LINEAR FINITE ELEMENTS WITH HIGH ACCURACY*

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§1. Introduction

This paper is concerned with the superconvergence and the acceleration of finite element methods. We start with the simplest finite element method, namely the linear elements. By using the piecewise strongly regular triangulation (see Definition 3) we find that the stress in the given domain can be approximated with the accuracy $O\left(h^2\log\frac{1}{h}\right)$ (see Theorem 3.2). Furthermore, higher accuracy, like $O\left(h^3\log^2\frac{1}{h}\right)$ or $O\left(h^4\log^2\frac{1}{h}\right)$, can be achieved if the extrapolation method is adopted. It seems that the linear elements are good enough for achieving higher accuracy in some cases.

As a by-product, some posteriori error estimates for finite elements are obtained in the two-dimensional case.

The paper is built upon the previous works by Lin, Lu, Xu, Zhu (see [11—15, 22—26]). A number of important related works which have influenced our analysis are included in the bibliography.

We clarify the analysis and generalize the ideas in [11—15, 22—24]. New results as well as shorter and more revealing proofs of known results are obtained. For the sake of expository continuity, the paper is essentially self-contained.

§2. Some Superconvergence Estimates

For simplicity, Let us consider the model problem: Find $u\in H_0^1(\Omega)$ such that

$$-\Delta u=f \quad \text{in } \Omega, \tag{2.1}$$

where $\Omega\subset R^2$ is a convex domain with Lipschitze continuous boundary.

We will approximate (2.1) by the simplest finite element method, namely linear elements. For this, let $T_h=\{K\}$, $0<h<h_0<1$, be a finite triangulation, which is supposed to be quasi-uniform, i.e. it satisfies the following condition: Each triangle $K\in T_h$ contains a circle of radius c_1h and is contained in a circle of radius c_2h, where the constants c_1, c_2 do not depend on K or h.

For clarity, let us introduce the definitions of some special kinds of quasi-uniform triangulation:

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Definition 1. *A triangulation $T_h=\{K\}$ is called strongly regular if any two adjacent triangles of T_h form an approximate parallelogram, i.e. there exists a constant c independent of h, K, such that (see Fig. 1)*

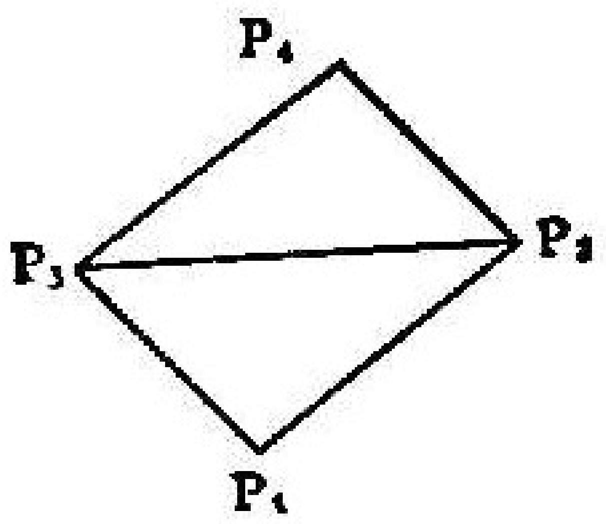

Fig. 1

$$|\overrightarrow{P_1P_2}-\overrightarrow{P_3P_4}|\leqslant ch^2. \tag{2.2}$$

Definition 2. *A triangulation is called completely regular if any two adjacent elements form an exact parallelogram.*

Generally speaking, the strong regularity condition is hard to be thoroughly satisfied over all the given region. For instance, a generic polygonal domain seemingly cannot be triangulated in the sense of strong regularity. But it is easy to observe that the strongly regular triangulation can be achieved over any quadrilateral or triangular region. For this reason we introduce the following

Definition 3. *A triangulation $T_h=\{K\}$ on the polygonal domain is called piecewise strongly regular, if Ω is divided into some quadrilateral or triangular subdomains with the vertices at the boundary, and the triangulation restricted on each such subdomain is strongly regular.*

Let S^h be a piecewise linear finite element space on Ω_h with zero on $\Omega\backslash\Omega_h$, and $u^h\in S^h$ the finite element approximation satisfying

$$a(u^h,\ v^h)=(f,\ v^h),\quad \forall v^h\in S^h. \tag{2.3}$$

For any fixed $z_0\in\Omega$, the Green function $G_{z_0}\in H_0^{1,1}(\Omega)$ is defined by

$$a(G_{z_0},\ v)=v(z_0),\quad \forall v\in C_0^\infty(\Omega) \tag{2.4}$$

and its finite element approximation $G_{z_0}^h\in S^h$ by

$$a(G_{z_0}^h,\ v^h)=v^h(z_0),\quad \forall v^h\in S^h. \tag{2.5}$$

The following estimates (for quasi-uniform triangulation) are shown by Frehse, Rannacher and Scott[6,16], Schatz and Wahlbin[17], Zhu[24]:

$$\|G_{z_0}-G_{z_0}^h\|_{1,1,\Omega}\leqslant ch\log\frac{1}{h}, \tag{2.6}$$

$$\|G_{z_0}-G_{z_0}^h\|_{0,1,\Omega}\leqslant ch^2\log^2\frac{1}{h}, \tag{2.7}$$

$$|(G_{z_0}-G_{z_0}^h)(z_1)|\leqslant ch^2\left|\log\frac{|z_1-z_0|}{h}\right|\Big/|z_1-z_0|^2, \tag{2.8}$$

$$|G_{z_1}^h-G_{z_2}^h|_{1,1,\Omega}\leqslant ch\log\frac{1}{h}, \tag{2.9}$$

$$\|G_{z_0}^h\|_{0,\infty,\Omega}+\left(\log\frac{1}{h}\right)^{\frac{1}{2}}\|G_{z_0}^h\|_{1,2,\Omega}\leqslant c\log\frac{1}{h}, \tag{2.10}$$

where $z_1,\ z_2\in\Omega$ with $|z_1-z_2|=O(h)$.

It is also known that there exists $q_0=q_0(\Omega)\in(2,\infty)$ such that if $q\in[2,\ q_0)$, then for all $F\in L^q(\Omega)$, there exists $v\in H^{2,q}(\Omega)\cap H_0^1(\Omega)$ such that

$$-\Delta v=F\quad \text{in } \Omega,$$

$$\|v\|_{2,q,\Omega}\leqslant C\|F\|_{0,q,\Omega}.$$

Lemma 2.1. *For each $p\in[1,\ 2)$ there exists a constant $c(p)>0$ such that*

$$\|G^h_{z_1}-G^h_{z_2}\|_{0,p,\Omega}\leqslant ch$$

for any $z_1,\ z_2\in\Omega$ *with* $|z_1-z_2|=O(h)$.

Proof. Let $v^h\in S^h$ be the finite element approximation of v. With the help of the result given in [16] and the inverse inequality one has

$$\|v^h-v\|_{0,\infty,\Omega}\leqslant ch^{2-\frac{2}{q}}\|v\|_{2,q,\Omega}.$$

Therefore

$$(G^h_{z_1}-G^h_{z_2},\ F)=a(G^h_{z_1}-G^h_{z_2},\ v)=a(G^h_{z_1}-G^h_{z_2},\ v^h)=v^h(z_1)-v^h(z_2)$$
$$\leqslant 2\|v-v^h\|_{0,\infty,\Omega}+|v(z_1)-v(z_2)|\leqslant ch\|F\|_{0,q,\Omega}.$$

Using the same trick we can obtain

Lemma 2.2. $\|G_{z_0}-G^h_{z_0}\|_{0,2,\Omega}\leqslant ch.$

In order to stress the key idea in [11—14], we confine ourselves to the case of parallelogramic or piecewise parallelogramic domain.

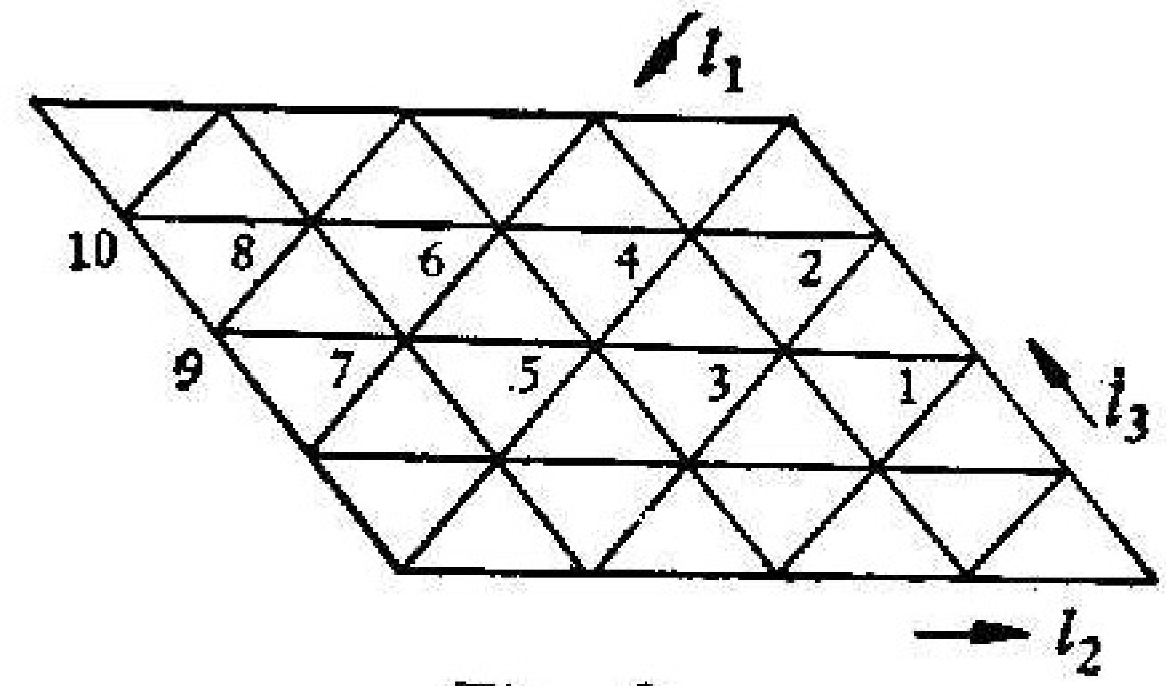

Fig. 2

Let D be a parallelogramic domain with the completely regular triangulation, as is shown in Fig. 2. We introduce the symbols l_1, l_2, l_3, which represent both the unit vectors in the directions indicated in Fig. 2 and the mesh sizes of the triangulation in the relative directions as the scalars.

