A MONOTONICALLY CONVERGENT ITERATIVE METHOD FOR LARGE SPARSE NONLINEAR EQUATIONS*

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§ 0. Introduction

The bisection method is a well-known method for the numerical solution of a single nonlinear equation. This method is effective and simple in finding the real root of a single nonlinear equation, and only requires that the function be continuous. Therefore it has a wide range of applications. This paper intends to extend this method to the case of nonlinear systems. Although not all nonlinear systems can be solved by the bisection method, there does exist some class of nonlinear systems of equations which can be solved by the bisection method. And this class of systems of equations can be obtained by approximating partial differential equations using the finite element or difference method. For some classes of nonlinear systems the bisection method is simple, safe and reliable[6]. By "safe and reliable" is meant that the desired solution can always be found (in the sence of global convergence).

The paper is built up as follows.

In the first section some definitions and notations are given. Section 2 describes the bisection method and gives its algorithm and its error estimate. Finally this method is applied to the minimal surface problem and some numerical results are given.

§ 1. Definition and Notation

We consider the following nonlinear system of equations

$$Fx=0, \tag{1.1}$$

where

$$Fx=\begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix}, \quad x=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

$$x\in D\subset R^n\to R^n.$$

Recall that the natural partial ordering on R^n is defined by $x\leqslant y$ $(x<y)$, for any

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$x, y \in R^n$, if and only if $x_i \leqslant y_i (x_i < y_i)$, $i = 1, 2, \cdots, n$.

Definition 1. *The mapping F: $D \subset R^n \to R^m$ is isotone (on D) if for any $x, y \in D$, $x \leqslant y$, implies that $Fx \leqslant Fy$. An isotone mapping F is strictly isotone if it follows from $x < y$, for any $x, y \in D$, that $Fx < Fy$.*

Definition 2. *The mapping F: $D \subset R^n \to R^m$ is antitone (on D) if for any $x, y \in D$, $x \leqslant y$, implies that $Fx \geqslant Fy$. An antitone mapping F is strictly antitone if it follows from $x < y$, for any $x, y \in D$, that $Fx > Fy$.*

Definition 3[2]**.** *A mapping F: $R^n \to R^n$ is diagonally isotone if, for any $x \in R^n$, the n functions*

$$\psi_{ii}: R^1 \to R^1,\ \psi_{ii}(t) = f_i(x + te^i), \quad i = 1, 2, \cdots, n \tag{1.2}$$

are isotone, where e^i are unit vectors. The function F is strictly diagonally isotone if, for any $x \in R^n$, $\psi_{ii} (i = 1, 2, \cdots, n)$ are strictly isotone.

Definition 4. *A mapping F: $R^n \to R^n$ is off-diagonally antitone if, for any $x \in R^n$, the functions*

$$\psi_{ij}: R^1 \to R^1,\ \psi_{ij}(t) = f_i(x + te^j), \quad i \neq j,\ i, j = 1, 2, \cdots, n \tag{1.3}$$

are antitone.

§ 2. Bisection Method

In Fig. 1, assuming $\varphi_i(x_i^{(k)}) < 0$, $\psi_i(y_i^{(k)}) > 0$, we first define a sequence of intervals:

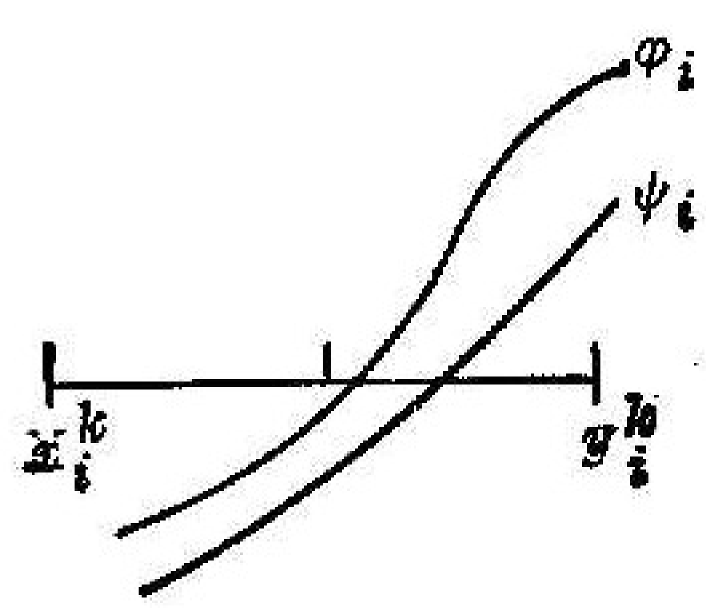


Fig. 1

$$I_{ij}^{(k)} = (x_i^{(k)}, x_{i,j+1}^{(k)}),$$

where

$$x_{ij}^{(k)} = \frac{x_i^{(k)} + x_{i,j-1}^{(k)}}{2}, \quad j = 1, 2, \cdots, m_i,\ x_{i0}^{(k)} := y_i^{(k)}. \tag{2.1}$$

Continue this process until

$$\varphi_i(x_i^{(k)})\, \varphi_i\left(\frac{x_i^{(k)} + x_{ij}^{(k)}}{2}\right) > 0. \tag{2.2}$$

Let m_i be the first index for $\varphi_i(x_{ij}^{(k)}) < 0$, and set

$$x_{i,m_i}^{(k)} = x_i^{(k+1)},$$

$$\left(x_i^{(k)}, \frac{x_i^{(k)} + y_i^{(k)}}{2}\right) - I_{i,m_i}^{(k)} = I_i^{(k)}. \tag{2.3}$$

Analogously, a sequence of intervals

$$R_{ij}^{(k)} = (y_{i,j+1}^{(k)}, y_i^{(k)}),$$

where

$$y_{ij}^{(k)} = \frac{y_{ij-1}^{(k)} + y_i^{(k)}}{2},\ j = 1, 2, \cdots, n_i,\ y_{i0}^{(k)} := x_i^{(k)}, \tag{2.4}$$

can be defined until

$$\psi_i(y_i^{(k)})\, \psi_i\left(\frac{y_{ij}^{(k)} + y_i^{(k)}}{2}\right) > 0. \tag{2.5}$$

Let n_i be the first index for $\psi_i(y_{ij}^{(k)}) > 0$, and set

$$y_{i,n_i}^{(k)} = y_i^{(k+1)},$$

$$\left(\frac{x_i^{(k)}+y_i^{(k)}}{2},\ y_i^{(k)}\right)-R_{i,n_i}^{(k)}=R_i^{(k)}. \tag{2.6}$$

Furthermore, we set $I_i^{(k)}+R_i^{(k)}=L_i^{(k)}$.

