ON THE CARDINALITIES OF INTERPOLATING RESTRICTED RANGES*

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Abstract

The cardinalities of interpolating restricted range $\boldsymbol{R}_1(m, n)$ for best rational approximation are discussed. The conditions under which the cardinality of $\boldsymbol{R}_1(m, n)$ is 0 or 1 or ∞ are established by using the results obtained in [3, 4].

The problem of best rational approximation with interpolating constraints proposed by Wang (cf. [1], p. 69) can be transformed into an equivalent form by means of Leibniz Rule for derivatives. The problem can be stated as follows.

Assume that $f\in C[a, b]$, $w\in C^t[a, b]$ with $w>0$, and $\boldsymbol{R}(m, n)$ is the class of rational functions

$$\boldsymbol{R}(m, n)=\{R:\ R=P/Q,\ P\in H_m,\ Q\in H_n\backslash\{0\}\}.$$

Let a set of $s+1$ distinct points in $[a, b]$

$$a\leqslant x_0<x_1<\cdots<x_s\leqslant b$$

and a corresponding set of real numbers

$$y_i^{(j)}\quad (i=0, 1, \cdots, s;\ j=0, 1, \cdots, k_i)$$

be given, where

$$0\leqslant k_i\leqslant t,\quad i=0, 1, \cdots, s,\quad k\equiv\sum_{i=0}^{s}(k_i+1)\leqslant m+n.$$

Find a rational function $R\in\boldsymbol{R}_1(m, n)$ such that

$$\|f-wR\|_{[a,b]}=\inf\{\|f-wr\|_{[a,b]}:\ r\in\boldsymbol{R}_1(m, n)\},\tag{1}$$

where $\boldsymbol{R}_1(m, n)\subset\boldsymbol{R}(m, n)$ is the set consisting of all irreducible rational functions satisfying the following constraints

$$\boldsymbol{R}^{(j)}(x_i)\equiv\frac{d^jR(x)}{dx^j}\bigg|_{x=x_i}=y_i^{(j)},\quad i=0, 1, \cdots, s;\ j=0, 1,\cdots, k_i.\tag{2}$$

$\boldsymbol{R}_1(m, n)$ is called the interpolating restricted range.

Solvability of this problem has already been investigated. Especially, the existence, uniqueness and characteristics of its solution have been discussed. It would be worthwhile to call attention to the fact that all the related discussions follow as results from a tacit assumption, namely, $\boldsymbol{R}_1(m, n)$ remains nonempty. However, it turns out that this is not always the case. A simple example can illustrate it well. Taking $k_i=0$ $(i=0, \cdots, s)$, $k>m+1$ and

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$$y_i^{(0)}=\begin{cases}0, & i=0, 1, \cdots, m,\\ 1, & i=m+1, \cdots, k-1,\end{cases}$$

it is obvious that, for any aforesaid set of the x_i's $\in[a, b]$, there is no rational function $R\in \boldsymbol{R}(m, n)$ satisfying

$$R(x_i)=y_i^{(0)}, \quad i=0, 1, \cdots, k-1,$$

i.e., $\boldsymbol{R}_1(m, n)=\emptyset$, the empty set.

The aim of our paper is simply to propound as a top priority regarding best rational approximation the fundamental question of under what conditions that $\boldsymbol{R}_1(m, n)$ will or will not be empty. We note that such a proposal is equivalent to considering the solvability of an under determined rational interpolation problem, which is clearly expressed by (2). Here "under determined" means that the number of interpolating conditions is less than that of the parameters to be determined. In what follows all our discussions and conclusions are based upon [4]. For the sake of brevity and convenience, we shall follow closely the notations used in [4] without any specification.

We first state a lemma concerning the interpolating properties of quasi-rational interpolant r_{mn}^*. This lemma includes both Theorem 3.2 and Theorem 3.5 given in [4] as its special cases.

Lemma 1. *Assume that* $R\in \boldsymbol{R}_0(m, n)$, $0\leqslant l_i\leqslant k_i+1$ $(i=0, 1, \cdots, s)$, $\mu\leqslant p\leqslant m$, $\nu\leqslant q\leqslant n$. *Then* $x_i\in X$ *is an* l_i*-fold unattainable point of* R *if and only if*

(i) rank $C_i^{(j)}(p-1, q-1)<$ rank $C(p-1, q-1)$, $j=k_i, k_i-1, \cdots, k_i-l_i+1$;

(ii) rank $C_i^{(k_i-l_i)}(p-1, q-1)=$ rank $C(p-1, q-1)$.

