1}{6}u''(x)\right)^2 \\ \qquad +\dfrac{\partial^2 f}{\partial y\partial z}(x,\ u(x),\ u'(x))\left(d_1(x)+\dfrac{1}{6}u''(x)\right)\cdot\left(d_1'(x)+\dfrac{1}{6}u'''(x)\right) \\ \qquad +\dfrac{1}{2}\dfrac{\partial^2 f}{\partial z^2}(x,\ u(x),\ u'(x))\cdot\left(d_1'(x)+\dfrac{1}{6}u'''(x)\right)^2, \\ d_2(x_0)=-\dfrac{1}{6}d_1''(x_0)-\dfrac{1}{72}u^{(4)}(x_0), \\ d_2(x_N)=-\dfrac{1}{6}d_1''(x_N)-\dfrac{1}{72}u^{(4)}(x_N). \end{cases}$$

Now

$$s(x_i)=\frac{\alpha_{i+1}+4\alpha_i+\alpha_{i-1}}{6}=\sum_{j=0}^{l}h^{2j}\sum_{k=0}^{j}\frac{d_{j-k}^{(2k)}(x_i)}{3\cdot(2k)!}+O(h^r),\quad i=1,\ 2,\ \cdots,N-1,$$

$$s(x_0)=s(x_N)=0.$$

So if we assume

$$e_j(x)=\sum_{k=0}^{j}\frac{d_{j-k}^{(2k)}(x)}{3\cdot(2k)!},\quad j=0,\ 1,\ \cdots,\ l, \tag{9}$$

then

$$s(x_i)=u(x_i)+\sum_{j=1}^{l}h^{2j}e_j(x_i)+O(h^r),\quad i=1,\ 2,\ 3,\ \cdots,\ N-1,$$

$$e_j(x_0)=e_j(x_N)=0.$$

In the following we shall deduce the equation which $e_j(x)$ must satisfy. From (9) we have

$$\begin{pmatrix} 1 & & & 0 \\ \frac{1}{3\cdot 2!} & 1 & & \\ \frac{1}{3\cdot 4!} & \frac{1}{3\cdot 2!} & 1 & \\ & & & \ddots \\ \frac{1}{3\cdot 2j!} & \frac{1}{3\cdot (2j-2)!} & & & 1 \end{pmatrix} \begin{pmatrix} d_0^{(2j+1)}(x) \\ d_1^{(2j-1)}(x) \\ \vdots \\ d_j'(x) \end{pmatrix} = \begin{pmatrix} e_0^{(2j+1)}(x) \\ e_1^{(2j-1)}(x) \\ \vdots \\ e_j'(x) \end{pmatrix}.$$

Let

$$A^{(j)} = \begin{pmatrix} 1 & & & \\ \frac{1}{3\cdot 2!} & 1 & & \\ \frac{1}{3\cdot 4!} & \frac{1}{3\cdot 2!} & 1 & \\ \cdots\cdots & \cdots\cdots & & \ddots \\ \frac{1}{3\cdot 2j!} & \frac{1}{3\cdot (2j-2)!} & \cdots\cdots & 1 \end{pmatrix}^{-1} \triangleq (a_{ik}^{(j)})_{i=0}^{j}{}_{k=0}^{j}.$$

Then

$$\begin{pmatrix} d_0^{(2j+1)}(x) \\ d_1^{(2j-1)}(x) \\ \vdots \\ d_j'(x) \end{pmatrix} = A^{(j)} \begin{pmatrix} e_0^{(2j+1)}(x) \\ e_1^{(2j-1)}(x) \\ \vdots \\ e_j'(x) \end{pmatrix},$$

$$\begin{pmatrix} d_0^{(2j+2)}(x) \\ d_1^{(2j)}(x) \\ \vdots \\ d_j''(x) \end{pmatrix} = A^{(j)} \begin{pmatrix} e_0^{(2j+2)}(x) \\ e_1^{(2j)}(x) \\ \vdots \\ e_j''(x) \end{pmatrix}.$$