Theorem A. *Let F: $D\subset R^N\to R^N, F\in C^0(D)$, be an off-diagonally antitone and strictly diagonally isotone mapping. Assume that there exist points x^0, $y^0\in R$ such that*

$$x^{(0)}\leqslant y^{(0)},\quad \langle x^0, y^0\rangle\subset D,\quad Fx^{(0)}\leqslant 0\leqslant Fy^{(0)}. \tag{2.7}$$

Then the sequences starting from $x^{(0)}$ and $y^{(0)}$ respectively are well defined and the sequences $\{x^k\}$ in (2.3) and $\{y^k\}$ in (2.6) have the properties

$$x^{(k)}\uparrow x^*,\ y^{(k)}\downarrow y^* \quad as\ k\to\infty$$

with

$$x^{(0)}\leqslant x^*\leqslant y^*\leqslant y^{(0)},\quad Fx^*=Fy^*=0.$$

Proof. We only give the proof for the Gauss–Seidel iteration (1.2); that for the Jacobi iteration (1.3) is analogous.

By induction hypothesis, suppose that for some $k\geqslant 0$ and $i\geqslant 1$

$$x^{(0)}\leqslant x^{(k)}\leqslant y^{(k)}\leqslant y^{(0)},\quad Fx^{(k)}\leqslant 0\leqslant Fy^{(k)}, \tag{2.8}$$

$$x_j^{(k)}\leqslant x_j^{(k+1)}\leqslant y_j^{(k+1)}\leqslant y_j^{(k)},\quad j=1, 2, \cdots, i-1, \tag{2.9}$$

where for $i=1$ the set of j satisfying (2.9) is empty. Obviously (2.8) and (2.9) hold for $k=0$, $i=1$. By off-diagonal antitonicity, it follows that equations

$$\varphi_i(s)=f_i(x_1^{k+1}, \cdots, x_{i-1}^{k+1}, s, x_{i+1}^k, \cdots, x_n^k), \tag{2.10}$$

$$\psi_i(s)=f_i(y_1^{k+1}, \cdots, y_{i-1}^{k+1}, s, y_{i+1}^k, \cdots, y_n^k) \tag{2.11}$$

satisfy

$$\psi_i(s)\leqslant\varphi_i(s),\quad \forall s\in D_i^k\quad (D_i^k{:}\langle x_i^k, y_i^k\rangle). \tag{2.12}$$

Evidently

$$\psi_i(x_i^{(k)})\leqslant\varphi_i(x_i^{(k)})\leqslant f_i(x^{(k)})\leqslant 0\leqslant f_i(y^{(k)})\leqslant\psi_i(y_i^{(k)})\leqslant\varphi_i(y_i^{(k)}). \tag{2.13}$$

By the continuity and strict isotonicity of $\psi_i(s)$ and $\varphi_i(s)$, there exist unique $\hat{y}_i^{(k)}$ and $\hat{x}_i^{(k)}$ satisfying

$$\psi_i(\hat{y}_i^{(k)})=0=\varphi_i(\hat{x}_i^{(k)}),$$

$$x_i^{(k)}\leqslant\hat{x}_i^{(k)}\leqslant\hat{y}_i^{(k)}\leqslant y_i^{(k)},$$

where $\hat{x}_i^{(k)}\leqslant\hat{y}_i^{(k)}$ is a consequence of (2.12).

We have

$$y_i^{(k)}\geqslant y_i^{(k+1)}=y_i^{(k)}-|R_{i,m_i}^{(k)}|\geqslant\hat{y}_i^{(k)}\geqslant\hat{x}_i^{(k)}\geqslant x_i^{(k+1)}=x_i^{(k)}+|I_{i,n_i}^{(k)}|\geqslant x_i^{(k)}, \tag{2.14}$$

where $|R_{i,m_i}^{(k)}|\geqslant 0$ and $|I_{i,n_i}^{(k)}|\geqslant 0$ are the distances of intervals $R_{i,m_i}^{(k)}$, $I_{i,n_i}^{(k)}$ respectively. It is obvious that $|R_{i,m_i}^{(k)}|\geqslant 0$ and, $|I_{i,n_i}^{(k)}|\geqslant 0$, and because of the structure of the algorithm, $R_{i,m_i}^{(k)}=l_{i,n_i}^{(k)}=0$ holds only when $f_i(x_i^{(k)})=f_i(y_i^{(k)})=0$. (2.14) shows that (2.9) holds for $i=1, 2, \cdots, n$, that is

$$x^{(k)}\leqslant x^{(k+1)}\leqslant y^{(k+1)}\leqslant y^{(k)}.$$

The off-diagonal antitonicity leads to

$$F_i(x^{(k+1)})\leqslant F_i(x_1^{k+1}, \cdots, x_{i-1}^{k+1}, x_i^{k+1}, x_{i+1}^k, \cdots, x_N^k)=\varphi_i(x_i^{k+1})\leqslant 0.$$

The last inequality is due to the structure of the algorithm.

