SPECTRAL METHOD FOR SOLVING THE RLW EQUATION*

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§ 1. Introduction

The earliest work for the numerical solution of the RLW equation was due to Peregrine[15] and following that Abdulloev, Bogolubsky and Makhankov[1] showed in their numerical experiments the inelastic interaction of soliton-like waves, which is different from that of the KDV equation. Later, Olver[14] proved that the RLW equation possesses only three conserved quantities justifying the inelasticity of the interactions. A lot of numerical work has been done on this equation and the interested reader is referred to Eilbeck and McGuire[5,6] and Alexander and Morris[3]. Recently Wu Hua-mo and Guo Ben-yu[19] have proposed a new high order accurate difference scheme for the KDV-Burgers-RLW equation with a strict error estimation from which the convergence followed. However, when using finite difference schemes or finite element schemes we get only implicit methods and the accuracy of the approximate solution is limited for a fixed scheme, even though the solution of the RLW equation is very smooth.

On the other hand, the above deficiency may be remedied by the use of spectral methods for such problems. In the past several authors (Gazdag[8], Tappert[17], Schamel and Elsässer[16], Canosa and Gazdag[4], Watanabe, Ohishi and Tanaca[18], Fornberg and Whitham[7] and Abe and Inoue[2]) have used spectral methods for such equations and in some recent papers Guo Ben-yu[10,11] has proposed a technique to strictly estimate the error of a spectral method for nonlinear partial differential equations.

This paper is devoted to the use of a spectral method for solving the RLW equation. In Sections 2 and 3, we consider the linear and the nonlinear problems respectively. The corresponding schemes are explicit and the smoother the solution of the RLW equation, the more accurate the approximate solutions are. In Section 4, we report the numerical results obtained for the solutions of the linear and the nonlinear problems and also compare some of these results with those obtained for finite difference and finite element schemes.

§ 2. The Linear RLW Equation

Firstly, we consider the linear RLW equation

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$$\begin{cases} \dfrac{\partial U}{\partial t}+\alpha\dfrac{\partial U}{\partial x}-\delta\dfrac{\partial^3 U}{\partial t\,\partial x^2}=f(x,\,t), & 0<x<2,\ 0<t\leqslant T, \\ U(0,\,t)=U(2,\,t), & 0\leqslant t\leqslant T, \\ U(x,\,0)=U_0(x), & 0<x<2, \end{cases} \tag{1}$$

where α and δ (>0) are constants and $f(0,\,t)=f(2,\,t)$. Let C_i denote a positive constant and

$$A_\alpha=\{U(x,\,t)/|U(x+y,\,t)-U(x,\,t)|\leqslant C_1|y|^\alpha,\quad 0\leqslant t\leqslant T\}.$$

We suppose that (1) has a unique classical solution such that

$$\partial^p U/\partial x^p\in A_\alpha. \tag{2}$$

Put

$$U(x,\,t)=A_0(t)/2+\sum_{l=1}^{\infty}(A_l(t)\cos l\pi x+B_l(t)\sin l\pi x),$$

$$f(x,\,t)=f_0(t)/2+\sum_{l=1}^{\infty}(f_l(t)\cos l\pi x+g_l(t)\sin l\pi x),$$

$$U^{(n)}(x,\,t)=A_0(t)/2+\sum_{l=1}^{n}(A_l(t)\cos l\pi x+B_l(t)\sin l\pi x),$$

and

$$f^{(n)}(x,\,t)=f_0(t)/2+\sum_{l=1}^{n}(f_l(t)\cos l\pi x+g_l(t)\sin l\pi x).$$

Let

$$R^{(n)}(U(x,\,t))=U(x,\,t)-U^{(n)}(x,\,t)$$

and

$$R^{(n)}(f(x,\,t))=f(x,\,t)-f^{(n)}(x,\,t).$$

From Jackson's theorem and Lebésque's theorem (see [20]), we have

$$|R^{(n)}(\partial^r U/\partial x^r)|\leqslant C_2\frac{\ln n}{n^{p+\alpha-r}},\ \text{for}\ \alpha>0. \tag{3}$$

Let τ be the mesh spacing in time and

$$\eta_t(x,\,k\tau)=\frac{1}{\tau}(\eta(x,\,k\tau+\tau)-\eta(x,\,k\tau)),\quad k\geqslant 0.$$