Above all, les us study the integral $\iint\limits_D\nabla(u-u^I)\nabla v^h\,dx\,dy$ for any fixed $v^h\in S^h$. By the Green formula,

$$\iint\limits_D\nabla(u-u^I)\nabla v^h\,dx\,dy=\sum_{K\subset D}\int_{\partial K}(u-u^I)\frac{\partial v^h}{\partial n}\,ds. \tag{2.11}$$

For brevity, we denote Σ_j as the contribution of the line integrations in (2.11) in the direction $l_j(j=1,\ 2,\ 3)$. Without loss of generality, we study Σ_1. Obviously,

$$\Sigma_1=\left(\int_{23}+\int_{45}+\int_{67}+\int_{89}\right)(u-u^I)\frac{\partial}{\partial n}(v^h-\bar{v}^h)ds=\cdots,$$

where $\frac{\partial v^h}{\partial n}$ and $\frac{\partial\bar{v}^h}{\partial n}$ represent the values taken on the two adjacent elements K and K' respectively. We have written out only the representative terms and the remaining ones are quite similar.

Note that $\frac{\partial}{\partial n}=-\sin\theta\frac{\partial}{\partial x}+\cos\theta\frac{\partial}{\partial y}$ (θ is the directional angle of l_1 and evidently independent of h). By the symmetric positions of x and y, we only ought to investigate the following integral

$$J=\left(\int_{23}+\int_{45}+\int_{67}+\int_{89}\right)(u-u^I)\frac{\partial}{\partial x}(v^h-\bar{v}^h). \tag{2.12}$$

Let S be the area of triangle K_{123}; then

$$\frac{\partial v^h}{\partial x}=\frac{1}{2S}[(y_2-y_3)v^h_1+(y_3-y_1)v^h_2+(y_1-y_2)v^h_3]\quad\text{on } K_{123},$$

$$\frac{\partial v^h}{\partial x}=\frac{1}{2S}\left[(y_2-y_1)v_2^h+(y_3-y_2)v_4^h+(y_1-y_3)v_3^h\right] \quad \text{on } K_{123},$$

where $v_j^h=v^h(x_j,\ y_j)$. Consequently

$$\frac{\partial}{\partial x}(v^h-\bar{v}^h)=\frac{y_2-y_3}{2S}(v_1^h-v_2^h-v_3^h+v_4^h) \quad \text{on } l_{23}.$$

Substituting it and the analogues into (2.12) we have the following supperconvergence estimate

$$J=\frac{y_2-y_3}{2S}\Big[(v_1^h-v_2^h)\int_{23}+(v_3^h-v_4^h)\Big(\int_{45}-\int_{23}\Big)+(v_5^h-v_6^h)\Big(\int_{67}-\int_{45}\Big)$$
$$+(v_7^h-v_8^h)\Big(\int_{89}-\int_{67}\Big)-(v_9^h-v_{10}^h)\int_{89}\Big](u-u^I)ds. \tag{2.13}$$

Lemma 2.3.

$$\Big(\int_{23}-\int_{45}\Big)f\,ds=\frac{l_1l_2}{2S}\iint_{P_{2345}}\frac{\partial f}{\partial l_2}\,dx\,dy,$$

$$l_1\int_{12}f\,ds-l_3\int_{23}f\,ds=\frac{l_1l_2l_3}{2S}\iint_{K_{123}}\frac{\partial f}{\partial l_2}\,dx\,dy.$$

Corollary 1. For the strongly regular triangulation and $p,\ q\in[1,\ \infty]$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$a(u-u^I,\ v^h)\leqslant ch^2\|u\|_{3,p,\Omega}\|v^h\|_{1,q,\Omega}; \tag{2.14}$$

furthermore

$$|u^h-u^I|_{1,\infty,\Omega}\leqslant ch^2\log\frac{1}{h}\|u\|_{3,\infty,\Omega}. \tag{2.15}$$

In fact, one has

$$\int_a^b f(x)dx-\frac{b-a}{2}(f(a)+f(b))=-\frac{1}{2}\int_a^b(x-a)(b-x)f''(x)dx.$$

Hence

$$\Big(\int_{45}-\int_{23}\Big)(u-u^I)ds=\frac{1}{2}\Big(\int_{23}s_0s_1\frac{\partial^2u}{\partial l_1^2}ds-\int_{45}s_0s_1\frac{\partial^2u}{\partial l_1^2}ds\Big)$$
$$=\frac{l_1l_2}{4S}\iint_{P_{2345}}s_0s_1\frac{\partial^3u}{\partial l_1^2\partial l_2}\,dx\,dy=O(h^2)\iint_{P_{2345}}\left|\frac{\partial^3u}{\partial l_1^2\partial l_2}\right|dx\,dy.$$

Note that

$$v_3^h-v_4^h=-l_3\frac{\partial v^h}{\partial l_3}=O(h)\left|\frac{\partial v^h}{\partial l_3}\right|,$$

....

$$v_1^h=v_2^h=v_9^h=v_{10}^h=0.$$

By (2.13), we have

$$|J|\leqslant ch^2\iint_{P_{12\cdots10}}\left|\frac{\partial^3u}{\partial l_1^2\partial l_2}\right|\left|\frac{\partial v^h}{\partial l_3}\right|dx\,dy.$$

Thus (2.14) follows from

$$a(u-u^I,\ v^h)\leqslant ch^2\iint_\Omega|D^3u|\,|Dv^h|dx\,dy\leqslant ch^2|u|_{3,p,\Omega}|v^h|_{1,q,\Omega}.$$

(2.15) is the consequence of (2.10) and (2.14).

Corollary 2. If the triangulation on Ω is completely regular and $u\in H^{4,2+\varepsilon}(\Omega)$ $(\varepsilon>0)$, then

$$|u^h-u^I|_{1,2+\varepsilon,\Omega}\leqslant ch^2\|u\|_{4,2+\varepsilon,\Omega}. \tag{2.16}$$

In fact, from the Euler–Maclaurin formula

$$\int_{23}(u-u^I)ds=-\frac{l_1^2}{12}\int_{23}\frac{\partial^2 u}{\partial l_1^2}ds+O(l_1^3)\int_{23}p(s)\frac{\partial^3 u}{\partial l_1^3}ds$$

one has, from Lemma 2.3,

$$\left(\int_{45}-\int_{23}\right)\frac{\partial^2 u}{\partial l_1^2}ds=\frac{l_1 l_2}{2S}\iint_{P_{2345}}\frac{\partial^3 u}{\partial l_1^2\partial l_2}dx\,dy, \tag{2.17}$$

$$v_3^h-v_4^h=-l_3\frac{\partial v^h}{\partial l_3}. \tag{2.18}$$

Consequently

$$\begin{aligned}(v_3^h-v_4^h)&\left(\int_{45}-\int_{23}\right)(u-u^I)ds\\ &=-\frac{l_1^3 l_2 l_3}{24S}\iint_{P_{2345}}\frac{\partial^3 u}{\partial l_1^2\partial l_2}\frac{\partial v^h}{\partial l_3}dx\,dy+O(h^4)\iint_{P_{2345}}|\nabla^4 u|\,|\nabla v^h|\,dx\,dy.\end{aligned} \tag{2.19}$$

Note that

$$\frac{\partial v^h}{\partial l_3}=0\quad\text{on } K_{123}\cup K_{8\overline{10}9}. \tag{2.20}$$

Thus from (2.15)—(2.20) we can conclude that

$$\Sigma_1=ch^2\iint_D\frac{\partial^3 u}{\partial l_1^2\partial l_2}\frac{\partial v^h}{\partial l_3}dx\,dy+O(h^3)\iint_D|\nabla^4 u|\,|\nabla v^h|\,dx\,dy, \tag{2.21}$$

where c is a constant independent of h, u and v^h.

Now let $D=\Omega$, z_1, z_2 be any two adjacent nodes of the mesh, $w^h=u^h-u^I$, $v^h=G_{z_1}^h-G_{z_2}^h$,

$$w^h(z_1)-w^h(z_2)=a(G_{z_1}^h-G_{z_2}^h,\ w^h)=a(u-u^I,\ v^h)=\sum_{j=1}^{3}\Sigma_j,$$

where Σ_1 is as shown in (2.21), and Σ_2, Σ_3 are similar.