Similarly, we have

$$F_i(y^{k+1}) \geqslant F_i(y_1^{k+1}, \cdots, y_{i-1}^{k+1}, y_i^{k+1}, y_{i+1}^k, \cdots, y_n^k) \geqslant 0, \quad i=1, 2, \cdots, n.$$

This completes the induction and hence the proof of (2.8).

As monotone and bounded sequences, $\{x_i^{(k)}\}$ and $\{y_i^{(k)}\}$ have the limits x_i^* and y_i^* satisfying $x^* \leqslant y^*$. Moreover, let

$$\omega_i^{(k)} = \frac{|I_{i,n_i}^{(k)}|}{(\hat{x}_i^k - x_i^k)}, \quad \bar{\omega}_i^{(k)} = \frac{|R_{i,m_i}^{(k)}|}{(y_i^k - \hat{y}_i^k)},$$

$$0 < \frac{1}{2} \leqslant \omega_i^{(k)} < 1, \quad \forall k, i, \ 0 < \frac{1}{2} \leqslant \bar{\omega}_i^{(k)} < 1, \quad \forall k, i,$$

where $\omega_i^{(k)}$, $\bar{\omega}_i^{(k)} \neq 0$ because $|I_{i,n_i}^{(k)}| \neq 0 (|I_{i,n_i}^{(k)}| = 0$ only for $f_i(x_i^{(k)}) = f_i(y_i^{(k)}) = 0)$ and $\omega_i^{(k)}$, $\bar{\omega}_i^{(k)} \geqslant \frac{1}{2}$ because functions $\varphi_i(x_i^{(k)})$, $\psi_i(y_i^{(k)})$ are strictly monotone functions.

Therefore, we have

$$\lim_{k\to\infty} \hat{x}_i^{(k)} = \lim_{k\to\infty} \frac{1}{\omega_i^{(k)}}(x_i^{(k+1)} - x_i^{(k)}) + \lim_{k\to\infty} x_i^{(k)} = x_i^*, \quad i=1, 2, \cdots, n,$$

$$\lim_{k\to\infty} \hat{y}_i^{(k)} = \lim_{k\to\infty} \frac{1}{\omega_i^{(k)}}(y_i^{(k+1)} - y_i^{(k)}) + \lim_{k\to\infty} y_i^{(k)} = y_i^*, \quad i=1, 2, \cdots, n.$$

Consequently $Fx^* = Fy^* = 0$ follows from the continuity of F. By a rather formal notation the algorithm can be described in the following procedure.

Procedure BIS (N, x, y, s, f); integer N; array x, y, f; value s

comment N: number of equations, dimension of space

x: input $x_i^{(0)}$, output x_i^*

y: input $y_i^{(0)}$, output y_i^*

f: function of the equation to be solved

s: desired accuracy

begin integer k; array l, R, ll, RR;

for k: $=1$ step 1 do

begin integer i; $0 \Rightarrow l \Rightarrow R \Rightarrow ll \Rightarrow RR$;

for $i=1$ step 1 until N do

begin integer j;

$x[i] \Rightarrow a$; $y[i] \Rightarrow b$;

for $J=1$ step 1 do

If $\varphi_i(a)\varphi_i\left(\frac{a+b}{2}\right) < 0$ then begin $\left(a, \frac{a+b}{2}\right) \rightarrow l[i]$;

$l[i] + ll[i] \Rightarrow ll[i]$; $\frac{(a+b)}{2} \Rightarrow b$ end

else begin $ll[i] + \frac{(a+b)}{2} \Rightarrow ll[i]$;

$\frac{(a+b)}{2} \Rightarrow x^*[i]$; end

$x[i] \Rightarrow a$; $y[i] \rightarrow b$;