The proof of this lemma is similar to that of Theorem 3.5 in [4]; the only thing we have to do is to replace m and n therein with p and q respectively. We note that the case $(p, q)=(\mu, \nu)$ corresponds to Theorem 3.2, and likewise, $(p, q)=(m, n)$ to Theorem 3.5.

Let us now consider the cardinality of $\boldsymbol{R}_1(m, n)$. To start with, we introduce the following

Lemma 2. *Let* P_1/Q_1, $P_2/Q_2\in \boldsymbol{R}_1(m, n)$ *with* $P_1/Q_1\neq P_2/Q_2$. *For any pair* (α, β) *of constants, if*

$$\alpha Q_1(x_i)+\beta Q_2(x_i)\neq 0, \quad i=0, 1, \cdots, s,$$

then

$$\frac{\alpha P_1+\beta P_2}{\alpha Q_1+\beta Q_2}\in \boldsymbol{R}_1(m, n).$$

Proof. Let $g\in C^t[a, b]$ satisfy

$$g^{(j)}(x_i)=y_i^{(j)}, \quad i=0, 1, \cdots, s; \quad j=0, 1, \cdots, k_i.$$

Then, from

$$(P_r-gQ_r)^{(j)}(x_i)=0, \quad i=0, 1, \cdots, s; j=0, 1, \cdots, k_i; r=1, 2,$$

we have

$$[(\alpha P_1+\beta P_2)-g(\alpha Q_1+\beta Q_2)]^{(j)}(x_i)=0, \quad i=0, 1, \cdots, s; j=0, 1, \cdots, k_i.$$

Since $\alpha Q_1(x_i)+\beta Q_2(x_i)\neq 0$ $(i=0, 1, \cdots, s)$, it follows that

$$\left(\frac{\alpha P_1+\beta P_2}{\alpha Q_1+\beta Q_2}\right)^{(j)}(x_i)=g^{(j)}(x_i)=y_i^{(j)}, \quad i=0, 1, \cdots, s; j=0, 1, \cdots, k_i.$$

This means

$$(\alpha P_1+\beta P_2)/(\alpha Q_1+\beta Q_2)\in \boldsymbol{R}_1(m,n).$$

By Lemma 2, we know that there are only three different possibilities about the cardinality of $\boldsymbol{R}_1(m,n)$, i.e.

$$\operatorname{card}(\boldsymbol{R}_1(m,n))=0,\ 1 \text{ or } \infty.$$

Next, we exclude the trivial cases, namely, $\sum_{i=0}^{s} m_i \geqslant m+1$ where m_i is defined as in [4], § 2, for it is evident that $\boldsymbol{R}_1(m,n)=\emptyset$ if $\sum_{i=0}^{s} m_i<k$ (as shown in the example given above) and that $0\in \boldsymbol{R}_1(m,n)$ is the only element if $\sum_{i=0}^{s} m_i=k$. Hereafter we assume that $\sum_{i=0}^{s} m_i<m+1$. The main results can be expressed as the following

Theorem. *Let* $p=m+n+1-k$.

(a) *If* $p>\min\{m,n\}$, *then* $\operatorname{card}(\boldsymbol{R}_1(m,n))=\infty$.

(b) *If* $p\leqslant\min\{m,n\}$, *then*

(i) $\operatorname{card}(\boldsymbol{R}_1(m,n))=0$, *if the matrix* $C(m-p,n-p)$ *is not of full rank in column and if there exists some index* $i\,(0\leqslant i\leqslant s)$ *such that*

$$\operatorname{rank} C_i^{(k_i)}(m-p-1,n-p-1)<\operatorname{rank} C(m-p-1,n-p-1);$$

(ii) $\operatorname{card}(\boldsymbol{R}_1(m,n))=1$, *if* $C(m-p,n-p)$ *is not of full rank in column and if the relation*

$$\operatorname{rank} C_i^{(k_i)}(m-p-1,n-p-1)=\operatorname{rank} C(m-p-1,n-p-1)$$

holds for each $i=0,1,\cdots,s$;