By the Green formula and Lemma 2.1,

$$\left|\iint_\Omega\frac{\partial^3 u}{\partial l_1^2\partial l_2}\frac{\partial v^h}{\partial l_3}\right|=\left|\iint_\Omega\frac{\partial^4 u}{\partial l_1^2\partial l_2\partial l_3}v^h\,dx\,dy\right|\leqslant ch|u|_{4,2+\varepsilon}.$$

By (2.10) and the inverse inequality, $\iint_D|\nabla^4 u|\,|\nabla v^h|\,dx\,dy\leqslant c(\varepsilon)|u|_{4,2+\varepsilon,\Omega}$. Hence we obtain from (2.21) that

$$|\Sigma_1|\leqslant ch^3\|u\|_{4,2+\varepsilon}.$$

Thus we have

$$|w^h(z_1)-w^h(z_2)|\leqslant ch^3\|u\|_{4,2+\varepsilon}.$$

This implies (2.16).

Corollary 3. If the triangulation on the polygonal domain Ω is piecewise strongly regular, and $u\in H^{4,2+\varepsilon}(\Omega)$, then

$$|u^h-u^I|_{1,\infty,\Omega}\leqslant ch^2\log\frac{1}{h}\|u\|_{4,2+\varepsilon,\Omega}.$$

In fact,

$$w^h(z_1)-w^h(z_2)=\sum_{D\subset\Omega}\iint_D \nabla(u-u^I)\nabla v^h\,dx\,dy,$$

where $v^h=G^h_{z_1}-G^h_{z_2}$, and D is the subdomain of Ω which is quadrilateral or triangular and strongly regularly triangulated.

For each D, instead of (2.21), we have, by means of Lemma 2.3,

$$\Sigma_1=c_0h^2\iint\frac{\partial^3 u}{\partial l_1^2\partial l_2}\frac{\partial v^h}{\partial l_3}\,dx\,dy+c_1h^2\int_{AB}\frac{\partial^2 u}{\partial l_1^2}\frac{\partial v^h}{\partial l_3}\,ds$$

$$+c_2h^2\int_{CD}\frac{\partial^2 u}{\partial l_1^2}\frac{\partial v^h}{\partial l_3}\,ds+O(h^3)\|u\|_{4,2+\varepsilon,\Omega}.$$

Integration by parts shows that

$$\left|\int_{AB}\frac{\partial^2 u}{\partial l_1^2}\frac{\partial v^h}{\partial l_3}\,ds\right|=\left|\int_{AB}\frac{\partial^3 u}{\partial l_1^2\partial l_3}v^h\,ds\right|\leqslant\int_{AB}|v^h|\,ds\,|u|_{3,\infty,\Omega}.$$

Since

$$\int_{AB}|v^h|\,ds\leqslant\|v^h\|_{1,1,\Omega},\tag{2.22}$$

taking (2.6) and (2.10) into account yields

$$\left|\int_{AB}\frac{\partial^2 u}{\partial l_1^2}\frac{\partial v^h}{\partial l_3}\,ds\right|\leqslant ch\log\frac{1}{h}\|u\|_{3,2+\varepsilon,\Omega}.$$

Similarly,

$$\left|\int_{CD}\frac{\partial^2 u}{\partial l_1^2}\frac{\partial v^h}{\partial l_3}\,ds\right|\leqslant ch\log\frac{1}{h}\|u\|_{3,2+\varepsilon,\Omega}.$$

Then, arguing as before, we deduce the desired results.

Remark. The idea stated above can be generalized to the case of curved boundary and we will discuss it in detail in a separate paper.

§ 3. High Accuracy Approximation to Stress in the Whole Region

On the superconvergence of linear triangular elements, a main conclusion is that the midpoint of the common side of two adjacent elements can be approximated with nearly two order accuracy[11]. By means of this result and the interpolation Krizek and Neittaanmaki[8] have proved that the stress at every point in any interior domain of Ω can be approximated with the accuracy $O(h^2\log h)$. In this section, we will generalize the said results.

First of all, one can easily prove (see also Lemma 4.1)

Lemma 3.1. *Let $M(h)$ be the set of all the midpoints of common sides of any two adjacent elements which form an approximate parallelogram, and $N(h)$ the set of such interior nodes that adjoin six elements among which any two adjacent ones form an approximate parallelogram. If $u\in H^{3,\infty}(\Omega)$, then*

$$\max_{M\in M(h)}|\bar\nabla(u-u^I)(M)|=O(h^2),\quad M\in M(h),$$

$$\max_{N\in N(h)}|\bar\nabla(u-u^I)(N)|=O(h^2),\quad N\in N(h),$$

where $\bar\nabla$ represents the arithmetic mean of gradients on the relative adjacent elements.

The above lemma together with estimate (2.15) leads to

Theorem 3.1. *For the (piecewise) strongly regular triangulation,*

$$\max_{M\in M(h)}|\overline{\nabla}(u-u^h)(M)|=O\left(h^2\log\frac{1}{h}\right),\quad M\in M(h),$$

$$\max_{N\in N(h)}|\overline{\nabla}(u-u^h)(N)|=O\left(h^2\log\frac{1}{h}\right),\quad N\in N(h),$$

moreover the factor $\log\frac{1}{h}$ *may be removed sometimes (cf. Corollary 2 in § 2).*

It is also easy to verify

Lemma 3.2. *Let* $P_i(i=1,2,3,4)$ *be four points on* R^2, P_0 the intersection point of P_1P_2, P_3P_4. $|P_1P_2|,|P_3P_4|=O(h)$, $l_1=\overrightarrow{P_1P_2}/|P_1P_2|$, $l_2=\overrightarrow{P_3P_4}/|P_3P_4|$. *If* A_i, B_i $(i=1,2)$ *areknown and satisfy*

$$\left|\frac{\partial u}{\partial l_2}(P_i)-A_i\right|+\left|\frac{\partial u}{\partial l_2}(P_{i+2})-B_i\right|=O(h^2|\log h|^\sigma)$$

for $i=1,2$ *and some constant* $\sigma\geqslant 0$, *then two such numbers* A_0, B_0 *can be determined by* A_i, B_i $(i=1,2)$ *that*

$$\left|\nabla u(P_0)-\begin{pmatrix}A_0\\B_0\end{pmatrix}\right|=O(h^2|\log h|^\sigma).$$

Using the techniques of extrapolation and interpolation, we can obtain from Theorem 3.1 and Lemma 3.2 the following main result.

Theorem 3.2. *For (piecewise) strongly regular triangulation, a piecewise linear function* $D^hu\in H^1(\Omega)\times H^1(\Omega)$ *can be constructed by* u^h *such that*

$$\|\nabla u-D^hu\|_{0,\infty,\Omega}=O(h^2|\log h|).$$

§ 4. The Asymptotic Expansion and Extrapolation for Finite Elements

1) The case of parallelogramic domain.

Let Ω be a parallelogramic domain which is first divided into several triangles as is shown in Fig. 3. Starting from these triangles, a successively refined triangulation is constructed, i.e. the k-th refinement is obtained by connecting every midpoint of the edges of all triangles of the $(k-1)$-th refinement. Having refined for s times and setting $h=1/2^s$, we then obtain a family of successively refined triangulations $\{T_h\}$. Corresponding to T_h and $T_{\frac{h}{2}}$, the finite element approximations and interpolations of u will be denoted by u^h, $u^{\frac{h}{2}}$ and u^I, $u^{\frac{I}{2}}$ respectively.

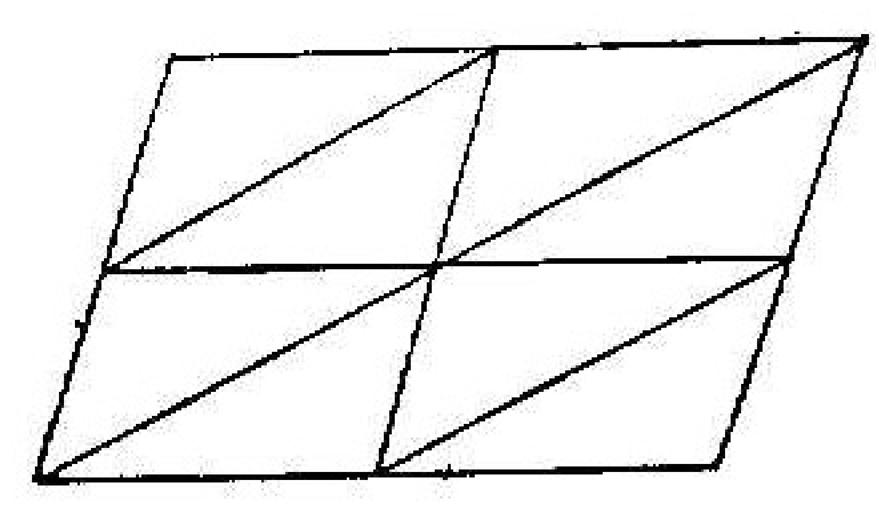

Fig. 3

Theorem 4.1. i) *If* $u\in H^{4,2}(\Omega)$,

$$(u^h-u^I)(z)=w(z)h^2+O\left(h^3\left(\log\frac{1}{h}\right)^{\frac{1}{2}}\right),\quad \forall z\in\Omega; \tag{4.1}$$

ii) *if* $u\in H^{4,\infty}(\Omega)$,

$$\nabla(u^h-u^I)(z)=\nabla w(z)h^2+O(h^{3-\frac{2}{q}}),\quad \forall z\in\Omega, \tag{4.2}$$

$$\nabla(u^h-u^I)(z)=\nabla w(z)h^2+O\left(h^3\log^2\frac{1}{h}\right),\quad \forall z\in\tilde{\Omega}; \tag{4.3}$$

iii) *if* $u\in H^{5,\infty}(\Omega)\cup H^{6,1}(\Omega)$,

$$(u^h-u^I)(z)=w(z)h^2+O\left(h^4\log^2\frac{1}{h}\right),\quad \forall z\in\tilde{\Omega}, \tag{4.4}$$

where $\tilde{\Omega}\subset\subset\Omega$, $2<q<q_0(\Omega)$ (*see* § 2), $w(z)$ *is the solution of the auxiliary problem*

$$-\Delta w=\tilde{D}^4u \quad in\ \Omega, \tag{4.5}$$

$$w=0 \quad on\ \partial\Omega, \tag{4.6}$$

where the differential operator $\tilde{D}^4$ *will be described below* (*see* (4.8)).