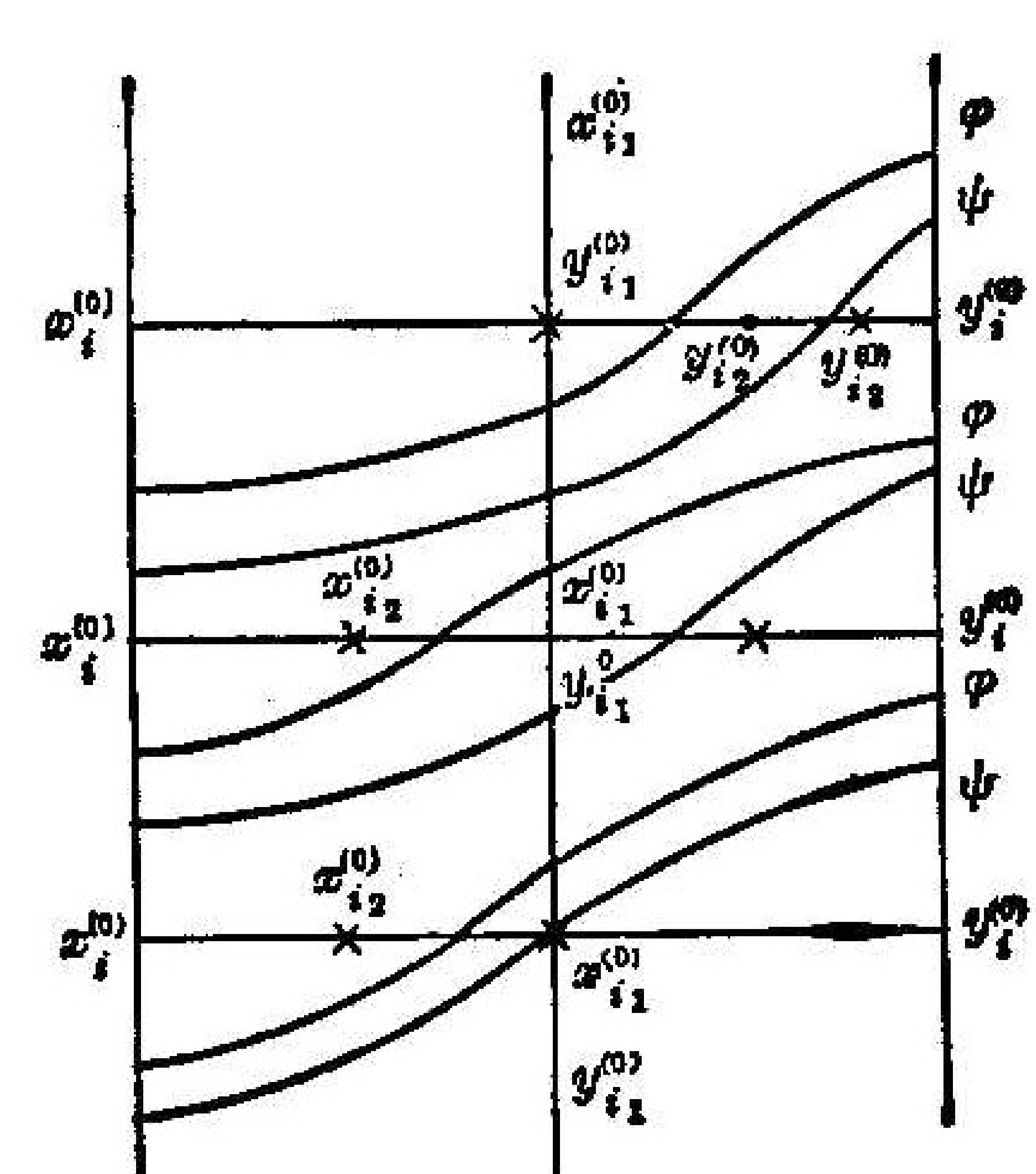


Fig. 2

for $J=1$ step 1 do

If $\psi_i(b)\psi_i\left(\frac{a+b}{2}\right)<0$ then begin $\left(\frac{a+b}{2},\ b\right)\Rightarrow R[i]$;

$R[i]+RR[i]\Rightarrow RR[i]$; $\frac{(a+b)}{2}\Rightarrow a$ end

else begin $\frac{(a+b)}{2}\Rightarrow y^*[i]$; $RR[i]+\frac{(a+b)}{2}\Rightarrow RR[i]$ end

end;

end; $x^*[i]\Rightarrow x[i]$; $y^*[i]\Rightarrow y[i]$;

If length $\sum_{i=1}^{N} l[i]+R[i]<\varepsilon$ then stop else

end;

Now we show

Lemma 1.

$$(y_i^{(k+1)}-x_i^{(k+1)})=\left(1-\frac{2^{n_i^{(k)}}+2^{m_i^{(k)}}}{2^{n_i^{(k)}+m_i^{(k)}}}\right)(y_i^{(k)}-x_i^{(k)}). \tag{2.15}$$

Proof. It is not difficult to show that

$$x_{ij}^{(k)}=\frac{x_i^{(k)}(2^j-1)+y_i^{(k)}}{2^j}$$

$$y_{ij'}^{(k)}=\frac{y_i^{(k)}(2^{j'}-1)+x_i^{(k)}}{2^{j'}}.$$

Therefore for $j=m_i$ and $j'=n_i$, we have

$$x_{i,m_i}^{(k)}=\frac{x_i^{(k)}(2^{m_i^{(k)}}-1)+y_i^{(k)}}{2^{m_i^{(k)}}},$$

$$y_{i,n_i}^{(k)}=\frac{y_i^{(k)}(2^{n_i^{(k)}}-1)+x_i^{(k)}}{2^{n_i^{(k)}}}.$$

After rearrangement, we get

$$[y_i^{(k+1)}-x_i^{(k+1)}]=[y_{im_i}^{(k)}-x_{in_i}^{(k)}]=\left(1-\frac{2^{n_i^{(k)}}+2^{m_i^{(k)}}}{2^{n_i^{(k)}+m_i^{(k)}}}\right)(y_i^{(k)}-x_i^{(k)}).$$

Lemma 2.

$$(y_i^{(k+1)}-x_i^{(k+1)})=\left(1-\frac{2^{n_i^{(k)}}+2^{m_i^{(k)}}}{2^{n_i^{(k)}+m_i^{(k)}}}\right)\left(1-\frac{2^{n_i^{(k-1)}}+2^{m_i^{(k-1)}}}{2^{n_i^{(k-1)}+m_i^{(k-1)}}}\right)\cdots\left(1-\frac{2^{n_i^{(0)}}+2^{m_i^{(0)}}}{2^{n_i^{(0)}+m_i^{(0)}}}\right)(y_i^{(0)}-x_i^{(0)}). \tag{2.16}$$

Proof. Follows immediately from Lemma 1.

Evidently we have a decreasing sequence of bounded, closed, non-empty sets

$$l_i^{(0)}+R_i^{(0)}\supset l_i^{(1)}+R_i^{(1)}\cdots\supset l_i^{(k)}+R_i^{(k)}\cdots\supset l_i^{(*)}+R_i^{(*)}.$$

Using Contor's intersection theorem, we have $\bigcap_{k=0}^{\infty} l_i^{(k)}+R_i^{(k)}=[x_i^*,\ y_i^*]$. In reality, we get an interval version of the iterative method. $x^{k+1}=x^k\cap K(x^k)$, where K is an operator of bisection, and x^k is an interval vector. The solution of (1.1) in X^0 lies in $K(X^0)$, and therefore in $X^1=X^0\cap K(X^0)$ also. When every side of $K(X^0)$ is shorter than any side of X^0, then the solution of (1.1) is unique in X^0.