So

$$\sum_{k=0}^{j} \frac{d_{j-k}^{(2k+1)}(x)}{(2k+1)!} = \sum_{0<k<t<j} \frac{a_{tk}^{(j)}}{(2j-2t+1)!} e_k^{(2j-2k+1)}(x)$$
$$= e_j'(x) + \sum_{\substack{0<k<t<j \\ k\neq j}} \frac{a_{tk}^{(j)}}{(2j-2t+1)!} e_k^{(2j-2k+1)}(x),$$

$$\sum_{k=0}^{j} \frac{2\cdot d_{j-k}^{(2k+2)}(x)}{(2k+2)!} = \sum_{0<k<t<j} \frac{2a_{tk}^{(j)}}{(2j-2t+2)!} e_k^{(2j-2k+2)}(x)$$
$$= e_j''(x) + \sum_{\substack{0<k<t<j \\ k\neq j}} \frac{2a_{tk}^{(j)}}{(2j-2t+2)!} e_k^{(2j-2k+2)}(x).$$

Substituting the above into (4) we have

$$\begin{cases} e_j''(x) - \dfrac{\partial f}{\partial z}(x, u(x), u'(x))e_j'(x) - \dfrac{\partial f}{\partial y}(x, u(x), u'(x))e_j(x) \\ = -\displaystyle\sum_{\substack{0<k<t<j \\ k\neq j}} \frac{2a_{tk}^{(j)}}{(2j-2t+2)!} e_k^{(2j-2k+2)}(x) + \frac{\partial f}{\partial z}(x, u(x), u'(x)) \\ \times \displaystyle\sum_{\substack{0<k<t<j \\ k\neq j}} \frac{a_{tk}^{(j)}}{(2j-2t+1)!} e_k^{(2j-2k+1)}(x) + \sum_{2<s+m<j} \frac{1}{s!m!} \frac{\partial^{s+m} f}{\partial y^s \partial z^m}(x, u(x), u'(x)) \end{cases}$$

$$
\left.\begin{aligned}
&\times\left\{\sum_{\substack{t_1+\cdots+t_{s+m}=j\\ t_i\geq 1}}\prod_{k=1}^{s}e_{t_k}(x)\prod_{k=s+1}^{s+m}\left(\sum_{0\leq n\leq t\leq t_k}\frac{a_{tn}^{(t_k)}e_n^{(2t_k-2n+1)}(x)}{(2t_k-2t+1)!}\right)\right\},\\
&e_j(x_0)=e_j(x_N)=0.
\end{aligned}\right. \tag{10}
$$

Theorem 3. *Suppose that $s(x)$ is a unique collocation solution of problem* (1), $e_0(x)=u(x)$, *and $e_j(x)$, $j=1, 2, \cdots, l$, satisfy equation* (10). *Then the approximate solution can be expanded as*

$$s(x_i)=u(x_i)+\sum_{j=1}^{l}h^{2j}e_j(x_i)+O(h^r),\quad i=0, 1, 2, \cdots, N.$$

Here $l=\left[\frac{(r-1)}{2}\right]$ and the functions $e_j(x)$ belong to $C^{r+2-2j}[a, b]$.

Remark 2. When $j=1$, $e_1(x)$ satisfies the equation

$$
\begin{cases}
e_1''(x)-\dfrac{\partial f}{\partial z}(x, u(x), u'(x))e_1'(x)-\dfrac{\partial f}{\partial y}(x, u(x), u'(x))e_1(x)=\dfrac{1}{12}u^{(4)}(x),\\
e_1(x_0)=e_1(x_N)=0.
\end{cases}
$$

When $j=2$, $e_2(x)$ satisfies the equation

$$
\begin{cases}
e_2''(x)-\dfrac{\partial f}{\partial z}(x, u(x), u'(x))e_2'(x)-\dfrac{\partial f}{\partial y}(x, u(x)\ u'(x))e_2(x)\\
\quad =\dfrac{1}{12}e_1^{(4)}(x)-\dfrac{1}{360}u^{(6)}(x)-\dfrac{1}{180}\dfrac{\partial f}{\partial z}(x, u(x), u'(x))\cdot u^{(5)}(x)\\
\quad +\dfrac{1}{2}\dfrac{\partial^2 f}{\partial y^2}(x, u(x), u'(x))e_1^2(x)+\dfrac{\partial^2 f}{\partial y\partial z}(x, u(x), u'(x))\cdot e_1(x)\cdot e_1'(x)\\
\quad +\dfrac{1}{2}\dfrac{\partial^2 f}{\partial z^2}(x, u(x), u'(x))\cdot(e_1'(x))^2,\\
e_2(x_0)=e_2(x_N)=0.
\end{cases}
$$

For the boundary-value problems of the third kind, similar results can be obtained. We shall discuss it in another paper.

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