Proof. With the same arguments as in § 2, we can obtain

$$\begin{aligned}(u^h-u^I)(z)&=a(u-u^I,\ G_z^h)\\ &=h^2\iint_\Omega\left(c_1\frac{\partial^4u}{\partial l_1^2\,\partial l_2\,\partial l_3}+c_2\frac{\partial^4u}{\partial l_2^2\,\partial l_3\,\partial l_1}+c_3\frac{\partial^4u}{\partial l_3^2\,\partial l_1\,\partial l_2}\right)G_z^h\,dx\,dy\\ &\quad+O(h^3)\iint_\Omega|\nabla G_z^h|\,|\nabla^4u|\,dx\,dy.\end{aligned} \tag{4.7}$$

Set

$$\tilde{D}^4=c_1\frac{\partial^4}{\partial l_1^2\,\partial l_2\,\partial l_3}+c_2\frac{\partial^4}{\partial l_2^2\,\partial l_3\,\partial l_1}+c_3\frac{\partial^4}{\partial l_3^2\,\partial l_1\,\partial l_2}; \tag{4.8}$$

then (4.1) follows from (2.10), (4.7) and Lemma 2.2.

In virtue of the theory of P. D. E. (see also § 2),

$$w\in H^{2,q}(\Omega)\cap H^{2,\infty}(\tilde{\Omega}); \tag{4.9}$$

hence[18]

$$|w-w^I|_{1,\infty,\Omega}=O(h^{1-\frac{2}{q}}), \tag{4.10}$$

$$|w-w^I|_{0,\infty,\tilde{\Omega}}=O(h). \tag{4.11}$$

Let K_{123} be an element of T_h, λ_1, λ_2, λ_3 the relative barycentric coordinates, and $v^h=u^h-u^I$. For $z\in K_{123}$, we have

$$\begin{aligned}\nabla(u^h-u^I)(z)&=\sum_{j=1}^3 v^h(j)\nabla\lambda_j=(v^h(1)-v^h(3))\nabla\lambda_1+(v^h(2)-v^h(3))\nabla\lambda_2\\ &=a(v^h,\ G_1^h-G_3^h)\nabla\lambda_1+a(v^h,\ G_2^h-G_3^h)\nabla\lambda_2\\ &=a(u-u^I,\ G_1^h-G_3^h)\nabla\lambda_1+a(u-u^I,\ G_2^h-G_3^h)\nabla\lambda_2.\end{aligned} \tag{4.12}$$

As analogous to (4.7),

$$a(u-u^I,\ G_1^h-G_3^h)=h^2\iint_\Omega\tilde{D}^4u(G_1^h-G_3^h)\,dx\,dy+O(h^3)\iint_\Omega|\nabla(G_1^h-G_3^h)|\,dx\,dy. \tag{4.13}$$

From (2.6)—(2.9),

$$\begin{aligned}a(u-u^I,\ G_1^h-G_3^h)&=h^2\iint_\Omega\tilde{D}^4u(G_1-G_3)\,dx\,dy+O\left(h^4\log^2\frac{1}{h}\right)\\ &=h^2(w(1)-w(3))+O\left(h^4\log^2\frac{1}{h}\right).\end{aligned} \tag{4.14}$$

Likewise

$$a(u-u^I,\ G_2^h-G_3^h)=h^2(w(2)-w(3))+O\left(h^4\log^2\frac{1}{h}\right).$$

Since $|\nabla\lambda_1|$, $|\nabla\lambda_2|=O(h^{-1})$, therefore

$$\nabla(u^h-u^I)(z)=h^2((w(1)-w(3))\nabla\lambda_1+(w(2)-w(3))\nabla\lambda_2)+O\left(h^3\log^2\frac{1}{h}\right)$$
$$=h^2\nabla w^I(z)+O\left(h^3\log^2\frac{1}{h}\right).$$

Thus (4.2), (4.3) follow from (4.10), (4.11).

In order to prove (4.4), we take one more term in the Euler-Maclaurin formula if $u\in H^{5,\infty}(\Omega)$, i. e. instead of (2.16), we now have

$$\int_{23}(u-u^I)\,ds=-\frac{l_1^2}{12}\int_{23}\frac{\partial^2 u}{\partial l_1^2}ds+\frac{l_1^4}{720}\int_{23}\frac{\partial^4 u}{\partial l_1^4}ds+O(h^6).$$

Corresponding to (2.19),

$$(v_3^h-v_4^h)\left(\int_{45}-\int_{23}\right)(u-u^I)\,ds$$
$$=C_0h^3\iint_{P_{2354}}\frac{\partial^3 u}{\partial l_1^2\partial l_2}\frac{\partial v^h}{\partial l_3}dx\,dy+C_1h^5\iint_{P_{2354}}\frac{\partial^5 u}{\partial l_1^4\partial l_3}\frac{\partial v^h}{\partial l_3}dx\,dy$$
$$+O(h^5)\iint_{P_{2354}}\left|\frac{\partial v^h}{\partial l_3}\right|dx\,dy.$$

Thus, as analogous to (4.7), we can obtain, for $v^h\in S^h$,

$$a(u-u^I,\ v^h)=h^2\iint_\Omega \tilde{D}^4uv^h\,dx\,dy+O(h^4)\iint_\Omega|\nabla V^h|\,dx\,dy.$$

Taking $v^h=G_z^h$ and using (2.17) lead to (4.4).

If $u\in H^{6,1}$, we can obtain

$$a(u-u^I,\ G_z^h)=h^2\iint_\Omega\tilde{D}^4uG_z^h\,dx\,dy$$
$$+O(h^4)\iint_\Omega|\nabla^5u|\,|\nabla v^h|\,dx\,dy+O(h^5)\iint_\Omega|\nabla^6u|\,|\nabla v^h|\,dx\,dy.$$

Thus (4.4) follows from (2.9) and the imbedding relations $H^{6,1}\hookrightarrow H^{4,\infty}$ and $H^{6,1}\hookrightarrow H^{5,2}$.

Lemma 4.1. *Suppose $M(h)$, $N(h)$ are as described in Lemma 3.1. Let $B(h)$ be the set of the barycenters of all elements, $E(h)$ the set of points on the edges of all elements which are located in the middle between the points respectively in $N(h)$ and in $M(h)$. Then*

i) *if* $u\in H^{3,\infty}$,

$$\left(u-\frac{4u^{\frac{I}{2}}-u^I}{3}\right)(z)=O(h^3),\quad \forall z\in B(h),$$
$$\left(u-\frac{3u^{\frac{I}{2}}-u^I}{2}\right)(z)=O(h^3),\quad \forall z\in E(h);$$

ii) *if* $u\in H^{4,2}$,

$$\left(u-\frac{4u^{\frac{I}{2}}-u^I}{3}\right)(z)=O\left(h^3\left(\log\frac{1}{h}\right)^{\frac{1}{2}}\right),\quad \forall z\in B(h),$$
$$\left(u-\frac{3u^{\frac{I}{2}}-u^I}{2}\right)(z)=O\left(h^3\left(\log\frac{1}{h}\right)^{\frac{1}{2}}\right),\quad \forall z\in E(h);$$

iii) *if* $u\in H^{4,\infty}$,

$$\overline{\nabla}\left(u-\frac{4u^{\frac{I}{2}}-u^I}{3}\right)(z)=O(h^3),\quad z\in N(h).$$

For the proof, let us introduce two equalities (cf. [25])

$$(u-u^I)(p)=-\sum_{i=1}^{3}\lambda_i(p)\int_0^1 t\,\partial_i^2 u(M_i)dt, \tag{4.15}$$

$$\nabla(u-u^I)(p)=-\sum_{i=1}^{3}\nabla\lambda_i(p)\int_0^1 t\,\partial_i^2 u(M_i)dt, \tag{4.16}$$

where P_i are the vertices of an element, λ_i the barycentric coordinates, $M_i=P_i+(P-P_i)t$, $P=(x, y)$ the variable point.

i) and ii) can be easily proved by careful calculation with the aid of (4.15) and (4.16).

In order to prove ii), we need a result given in [26].

Lemma 4.2. *There exists a constant $c>0$ such that for any segment $\Gamma\subset\Omega$,*

$$\|u\|_{L^2(\Gamma)}\leqslant c\,|d\log d|^{1/2}\|u\|_{1,\Omega},\quad \forall u\in H^1(\Omega),$$

where $d=$length of $\Gamma\leqslant d_0<1$.