§ 3. Application and Numerical Computation

In the domain

$$G: \left\{c_1^2<x^2+y^2<c_2^2,\ \ c_1^2>\frac{1}{\sqrt{2}}(1+\sqrt{2})a^2,\ x,\ \ y\geqslant 0\right\}$$

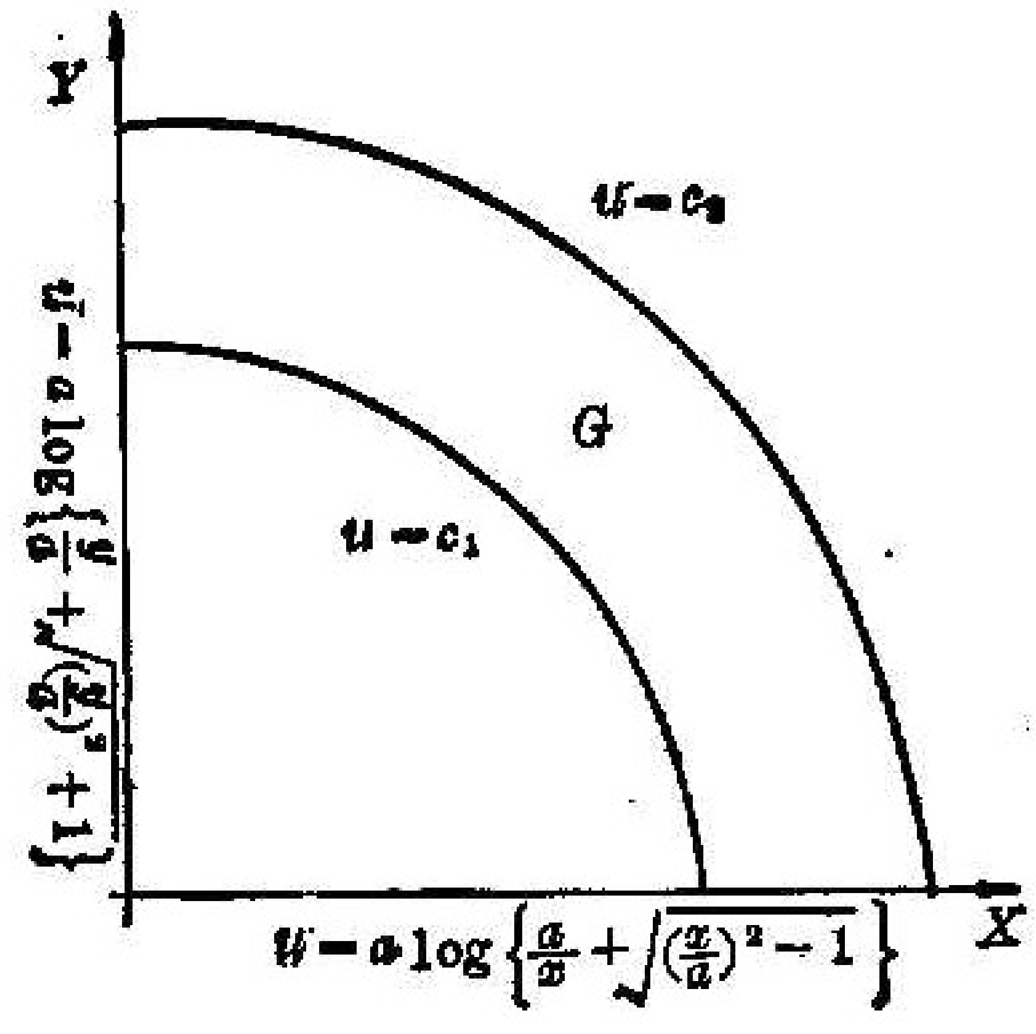


Fig. 3

we consider the following equation of minimal surface

$$(1+u_y^2)u_{xx}-2u_xu_yu_{xy}+(1+u_x^2)u_{yy}=0 \tag{3.1}$$

and the boundary condition

$$\begin{aligned} u(x,\ y)&=c_1 \text{ for } x^2+y^2=a^2\cosh^2\frac{c_1}{a},\\ u(x,\ y)&=c_2 \text{ for } x^2+y^2=a^2\cosh^2\frac{c_2}{a}, \end{aligned} \tag{3.2}$$

$u(x,0)=a\log\left\{\frac{x}{a}+\sqrt{\left(\frac{x}{a}\right)^2-1}\right\}$ for $a\cosh\frac{c_1}{a}\leqslant x \leqslant a\cosh\frac{c_2}{a}$, $u(0,\ \ y)=a\log\left\{\frac{y}{a}+\sqrt{\left(\frac{y}{a}\right)^2-1}\right\}$ for $a\cosh\frac{c_1}{a}\leqslant y\leqslant a\cosh\frac{c_2}{a}$. The solution satisfying equation (3.1) and boundary condition (3.2) is

$$u(x,\ y)=a\,\mathrm{Arsh}\left\{\sqrt{\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}\right)^2-1}\right\}\quad \forall x,\ y\in\bar{\Omega}.$$

Multiplying (3.1) by an integral factor $(\sqrt{1+u_x^2+u_y^2})^{-1}$ results in the variational problem

$$I(u)=\int_G \sqrt{1+u_x^2+u_y^2}\,dx\,dy. \tag{3.3}$$

The domain G is approximated by the quadrilateral element

$$G_h=\bigcup_{j=1}^{L} Q_j,\ Q_j \text{ quadrangle.} \tag{3.4}$$

We use continuous and piecewise linear functions to solve approximately the problem (3.3):

$$\begin{aligned} p:&=u_x=\frac{1}{h}(u_{i+1,j}-u_{ij})=\sum_{i=1}^{N}\alpha_i^{(1)}u_h(P_i),\\ q:&=u_y=\frac{1}{h}(u_{i,j+1}-u_{ij})=\sum_{i=1}^{N}\alpha_i^{(2)}u_h(P_i). \end{aligned} \tag{3.5}$$

Let $P_i(i=1,\ \cdots,\ M(M+1,\ \cdots,\ N))$ be the set of interior (boundary) vertices.