(iii) $\operatorname{card}(\boldsymbol{R}_1(m,n))=\infty$, if $C(m-p,n-p)$ *is of full rank in column.*

Moreover, the inverse conclusions of (i), (ii) *and* (iii) *are also valid.*

Proof. (a) If $p>\min\{m,n\}$, then $k<\max\{m+1,n+1\}$. Hence in the case $m\geqslant n$, the under determined polynomial interpolation problem: Find $P_m\in H_m$ such that

$$P_i^{(j)}(x_i)=y_i^{(j)},\quad i=0,1,\cdots,s;\ j=0,1,\cdots,k_i$$

has infinitely many solutions and $\boldsymbol{R}_1(m,n)$ is therefore an infinite set.

In the case $m<n$, we put $\omega(x)=\prod_{i=0}^{s}(x-x_i)^{m_i}$ and consider the interpolation problem: Find $Q_n\in H_n$ such that

$$\left(\frac{\omega}{Q_n}\right)^{(j)}(x_i)=y_i^{(j)},\quad i=0,1,\cdots,s;\ j=0,1,\cdots,k_i.$$

This can be reduced to an equivalent under determined polynomial interpolation problem: Find $Q_n\in H_n$ such that

$$Q_n^{(j)}(x_i)=\tilde{y}_i^{(j)},\quad i=0,1,\cdots,s;\ j=0,1,\cdots,k_i-m_i,$$

where the $\tilde{y}_i^{(j)}$'s are uniquely determined from $y_i^{(j)}$'s and $\omega(x)$. Since there are infinitely many such polynomials $Q_n\in H_n$, assertion (a) is thus proved.

(b) (i) Consider the following rational interpolation problem (non-under-determined): Find $R\in \boldsymbol{R}(m_1,n_1)$ such that

$$R^{(j)}(x_i)=y_i^{(j)},\quad i=0,1,\cdots,s;\ j=0,1,\cdots,k_i,\tag{3}$$

where $(m_1,n_1)\in \boldsymbol{N}\times\boldsymbol{N}$, and

$$m-p\leqslant m_1\leqslant m,\quad n-p\leqslant n_1\leqslant n,\quad m_1+n_1+1=k.$$

To this problem we are to apply the conclusions given in [4].

Since $C(m-p, n-p)$ is not of full rank in column, there exists a rational function $P/Q\in \boldsymbol{R}_0(m_1, n_1)$ such that

$$F^{-1}_{m-p,n-p}(P/Q)\in \mathfrak{N}(C(m-p, n-p))\setminus\{0\}.$$

Thus $\mu\leqslant m-p$, $\nu\leqslant n-p$. It follows from Lemma 1 that problem (3) is unsolvable.

Suppose that $\operatorname{card}(\boldsymbol{R}_1(m, n))>0$ and $P_1/Q_1\in \boldsymbol{R}_1(m, n)$. Let $P_2/Q_2=r^*_{m_1n_1}$ be the quasi-rational interpolant corresponding to problem (3). Then we have

$$\partial(P_2)\leqslant m-p,\ \partial(Q_2)\leqslant n-p,\ \partial(P_1)\leqslant m,\ \partial(Q_1)\leqslant n.$$

By the argument similar to the proof of Lemma 2.1 in [2, chap. 1] or [4], we know that $P_1Q_2-P_2Q_1$ has at least k zeros. In view of $\partial(P_1Q_2-P_2Q_1)\leqslant m+n-p=k-1$, $P_1Q_2=P_2Q_1$ follows. This indicates that P_2/Q_2 and P_1/Q_1 possess the same irreducible form. Hence $P_2/Q_2\in \boldsymbol{R}_0(m_1, n_1)\subset \boldsymbol{R}(m_1, n_1)$ satisfies (3), a contradiction. Therefore $\operatorname{card}(\boldsymbol{R}_1(m, n))=0$.