Now we confine ourselves to the proof of the first formula of ii). From (4.15) we can deduce that

$$\left|\left(u-\frac{4u^{\frac{I}{2}}-u^I}{3}\right)(z)\right|\leqslant c\sum_{i=1}^{3}\left(\left|\int_0^1 t^2\,\partial_i^3 u(M_i)dt\right|+\left|\int_0^1 t^2\partial_i^3 u(\widetilde{M}_i)dt\right|\right),$$

where $\widetilde{M}_i$, analogous to M_i, is defined in correspondence with $T_{\frac{h}{2}}$, but

$$\left|\int_0^1 t^2\partial_i^3 u(M_i)dt\right|\leqslant ch^3\int_0^1|\nabla^3 u(M_i)|dt\leqslant ch^2\int_{P_i}^{z}|\nabla^3 u(\xi,\ \eta)|ds$$
$$\leqslant ch^{\frac{5}{2}}\|\nabla^3 u\|_{L^2(\overline{zP_i})}\leqslant ch^3\left(\log\frac{1}{h}\right)^{1/2}\|u\|_{4,2,\Omega},$$

where we have used Lemma 4.2 in the last step.

The desired result is therefore proved.

Theorem 4.2. i) *If $u\in H^{4,2}$,*

$$\max_{z\in N(h)\cup B(h)}\left|\left(u-\frac{4u^{\frac{h}{2}}-u^h}{3}\right)(z)\right|=O\left(h^3\left(\log\frac{1}{h}\right)^{\frac{1}{2}}\right); \tag{4.17}$$

ii) *if $u\in H^{4,\infty}$,*

$$\max_{z\in N(h)}\left|\overline{\nabla}\left(u-\frac{4u^{\frac{h}{2}}-u^h}{3}\right)(z)\right|=O\left(h^3\log^2\frac{1}{h}\right); \tag{4.18}$$

iii) *if $u\in H^{5,\infty}\cup H^{6,1}$,*

$$\max_{z\in N(h)}\left|\left(u-\frac{4u^{\frac{h}{2}}-u^h}{3}\right)(z)\right|=O\left(h^4\log^2\frac{1}{h}\right). \tag{4.19}$$

Proof. Obviously, (4.17) and (4.19) are the consequences of Theorem 4.1 and Lemma 4.1.

Now we prove (4.18). Let $z\in N(h)$ be given and $\tilde{z}$ be any vertex, other than z, of the elements that z adjoins in $\frac{h}{2}$-mesh. Set $w^h=u^h-u^I$. From (4.7), we have

$$w^h(\tilde{z})-w^h(z)=a(u-u^I,\ G^h_{\tilde{z}}-G^h_z)$$
$$=h^2\iint_\Omega \tilde{D}^4u(G^h_{\tilde{z}}-G^h_z)\,dx\,dy+O(h^3)\iint_\Omega|\nabla(G^h_{\tilde{z}}-G^h_z)dx\,dy. \quad (4.20)$$

Using the estimates (2.7) and (2.9), we obtain

$$w^h(\tilde{z})-w^h(z)=h^2\iint_\Omega \tilde{D}^4u(G_{\tilde{z}}-G_z)\,dx\,dy+O\Big(h^4\log^2\frac{1}{h}\Big),$$

namely
$$\frac{w^h(\tilde{z})-w^h(z)}{|\tilde{z}-z|}=h^2\iint_\Omega \tilde{D}^4u\frac{G_{\tilde{z}}-G_z}{|\tilde{z}-z|}\,dx\,dy+O\Big(h^3\log^2\frac{1}{h}\Big).$$

Likewise

$$\frac{w^{\frac{h}{2}}(\tilde{z})-w^{\frac{h}{2}}(z)}{|\tilde{z}-z|}=\Big(\frac{h}{2}\Big)^2\iint_\Omega D^4u\frac{G_{\tilde{z}}-G_z}{|\tilde{z}-z|}\,dx\,dy+O\Big(h^3\log^2\frac{1}{h}\Big).$$

Thus, (4.18) follows from the above two equalities.

Remark 4.1. The factor $\left(\log\frac{1}{h}\right)^{1/2}$ in (4.17) can be removed when $u\in H^{4,2+\varepsilon}$ $(\varepsilon>0)$.

Remark 4.2. By using the technique of the regular Green function (see [6]) it may be proved that (4.17) is also valid when $u\in H^{3,\infty}$.

Theorem 4.3. *By the approximations u^h and $u^{h/2}$,*

i) *under the assumptions of* i), ii) *in Theorem* 4.2, *the piecewise quadric functions $\tilde{u}^h\in H^1(\Omega)$, $D^hu\in H^1(\Omega)\times H^1(\Omega)$ can be constructed such that*

$$\|u-\tilde{u}^h\|_{0,\infty,\Omega}=O\Big(h^3\Big(\log\frac{1}{h}\Big)^{1/2}\Big),$$
$$\|\nabla u-D^hu\|_{0,\infty,\Omega}=O\Big(h^3\log^2\frac{1}{h}\Big);$$

ii) *if $u\in H^{5,\infty}$, a piecewise cubic function $\tilde{u}^h\in H^1(\Omega)$ can be constructed such that*

$$\|u-\tilde{u}^h\|_{0,\infty,\Omega}=O\Big(h^4\log^2\frac{1}{h}\Big).$$

2) The case of generic polygonal domain.

Let Ω be a polygonal domain. First, we divide Ω into several triangular domains Ω_i $(i=1,2,\cdots,m)$ which meet at a point $A\in\Omega$ (see Fig. 4). Then, a successively refined triangulation T_h is constructed in the same way as is described in 1).

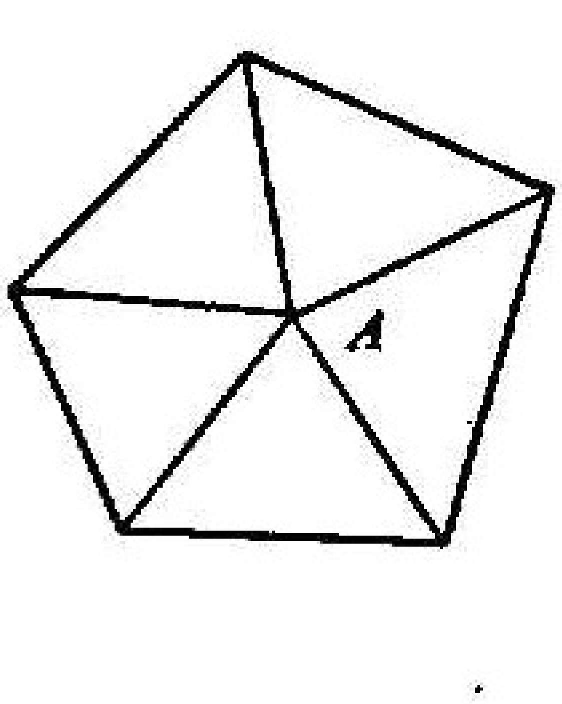

Fig. 4

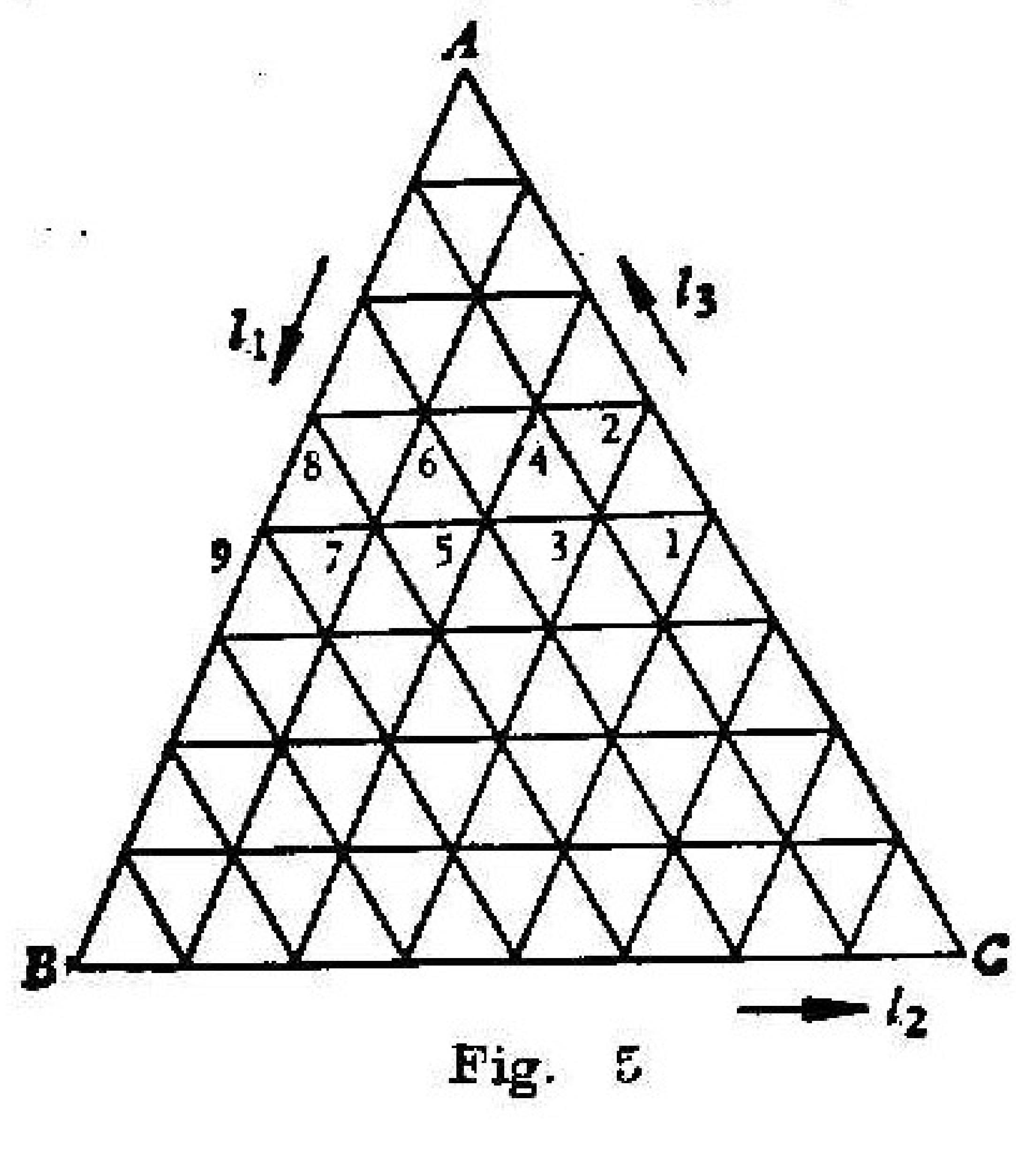

Fig. 5

Theorem 4.4. *Suppose* $\tilde{\Omega}=\bigcup_{i=1}^{m}\tilde{\Omega}_i(\tilde{\Omega}_i\subset\subset\Omega_i)$. *There exist two continuous functions* $K_0(z)$ *and* $K(z)\in H^{2,\infty}(\tilde{\Omega})$, *such that*