Using the method of discretization [3], we obtain a discretized system approximating (3.3)

$$A(v)v=b(v), \tag{3.6}$$

where

$$A(v)=H^TD(Hv+Bv_g)H, \tag{3.7}$$

$$H^T=\begin{pmatrix} -1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{3.8}$$

$$D(Hv+Bv_g)=\begin{pmatrix} r_1/d_1 & & & & & & & \\ & r_1/d_1 & & & & 0 & & \\ & & r_2/d_2 & & & & & \\ & & & r_2/d_2 & & & & \\ & & & & r_3/d_3 & & & \\ & & & & & r_3/d_3 & & \\ & & 0 & & & & r_4/d_4 & \\ & & & & & & & r_4/d_4 \end{pmatrix}, \tag{3.9}$$

$$D(Hv+Bv_g)H=\begin{pmatrix} -\frac{r_1}{d_1} & \frac{r_1}{d_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{r_1}{d_1} & 0 & 0 & \frac{r_1}{d_1} & 0 & 0 & 0 & 0 & 0 \\ \frac{r_2}{d_2} & 0 & 0 & 0 & 0 & -\frac{r_2}{d_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{r_2}{d_2} & -\frac{r_2}{d_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{r_3}{d_3} & \frac{r_3}{d_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{r_3}{d_3} & \frac{r_3}{d_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{r_4}{d_4} & \frac{r_4}{d_4} \\ \frac{r_4}{d_4} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{r_4}{d_4} & 0 \end{pmatrix}. \tag{3.10}$$

For the point 1 of Fig. 4, we obtain the following difference equation

$$\left(\frac{r_1}{d_1}+\frac{r_1}{d_1}+\frac{r_2}{d_2}+\frac{r_4}{d_4}\right)u_1-\frac{r_1}{d_1}u_2-\frac{r_1}{d_1}u_4-\frac{r_2}{d_2}u_6-\frac{r_4}{d_4}u_8=0, \tag{3.11}$$

where $d_i=\sqrt{1+p_i^2+q_i^2}$, $r_i=\frac{1}{h^2}$. Thus we have a formula

$$u_{ij-1}A_3+u_{i-1j}A_2+u_{i+1j}A_1+u_{ij+1}A_1+u_{ij}(-2A_1-A_2-A_3)=0$$

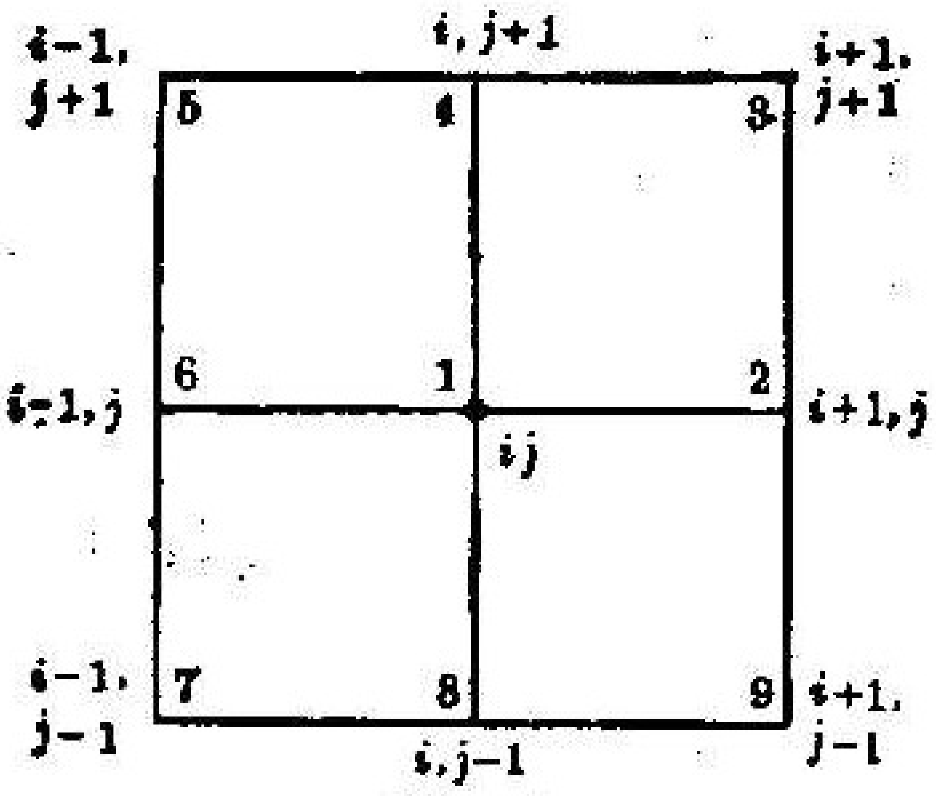


Fig. 4

with

$$A_1=-\left\{1+\frac{1}{h^2}(u_{i+1j}-u_{ij})^2+(u_{ij+1}-u_{ij})^2\right\}^{-\frac{1}{2}}<0,$$

$$A_2=-\left\{1+\frac{1}{h^2}(u_{ij}-u_{i-1j})^2+(u_{i-1j+1}-u_{i-1j})^2\right\}^{-\frac{1}{2}}<0, \quad (3.12)$$

$$A_3=-\left\{1+\frac{1}{h^2}(u_{i+1j-1}-u_{ij-1})^2+(u_{ij}-u_{ij-1})^2\right\}^{-\frac{1}{2}}<0.$$