(ii) From the hypotheses and the proof of (i), it follows that problem (3) is solvable. Hence $r^*_{m_1n_1}=P_2/Q_2\in \boldsymbol{R}_1(m, n)$. For any $P_1/Q_1\in \boldsymbol{R}_1(m, n)$, as mentioned above, we have $P_1Q_2=P_2Q_1$. Therefore $P_1/Q_1=P_2/Q_2$, namely, $\operatorname{card}(\boldsymbol{R}_1(m, n))=1$.

(iii) Consider again the rational interpolation problem (3). Since $C(m-p, n-p)$ is of full rank in column, there must be

$$\mu\geqslant m-p+1\quad \text{or}\quad \nu\geqslant n-p+1.$$

Let $r^*_{m_1n_1}=P^*_{m_1n_1}/Q^*_{m_1n_1}$. In addition to the points $x_0, x_1, \cdots, x_s$, we add p nodes

$$x_{s+1}, x_{s+2}, \cdots, x_{s+p}\in[a, b]$$

and the corresponding interpolation data $y^{(0)}_{s+i}$ (with $k_{s+i}=0$, $i=1, \cdots, p$) such that the x_i's are distinct from each other for $i=0, 1, \cdots, s+p$ and

$$\begin{aligned}&Q^*_{m_1n_1}(x_{s+i})\neq 0, && i=1, \cdots, p,\\ &y^{(0)}_{s+i}\neq r^*_{m_1n_1}(x_{s+i}), && i=1, \cdots, \alpha,\\ &y^{(0)}_{s+i}=r^*_{m_1n_1}(x_{s+i}), && i=\alpha+1, \cdots, p,\end{aligned}\tag{4}$$

where $\alpha=\min\{m-\mu, n-\nu\}$, i.e. $0\leqslant\alpha\leqslant p-1$. We are now to consider the following rational interpolation problem: Find $R\in \boldsymbol{R}(m, n)$ such that

$$R^{(j)}(x_i)=y_i^{(j)},\quad i=0, 1, \cdots, s+p;\ j=0, 1, \cdots, k_i.\tag{5}$$

By the definition of $r^*_{m_1n_1}$ and relations (4), it is easy to verify that

$$r^*_{mn}=P^*_{m_1n_1}\prod_{i=1}^{\alpha}(x-x_{s+i})/Q^*_{m_1n_1}\prod_{i=1}^{\alpha}(x-x_{s+i})$$

is the quasi-rational interpolant corresponding to problem (5) and that $d(r^*_{mn})=0$. Since $x_{s+i}(i=\alpha+1, \cdots, p)$ are attainable points of r^*_{mn}, similarly to the proof of Lemma 3.3 in [3], we can change at most one point among $(x_{s+i}, y^{(0)}_{s+i})$ $(i=\alpha+1, \cdots, p)$ so that problem (5) can be made solvable. Since there are infinitely many ways to do such a change, we conclude that $\operatorname{card}(\boldsymbol{R}_1(m, n))=\infty$.

For the remaining part of the theorem, we notice that the conditions of (i), (ii), (iii) in (b) are independent of each other and they include all the logical possibilities. Therefore the inverse conclusions of (i), (ii) and (iii) are also true.

The theorem is completely proved.

Finally, we point out an important property of $\boldsymbol{R}_1(m, n)$. If $\text{card}(\boldsymbol{R}_1(m, n))=\infty$, then $\boldsymbol{R}_1(m, n)$ may not be a complete set in the following sense: If $R_i=P_i/Q_i \in \boldsymbol{R}_1(m, n)$ with $\max\limits_{a<x<b}|R_i(x)|$ are bounded above, and if $P_i \to P \in H_m$, $Q_i \to Q \in H_n$, then P/Q may not be in the set $\boldsymbol{R}_1(m, n)$. We may see this from a simple example. Take

$$s=0,\ k_0=0,\ x_0=0,\ y_0=2,\ [a, b]=[-1, 1].$$

Then it is clear that

$$R_i=P_i/Q_i=\left(x^2+\frac{2}{i}\right)\Big/\left(x^2+\frac{1}{i}\right)\in \boldsymbol{R}_1(2, 2),$$

and

$$P_i \to P=x^2,\ Q_i \to Q=x^2 \quad \text{as } i\to\infty.$$

However, $P/Q=1$, which does not belong to $\boldsymbol{R}_1(2, 2)$. This remark merits attention in the discussion of the existence for best rational approximation with interpolating constraints.

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