i) *if* $u\in H^{4,\infty}$,

$$(u^h-u^I)(z)=K_0(z)h^2+O\Big(h^3\log\frac{1}{h}+h^4\left|\log\frac{|A-z|}{h}\right|\Big/|A-z|^2\Big)$$
$$\text{for } z\in\Omega\backslash\{A\}, \tag{4.21}$$

$$\nabla(u^h-u^I)(z)=\nabla K(z)h^2+O\Big(h^3\log^2\frac{1}{h}\Big)\quad \text{for } z\in\tilde{\Omega}; \tag{4.22}$$

ii) *if* $u\in H^{5,\infty}\cup H^{6,1}$,

$$(u^h-u^I)(z)=K(z)h^2+O\Big(h^4\log^2\frac{1}{h}\Big)\quad \text{for } z\in\tilde{\Omega}.$$

Proof. Since

$$(u^h-u^I)(z)=\sum_{i=1}^{m}\iint_{\Omega_i}\nabla(u-u^I)\nabla G_z^h\,dx\,dy,$$

taking fixed Ω_i, as is illustrated in Fig. 5, we study the integral

$$\iint_{\Omega_i}\nabla(u-u^I)\nabla G_z^h\,dz\,dy.$$

As analogous to (2.13), we have

$$J=\frac{y_2-y_3}{2S}\Big[(G_1^h-G_2^h)\int_{23}+(G_3^h-G_4^h)\Big(\int_{45}-\int_{23}\Big)$$
$$+(G_5^h-G_6^h)\Big(\int_{67}-\int_{45}\Big)+(G_7^h-G_8^h)\Big(\int_{89}-\int_{67}\Big)\Big](u-u^I)ds$$
$$+\frac{y_2-y_1}{2S}(G_8^h-G_9^h)\int_{89}(u-u^I)ds.$$

By the Euler-Maclaurin formula and Lemma 2.3,

$$\int_{23}(u-u^I)\,ds=-\frac{1}{12}l_1^2\int_{23}\frac{\partial^2u}{\partial l_1^2}\,ds+O(h^5)$$
$$=\frac{l_1^3l_2}{24S}\iint_{\Delta_{123}}\frac{\partial^3u}{\partial l_1^2\,\partial l_2}\,dx\,dy-\frac{l_1^3}{12l_3}\int_{12}\frac{\partial^2u}{\partial l_1^2}\,ds+O(h^5),$$

where $\overline{12}$, $\overline{23}$ are the midpoints of the relative edges of Δ_{123} and $\overline{123}$ the midpoint of the connection line of points $\overline{12}$ and $\overline{23}$.

Therefore, in correspondence with (2.21), we have

$$\Sigma_{i1}=h^2\Big(c_1\iint_{\Omega_i}\frac{\partial^3u}{\partial l_1^2\,\partial l_2}\frac{\partial G^h}{\partial l_3}\,dx\,dy+c_2\int_{AB}\frac{\partial^2u}{\partial l_1^2}\frac{\partial G^h}{\partial l_1}\,ds$$
$$+c_3\int_{CA}\frac{\partial^2u}{\partial l_1^2}\frac{\partial G^h}{\partial l_3}\,ds\Big)+O(h^3)\,|G^h|_{1,1,\Omega_i},$$

where the constants c_1, c_2, c_3 do not depend on h, u, z.

Integrating by parts the line integrations $\int_{AB}$ and $\int_{CA}$ we obtain

$$\Sigma_{i1}=h^2\Big(c_1\iint_{\Omega}\frac{\partial^3 u}{\partial l_1^2\partial l_2}\frac{\partial G^h}{\partial l_3}\,dx\,dy+(c_2+c_3)\frac{\partial^2 u}{\partial l_1^2}(A)G^h(A)$$
$$-c_2\int_{AB}G^h\frac{\partial^3 u}{\partial l_1^3}\,ds-c_3\int_{CA}G^h\frac{\partial^3 u}{\partial l_1^2\partial l_3}\,ds\Big)+O(h^3)\,|G^h|_{1,1,\Omega_i}. \tag{4.23}$$

Hence (4.21) follows from the estimates (2.6)—(2.8) and an inequality like (2.22).

In order to prove (4.22), we first make a further investigation into $a(u-u^I, G_z^h)$ as just studied, but we define $z\in\tilde{\Omega}$. Using the Green formula and estimates (2.7) and (2.8), we can obtain from (4.23)

$$\Sigma_{i1}=h^2\Big(A_i\iint_{\Omega_i}\frac{\partial^4 u}{\partial l_1^2\partial l_2\partial l_3}G_z\,dx\,dy+B_i\frac{\partial^2 u}{\partial l_1^2}(A)G_z(A)$$
$$+\int_{AB}P_i(u)G_z\,ds+\int_{CA}Q_i(u)G_z\,ds\Big)$$
$$+O\Big(h^4\log^2\frac{1}{h}\Big)+O(h^3)\,|G_z^h|_{1,1,\Omega_i},$$

where A_i, B_i, C_i are constants independent of h, u, z, $P_i(u)$, $Q_i(u)$ are linear combinations of some order ($\leqslant 3$) derivatives of u.

Defining the function F_1 such that

$$F_1(x, y)=A_i\frac{\partial^4 u(x, y)}{\partial l_1^2\partial l_2\partial l_3}\quad\text{if } (x, y)\in\Omega_i,$$

we obviously have $F_1\in L^\infty(\Omega)$. Consider an auxiliary problem

$$-\Delta\phi_1=F_1\quad\text{in } \Omega,$$
$$\phi_1=0\quad\text{on } \partial\Omega.$$

Then $\phi_1\in H^2(\Omega)\cap H^{2,\infty}(\tilde{\Omega})$.

On the other hand, set

$$\psi_1(z)=\sum_{i=1}^m\Big(B_i\frac{\partial^2 u}{\partial l_1^2}(A)G_z(A)+\int_{AB}P_i(u)G_z\,ds+\int_{CA}Q_i(u)G_z\,ds\Big).$$

From the property of the Green function, we see that

$$\psi_1(z)\in W^{2,\infty}(\tilde{\Omega}).$$

To sum up the above arguments,

$$\Sigma_1=\sum_{i=1}^m\Sigma_{i1}=h^2(\phi_1(z)+\psi_1(z))+O(h^3)\,|G_z^h|_{1,1,\Omega}+O\Big(h^4\log^2\frac{1}{h}\Big).$$

Likewise there exist other four functions $\phi_j(z)$, $\psi_j(z)\in W^{2,\infty}(\tilde{\Omega})$ ($j=2, 3$) such that

$$\Sigma_j=h^2(\phi_j(z)+\psi_j(z))+O(h^3)\,|G_z^h|_{1,1,\Omega}+O\Big(h^4\log^2\frac{1}{h}\Big).$$

Set

$$K(z)=\sum_{j=1}^3(\phi_j(z)+\psi_j(z))\in H^{2,\infty}(\tilde{\Omega}).$$

We therefore have

$$a(u-u^I, G_z^h)=h^2K(z)+O\Big(h^4\log^2\frac{1}{h}\Big).$$

Arguing as in 1), we deduce that

$$\nabla(u^h-u^I)(z)=\nabla K(z)h^2+O\left(h^3\log^2\frac{1}{h}\right).$$

The idea in proving ii) is the same and so the detail is omitted.

That completes the proof.