Now we calculate the elements of the functional matrix

$$\frac{\partial f}{\partial u_{ij}}=(-2A_1-A_2-A_3)+(u_{ij-1}-u_{ij})\frac{\partial A_3}{\partial u_{ij}}+(u_{i-1j}-u_{ij})-\frac{\partial A_2}{\partial u_{ij}}$$

$$+[(u_{i+1j}-u_{ij})+(u_{ij+1}-u_{ij})]\frac{\partial A_1}{\partial u_{ij}}$$

$$=\frac{2+(p_{ij}-q_{ij})^2}{\sqrt{(1+p_{ij}^2+q_{ij}^2)^3}}+\frac{1+q_{i-1j}^2}{\sqrt{(1+p_{i-1j}^2+q_{i-1j}^2)}}+\frac{1+p_{ij-1}^2}{\sqrt{(1+p_{ij-1}^2+q_{ij-1}^2)^3}}$$

$$\frac{\partial f}{\partial u_{ij-1}}=A_3+(u_{ij-1}-u_{ij})\frac{\partial A_3}{\partial u_{ij-1}}=-\frac{1+p_{ij-1}^2-p_{ij-1}q_{ij-1}}{\sqrt{(1+p_{ij-1}^2+q_{ij-1}^2)^3}}$$

$$\frac{\partial f}{\partial u_{i-1j}}=A_2-p_{i-1j}\frac{\partial A_2}{\partial u_{ij}}=-\frac{1+q_{i-1j}^2-p_{i-1j}q_{i-1j}}{\sqrt{(1+p_{i-1j}^2+q_{i-1j}^2)^3}}$$

$$\frac{\partial f}{\partial u_{i+1j}}=A_1+(u_{i+1j}-u_{ij})\frac{\partial A_1}{\partial u_{i+1j}}+(u_{ij+1}-u_{ij})\frac{\partial A_1}{\partial u_{i+1j}}=-\frac{1+q_{ij}^2-p_{ij}q_{ij}}{\sqrt{(1+p_{ij}^2+q_{ij}^2)^3}}$$

$$\frac{\partial f}{\partial u_{ij+1}}=A_1+(u_{i+1j}-u_{ij})\frac{\partial A_1}{\partial u_{ij+1}}+(u_{ij+1}-u_{ij})\frac{\partial A_1}{\partial u_{ij+1}}=-\frac{1+p_{ij}^2-p_{ij}q_{ij}}{\sqrt{(1+p_{ij}^2+q_{ij}^2)^3}}$$

$$\frac{\partial f}{\partial u_{i-1j+1}}=(u_{i-1j}-u_{ij})\frac{\partial A_2}{\partial u_{i-1j+1}}=\frac{-p_{i-1j}q_{i-1j}}{\sqrt{(1+p_{i-1j}^2+q_{i-1j}^2)^3}}$$

$$\frac{\partial f}{\partial u_{i+1j-1}}=(u_{ij-1}-u_{ij})\frac{\partial A_3}{\partial u_{i+1j-1}}=\frac{-q_{ij-1}p_{ij-1}}{\sqrt{(1+p_{ij-1}^2+q_{ij-1}^2)^3}}.$$

Obviously $\frac{\partial f}{\partial u_{ij}}>0$.

The inequality $1+q_{ij}^2-p_{ij}q_{ij}>0$, $p_{ij}q_{ij}>0$, in the above domain G holds[1]. Thus we get a functional matrix, which is an L-matrix. Furthermore, it is an irreducible diagonally dominant matrix. Of course, such a functional matrix is an M-matrix.

Lemma 3[4,5]. *Let F: $D\subset R^n\to R^n$ be continuously differentiable on the closed rectangle B and let the functional matrix be an L-matrix (M-matrix) for each $x\in D$. Then F is a continuous, off-diagonally antitone and strictly diagonally isotone function (M-function) on D.*

In another domain

$$G:\ 0<x<1,\ 0<y<1, \quad (3.13)$$

we consider the same equation (3.1) of minimal surface with boundary condition

$$\begin{aligned} &u(x,\ 0) = -\ln\cos x, \quad \forall x \in [0,\ 1], \\ &u(x,\ 1) = \ln(\cos 1) - \ln\cos x, \quad \forall x \in [0,\ 1], \\ &u(0,\ y) = \ln(\cos y), \quad \forall y \in [0,\ 1], \\ &u(1,\ y) = \ln(\cos y) - \ln(\cos 1), \quad \forall y \in [0,\ 1]. \end{aligned} \tag{3.14}$$