Theorem 4.5. *Under the assumptions of Theorem* 4.4,

i) *if* $u\in H^{4,\infty}$,

$$\left(u-\frac{4u^{\frac{h}{2}}-u^h}{3}\right)(z)=O\left(h^3\log\frac{1}{h}+h^4\left|\log\frac{|A-z|}{h}\right|\Big/|A-z|^2\right)$$

$$\text{for } z\in N(h)\cup B(h)\setminus\{A\}, \tag{4.24}$$

$$\nabla\left(u-\frac{4u^{\frac{h}{2}}-u^h}{3}\right)(z)=O\left(h^3\log\frac{1}{h}\right)\quad \text{for } z\in N(h)\cap\tilde{\Omega}; \tag{4.25}$$

ii) *if* $u\in H^{5,\infty}\cup H^{6,1}$,

$$\left(u-\frac{4u^{\frac{h}{2}}-u^h}{3}\right)(z)=O\left(h^4\log^2\frac{1}{h}\right)\quad \text{for } z\in N(h)\cap\tilde{\Omega}. \tag{4.26}$$

Theorem 4.6. *Under the assumptions of Theorem* 4.4 *and by the approximations* u^h *and* $u^{h/2}$,

i) *if* $u\in H^{4,\infty}$, *the piecewise quadric functions* $\tilde{u}^h\in H^1(\Omega)$, $D^h u\in H^1(\Omega)\times H^1(\Omega)$ *can be constructed such that*

$$(u-\tilde{u}^h)(z)=O\left(h^3\log\frac{1}{h}+h^4\left|\log\frac{|A-z|}{h}\right|\Big/|A-z|^2\right)\quad \text{for } z\in\Omega\setminus\{A\},$$

$$\|\nabla u-D^h u\|_{0,\infty,\tilde{\Omega}}=O\left(h^3\log^2\frac{1}{h}\right);$$

ii) *if* $u\in H^{5,\infty}$, *a piecewise cubic function* $\tilde{u}^h\in H^1(\Omega)$ *can be constructed such that*

$$\|u-\tilde{u}^h\|_{0,\infty,\tilde{\Omega}}=O\left(h^4\log^2\frac{1}{h}\right).$$

Remark 4.3. If the meeting point A is at the boundary of Ω, the terms with "singularities" appearing in (4.21), (4.24) and (4.27) (see also Theorem 5.1) can be removed.

§ 5. The Posteriori Error Estimates

As a by-product, the following posteriori error estimates can be obtained immediately from the results given in § 4.

Theorem 5.1. i) *Under the assumption of Theorem* 4.2,

$$(u-u^{\frac{h}{2}})(z)=\frac{(u^{\frac{h}{2}}-u^h)(z)}{3}+R(h)$$

for $z\in N(h)$, $R(h)=O\left(h^3\left(\log\frac{1}{h}\right)^{1/2}\right)$ *if* $u\in H^{4,2}$,

$$R(h)=O\left(h^4\log^2\frac{1}{h}\right)\quad \text{if } u\in H^{5,\infty}\cup H^{6,1},$$

$$\nabla(u-u^{\frac{h}{2}})(z)=\frac{\nabla(u^{\frac{h}{2}}-u^h)(z)}{3}+O\left(h^3\log^2\frac{1}{h}\right)$$

for $z\in N(h)$ *and* $u\in H^{4,\infty}$;

ii) *under the assumption of Theorem* 4.4,

$$(u-u^{\frac{h}{2}})(z)=\frac{(u^{\frac{h}{2}}-u^h)(z)}{3}+R(h)$$

for $z\in N(h)\setminus\{A\}$, $R(h)=O\left(h^3\log\frac{1}{h}+h^4\left|\log\frac{|A-z|}{h}\right|\Big/|A-z|^2\right)$ *if* $u\in H^{4,\infty}$, *for* $z\in N(h)\cap\tilde{\Omega}$, $R(h)=O\left(h^4\log\frac{1}{h}\right)$ *if* $u\in H^{5,\infty}\cup H^{6,1}$,

$$\nabla(u-u^{\frac{h}{2}})(z)=\frac{\nabla(u^{\frac{h}{2}}-u^h)(z)}{3}+O\left(h^3\log^2\frac{1}{h}\right)$$

for $z\in N(h)\cap\tilde{\Omega}$ *and* $u\in H^{4,\infty}$.

Furthermore, the main terms of the errors in i) or ii) satisfy

$$\frac{(u^{\frac{h}{2}}-u^h)(z)}{3},\quad \frac{\nabla(u^{\frac{h}{2}}-u^h)(z)}{3}=O(h^2)$$

for z in its relative range.

§ 6. Superconvergence, Extrapolation and Deferred Correction

Superconvergence, extrapolation and deferred correction can all be regarded as the methods for accelerating the convergence of finite elements. One may hope that some synthesizing utilizations of these methods will produce some new results. In this section, we would like to give some examples to demonstrate that there may be some potentialities in this respect.

Theorem 6.1. *Let Ω be triangulated in the sense of piecewise strong regularity, and $B_\sigma(h)=\{x\in\Omega\,|\,\mathrm{dist}(x,\partial\Omega)=O(h^\sigma)\}$ $(0\leqslant\sigma\leqslant1)$. Then*

$$\left(u-\frac{4u^{\frac{h}{2}}-u^h}{3}\right)(z)=O(h^{2+\sigma}\log h)$$

for $z\in(N(h)\cup B(h))\cap B_\sigma(h)$ *and*

$$\left(u-\frac{3u^{\frac{h}{2}}-u^h}{2}\right)(z)=O(h^{2+\sigma}\log h)$$

for $z\in E(h)\cap B_\sigma(h)$.

Proof. Let $z\in B_\sigma(h)$ and $x_0\in\partial\Omega$ such that $|x_0-z|=\mathrm{dist}(z,\partial\Omega)$. Using the superconvergence results in § 3, we have

$$\begin{aligned}|(u^h-u^I)(z)|&=|(u^h-u^I)(z)-(u^h-u^I)(x_0)|\\&\leqslant ch^\sigma|u^h-u^I|_{1,\infty,\Omega}\leqslant ch^{2+\sigma}|\log h|,\end{aligned}$$

namely $(u^h-u^I)(z)=O(h^{2+\sigma}\log h)$. Likewise $(u^{\frac{h}{2}}-u^{\frac{I}{2}})(z)=O(h^{2+\sigma}\log h)$.

Therefore the desired resultes follow from Lemma 4.1.

By Chen and Lin [27] more productions can be gained by means of the above ideas. In fact, we have

Theorem 6.2. *Let the assumptions of Theorem 4.4 be valid, $D_h(A)=\{x\in\Omega\,|\,|x-A|=O(h^{1/2})\}$ and z_1, $z_2\in N(h)\setminus D_h(A)$ be any two adjacent nodes with the* *midpoint* $z\in N\left(\frac{h}{2}\right)$. *Then*

$$u(z)=u^{\frac{h}{2}}(z)+\frac{1}{6}[(u^{\frac{h}{2}}(z_1)+u^{\frac{h}{2}}(z_2))-(u^h(z_1)+u^h(z_2))]+O(h^3\log h).$$

Proof. We can obtain the following superconvergence estimate analogous to Corollary 3 in § 2:

$$|u^h-u^I|_{1,\infty,\Omega\backslash D_h(A)}=O(h^2\log h),$$

therefore

$$(u^{\frac{h}{2}}-u^{\frac{I}{2}})(z)-(u^{\frac{h}{2}}-u^{\frac{I}{2}})(z_1)=O(h^3\log h),$$

i.e.

$$u(z)=u^{\frac{h}{2}}(z)-u^{\frac{h}{2}}(z_1)+u(z_1)+O(h^3\log h).$$

By Theorem 4.5,

$$u(z_1)=\frac{(4u^{\frac{h}{2}}-u^h)(z_1)}{3}+O(h^3\log h).$$

Consequently

$$u(z)=u^{\frac{h}{2}}(z)+\frac{(u^{\frac{h}{2}}-u^h)(z_1)}{3}+O(h^3\log h).$$

Similarly

$$u(z)=u^{\frac{h}{2}}(z)+\frac{(u^{\frac{h}{2}}-u^h)(z_2)}{3}+O(h^3\log h).$$

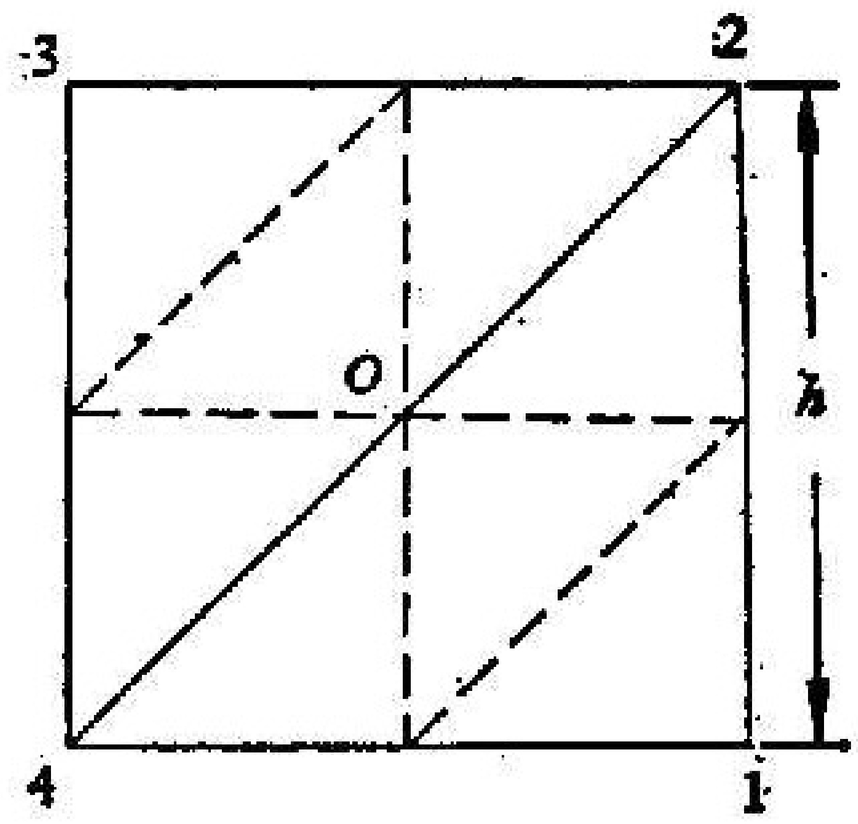

Fig. 6

The theorem is proved by combining the above two formulae.

Now, let us show how the deferred correction method may be used in the extrapolation. Taking two adjacent elements (in h-mesh) which form a square as is shown in Fig. 6, we have, by Taylor's formula,

$$u(0)=\frac{1}{4}\sum_{i=1}^{4}u(i)-\frac{1}{2}h^2\Delta u(0)+O(h^4),$$

$$\nabla u(0)=\frac{1}{4}\sum_{i=1}^{4}\nabla u(i)-\frac{1}{2}h^2\nabla\cdot\Delta u(0)+O(h^3).$$

Noting that $-\Delta u=f$ and using Theorem 4.2, we finally obtain Theorem 6.3. Under the assumptions of Theorem 4.2,

$$u(0)=\frac{1}{12}\sum_{i=1}^{4}(4u^{\frac{h}{2}}-u^h)(i)+\frac{1}{2}h^2f(0)+O(h^4\log^2 h),$$

$$\nabla u(0)=\frac{1}{12}\sum_{i=1}^{4}\nabla(4u^{\frac{h}{2}}-u^h)(i)+\frac{1}{2}h^2\nabla f(0)+O(h^3\log^2 h).$$

Remark 6.1. If two more elements are involved, similar results can be obtained with respect to the midpoints of the right angle edges (say, the segments $\overline{12}$, $\overline{23}$ etc.).

Remark 6.2. By Theorem 6.3, the approximations at the nodes of the refined mesh (i.e. $T_{h/2}$), those of T_h excluded, may be more accurate than those we had in Theorem 4.3 since less elements are locally taken into account here.

§ 7. General Linear Second Order Elliptic Problems

The results stated in § 2—§ 5 were all proved in accordance with the model problem, most of them are valid for the general problems. As the proof is slightly different and a little more difficult, in this section we make some investigations on the following problems

$$\begin{cases} Lu \equiv -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + \sum_{k=1}^{2} b_k \frac{\partial u}{\partial x_k} + c_0 u = f & \text{in } \Omega, \\ u=0, \quad \text{on } \partial\Omega, \end{cases}$$

where Ω, for simplicity, is supposed to be a parallelogramic domain, and the coefficients a_{ij}, b_k, c_0 are sufficiently smooth.

In order to outline the key idea, we merely prove the analogous (4.1) in § 4.

Let G_z^h be the Green function associated with the said problem for $z \in \Omega$, therefore

$$(u^h - u^I)(z) = A_1 + A_2 + A_3,$$

where

$$A_1 = \sum_{i,j=1}^{2} \iint_{\Omega} a_{ij} \frac{\partial(u-u^I)}{\partial x_i} \frac{\partial G_z^h}{\partial x_j}\, dx\, dy,$$

$$A_2 = \sum_{k=1}^{2} \iint_{\Omega} b_k \frac{\partial(u-u^I)}{\partial x_k} G_z^h\, dx\, dy,$$

$$A_3 = \iint_{\Omega} c_0 (u-u^I) G_z^h\, dx\, dy.$$

By the Green formula,

$$A_1 = \sum_{i,j=1}^{2}\Big[-\iint_{\Omega} (u-u^I)\frac{\partial a_{ij}}{\partial x_i}\frac{\partial G_z^h}{\partial x_j}\, dx\, dy + \sum_{K_e}\int_{\partial K} a_{ij}(u-u^I)\frac{\partial G_z^h}{\partial x_j}\cos\alpha_j\, ds\Big] \triangleq A_{11} + A_{12},$$

$$A_2 = -\sum_{k=1}^{2} \iint_{\Omega} (u-u^I)\frac{\partial (b_k G_z^h)}{\partial x_k}\, dx\, dy.$$

By the Euler-Maclaurin formula,

$$\int_{23} a_{ij}(u-u^I)\, ds = -\frac{l_1^2}{12}\int_{23} \frac{\partial^2 (a_{ij}(u-u^I))}{\partial l_1^2}\, ds + l_1^3 \int_{23} p(s)\frac{\partial^3(a_{ij}(u-u^I))}{\partial l_1^3}\, ds.$$

Hence A_{12} can be estimated in a way similar to that in § 4.

The estimates for A_{11}, A_2, A_3 are intrinsically the same, so we only study A_3 next.

From (4.15), we have

$$(u-u^I)(x, y) = -\frac{1}{2}\sum_{l=1}^{3} \lambda_l(x, y)\left[(x-x_l)^2 u_{xx}(x, y) + 2(x-x_l)(y-y_l)u_{xy}(x, y) + (y-y_l)^2 u_{yy}(x, y)\right] + \frac{1}{2}\sum_{l=1}^{3}\lambda_l(x, y)\int_0^1 t^2\, \partial_l^3 u(M_l)\, dt.$$

Therefore

$$A_3=-\sum_K\Big[\sum_{l=1}^3\frac{1}{|K|}\iint_K\lambda_l(x,\ y)\,(x-x_l)^2 v\,dx\,dy+\cdots\Big]+R_{31}+R_{32},$$

where $v(x,\ y)=c_0(x,\ y)u_{xx}(x,\ y)G_z^h(x,\ y)$,

$$R_{31}=-\frac{1}{2}\sum_K\Big\{\sum_{l=1}^3\iint_K\lambda_l(x,\ y)\,(x-x_l)^2-\frac{1}{|K|}\iint_K\lambda_l(\xi,\ \eta)\,(\xi-x_l)^2 v\,dx\,dy+\cdots\Big\},$$

$$R_{32}=\frac{1}{2}\sum_K\sum_{l=1}^3\iint_K\lambda_l(x,\ y)\int_0^1 t^2\,\partial_l^3 u(M_l)\,dt\,c_0G_z^h\,dx\,dy.$$

In virtue of Lemma 4.2, arguing as in § 4, we can prove that

$$|R_{32}|\leqslant ch^3|\log h|^{1/2}\|u\|_{4,2,\Omega}.$$

By means of the well known Bramble-Hilbert lemma, we can obtain

$$|R_{31}|\leqslant Ch^3|v|_{1,2,\Omega}.$$

Consequently

$$|R_3|\leqslant Ch^3|\log h|^{1/2}\|c_0\|_{1,\infty}\|u\|_{4,2,\Omega}.$$

Since it is easy to show that $\sum_{l=1}^3\frac{1}{|K|}\iint_K\lambda_l(x,\ y)\,(x-x_l)^2\,dx\,dy$ and the like are independent of K, we can set

$$C_1h^2=-\frac{1}{2}\sum_{l=1}^3\frac{1}{|K|}\iint_K\lambda_l(x,\ y)\,(x-x_l)^2\,dx\,dy,$$

....

To sum up, we obtain

$$A_3=h^2\iint_\Omega(c_1u_{xx}+c_2u_{xy}+c_3u_{yy})c_0G_z^h\,dx\,dy+O(h^3|\log h|^{1/2}).$$

The rest of the proof is trivial and the desired result is therefore obtained.

Remark. The Dirichlet boundary condition is not essential to our.

Proof. In fact, most of the results also hold for the problems with some more general boundary conditions.

References

[1] J. Bramble, J. Nitsche, A. Schatz, *Math. Comp.*, v. q, 1975.
[2] C. M. Chen, *Numer. Math. J. Chinese Univ.*, v. 2, 1980.
[3] J. Jr. Douglas, T. Dupont, M. Wheeler, *RAIRO Anal. Numer.*, v. 8, 1974.
[4] K. Feng, *Appl. Math., Comp. Math.*, v. 2, 1965.
[5] J. Frehse, Proc. China-France Symposium on FEM, Beijing, 1983.
[6] J. Frehse, R. Rannacher, Bonn. Math. Schrift., No. 89, 1976.
[7] H. C. Huang, Y. G. Liu, *Comp. Math.* (in Chinese), v. 1, 1979.
[8] M. Krizek, P. Neittaanmaki, Preprint 26, Univ. Jyvaskyla, 1984.
[9] P. Lesaint, M. Zlamal, *RAITO Anal. Numer.*, v. 13, 1979.
[10] N. Levine, Num. Anal. Report 6, Univ. Reading, 1983.
[11] Q. Lin, T. Lu, S. Shen, Research Report IMS-11, Chengdu Branch of Academia Sinica, 1983.
[12] Q. Lin, T. Lu, S. Shen, *J. Comp. Math.*, v. 1, 1983.
[13] Q. Lin, T. Lu, Sixth Int. Conference on CMASE, Versailles, 1983.
[14] Q. Lin, T. Lu, Bonn. Math. Schrift., 1984.

[15] Q. Lin, Q. D. Zhu, Submitted to J. Comp. Math.
[16] R. Rannacher, R. Scott, *Math. Comp.*, v. 38, 1982.
[17] A. Schatz, L. Wahlbin, *Math. Comp.*, v. 31, 1977.
[18] G. Strang, G. Fix, An analysis of FEM, Prentice Hall, Inc., 1973.
[19] V. Thomee, *Math. Comp.*, v. 31, 1977.
[20] M. Zlamal, Lecture Notes in Math., 606, 1977.
[21] M. Zlamal, *Math. Comp.*, v. 32, 1978.
[22] Q. D. Zhu, *Numer. Math. J. Chinese Univ.*, v. 3, 1981.
[23] Q. D. Zhu, *Numer. Math. J. Chinese Univ.*, v. 5, 1983.
[24] Q. D. Zhu, Proc. China-France Symposium on FEM, Beijing, 1983.
[25] J. C. Xu, *J. Xiangtan Univ.* (in Chinese), 1982.
[26] J. C. Xu, Submitted to Acta Math. Scientia.
[27] C. M. Chen, Q. Lin, Submitted to J. Comp. Math.