The solution satisfying equation (3.1) and boundary condition (3.2) is

$$u(x,\ y) = \ln(\cos y) - \ln(\cos x),\ \forall x,\ y \in G.$$

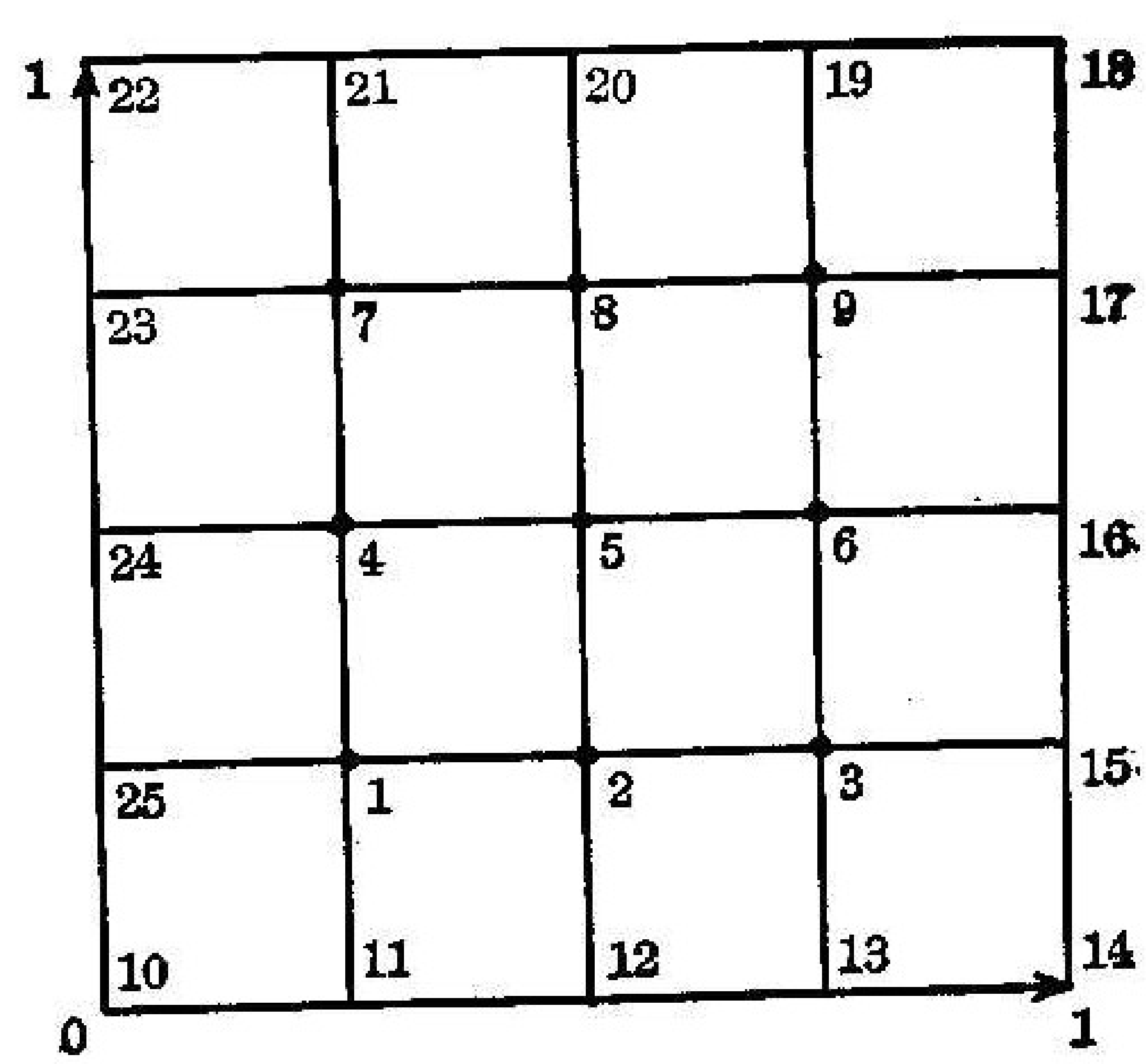
We again calculate the elements of the functional matrix. In this case

$$\frac{\partial f}{\partial u_{ij}} > 0, \quad \frac{\partial f}{u_{ij-1}} \leqslant 0, \quad \frac{\partial f}{\partial u_{i-1j}} \leqslant 0, \quad \frac{\partial f}{\partial u_{i+1j}} \leqslant 0, \quad \frac{\partial f}{\partial u_{ij+1}} \leqslant 0$$

but

$$\frac{\partial f}{\partial u_{i+1j+1}} \geqslant 0, \quad \frac{\partial f}{\partial u_{i+1j-1}} \geqslant 0.$$

Thus we get a functional matrix which is not an M-matrix. Nevertheless, we can still apply the bisection method. This means the condition in Theorem A that "F is off-diagonally antitone and strictly diagonally isotone" is too strong. We will show this by an interesting example from (3.1) with boundary condition (3.12). First we select the starting vectors $x^{(0)}$, $y^{(0)}$. We know that the maximum and minimum of the demanded function are always assumed on the boundary. So we may select the starting vectors $x^{(0)}$ and $y^{(0)}$ from the boundary condition (3.2). As $u(x,\ y) = \ln(\cos y) - \ln(\cos x)$, $x,\ y \in [0,\ 1]$ and $\cos x,\ \cos y \in [0.5403;\ 1]$, then $\ln(\cos x), \ln(\cos y) \in [-0.6156;\ 0]$. We can obtain the minimal value



$$\begin{aligned} u(0,\ 1) &= \ln(\cos 1) - \ln(\cos 0) \\ &= \ln 0.5403 = -0.6156 \end{aligned}$$

and the maximal value

$$u(1,\ 0) = -u(0,\ 1) = 0.6156.$$

Therefore, for the starting vectors $x^{(0)}$, $y^{(0)}$, we can select

$$x^{(0)} = (-0.62,\ -0.62,\ \cdots,\ -0.62)^T,$$
$$y^{(0)} = (0.62,\ 0.62,\ \cdots,\ 0.62)^T.$$

Obviously (3.13) satisfies condition (2.7).

Superimpose square grid sover the unit square with mesh size $h = 1/(N+1)$ for some positive integer N.

We calculate the solution of (3.1), (3.2) with the difference scheme (3.11) obtained by quadrilateral element. In our example we set $N=3$. Thus, we get a system of 9 nonlinear equations.

For the starting vectors $x^{(0)}$, $y^{(0)}$, we have

$x^{(0)}$:

0.615626574	0.584045529	0.485042334	0.303226650	0
0.312399924	−0.62	−0.62	−0.62	−0.303226650

0.130584240	−0.62	−0.62	−0.62	−0.485042334
0.031580988	−0.62	−0.62	−0.62	−0.584045529
0.0	−0.031580988	−0.130584240	−0.312399924	−0.615626574

$y^{(0)}$:

0.615626574	0.584045529	0.485042334	0.303226650	0
0.312399924	+0.62	+0.62	+0.62	−0.303226650
0.130584240	+0.62	+0.62	+0.62	−0.485042334
0.031580988	+0.62	+0.62	+0.62	−0.584045529
0.0	−0.031580988	−0.130584240	−0.312399924	−0.615626574

After 45 iterations we obtain the following result:

$x^{(45)}=y^{(45)}$:

0.615626574	0.584045529	0.485042334	0.303226650	0
0.312399924	0.285193711	0.185495973	−0.000000203	−0.303226650
0.130584240	0.100936592	−0.000000346	−0.185496509	−0.485042334
0.031580988	−0.000000212	−0.100937128	−0.285194099	−0.584045529
0.0	−0.031580988	−0.130584240	−0.312399924	−0.615626574

The convergence of this method is slow, as in the case of a single nonlinear equation, but the method is simple. Moreover, we can obtain its global convergence and error estimate.

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