Also let

$$u^{(n)}(x,\,k\tau)=a_0^{(n)}(k\tau)/2+\sum_{l=1}^{n}(a_l^{(n)}(k\tau)\cos l\pi x+b_l^{(n)}(k\tau)\sin l\pi x) \tag{4}$$

be the approximation of $U^{(n)}(x,\,k\tau)$ satisfying

$$\begin{cases} u_t^{(n)}(x,\,k\tau)+\alpha\dfrac{\partial u^{(n)}}{\partial x}(x,\,k\tau)-\delta\dfrac{\partial^2 u_t^{(n)}}{\partial x^2}(x,\,k\tau)=f^{(n)}(x,\,k\tau), & 0<x<2,\ \ k\geqslant 0, \\ u^{(n)}(0,\,k\tau)=u^{(n)}(2,\,k\tau),\quad k\geqslant 0, & \\ u^{(n)}(x,\,0)=U_0^{(n)}(x),\quad 0<x<2, & \end{cases} \tag{5}$$

i.e.

$$\begin{cases} \dfrac{a_l^{(n)}(k\tau+\tau)-a_l^{(n)}(k\tau)}{\tau}(1+\delta l^2\pi^2)+\alpha l\pi b_l^{(n)}(k\tau)=f_l^{(n)}(k\tau), & 0\leqslant l\leqslant n, \\ \dfrac{b_l^{(n)}(k\tau+\tau)-b_l^{(n)}(k\tau)}{\tau}(1+\delta l^2\pi^2)-\alpha l\pi a_l^{(n)}(k\tau)=g_l^{(n)}(k\tau), & 1\leqslant l\leqslant n, \\ a_l^{(n)}(0)=A_l(0),\quad 0\leqslant l\leqslant n, & \\ b_l^{(n)}(0)=B_l(0),\quad 1\leqslant l\leqslant n. & \end{cases} \tag{6}$$

If $a_l^{(n)}(k\tau)$ and $b_l^{(n)}(k\tau)$ are known, then we can calculate $a_l^{(n)}(k\tau+\tau)$ and $b_l^{(n)}(k\tau+\tau)$ explicitly. This is one of the advantages of the scheme (5).

Now, we consider the error estimation. We have

$$U_t^{(n)}(x, k\tau)+\alpha\frac{\partial U^{(n)}(x, k\tau)}{\partial x}-\delta\frac{\partial^2 U_t^{(n)}(x, k\tau)}{\partial x^2}$$
$$=f^{(n)}(x, k\tau)+\widetilde{E}_1(x, k\tau)+\widetilde{E}_2(x, k\tau), \tag{7}$$

where

$$\widetilde{E}_1(x, k\tau)=U_t(x, k\tau)-\frac{\partial U}{\partial t}(x, k\tau)-\delta\frac{\partial^2}{\partial x^2}[U_t(x, k\tau)]+\delta\frac{\partial^3 U}{\partial t\,\partial x^2}(x, k\tau)$$

and

$$\widetilde{E}_2(x, k\tau)=-R^{(n)}(U_t(x, k\tau))-\alpha R^{(n)}\Big(\frac{\partial U}{\partial x}(x, k\tau)\Big)$$
$$+\delta R^{(n)}\Big(\frac{\partial^2 U_t}{\partial x^2}(x, k\tau)\Big)+R^{(n)}(f(x, k\tau)).$$

Let

$$\tilde{u}^{(n)}(x, k\tau)=u^{(n)}(x, k\tau)-U^{(n)}(x, k\tau)$$
$$=\tilde{a}_0^{(n)}(k\tau)/2+\sum_{l=1}^{\infty}(\tilde{a}_l^{(n)}(x, k\tau)\cos l\pi x+\tilde{b}_l^{(n)}(x, k\tau)\sin l\pi x).$$

Then, from (5) and (7) we have

$$\begin{cases}\tilde{u}_t^{(n)}(x, k\tau)+\alpha\dfrac{\partial\tilde{u}^{(n)}(x, k\tau)}{\partial x}-\delta\dfrac{\partial^2\tilde{u}_t^{(n)}(x, k\tau)}{\partial x^2}\\ \quad=-\widetilde{E}_1(x, k\tau)-\widetilde{E}_2(x, k\tau), \quad 0<x<2,\ k\geqslant 0,\\ \tilde{u}^{(n)}(x, 0)=0, \quad 0<x<2,\end{cases} \tag{8}$$

i.e. Then

$$\begin{aligned}&(1+\delta l^2\pi^2)\tilde{a}_l^{(n)}(k\tau+\tau)=(1+\delta l^2\pi^2)\tilde{a}_l^{(n)}(k\tau)-\alpha\tau l\pi\tilde{b}_l^{(n)}(k\tau)+\tau\tilde{\sigma}_l(x, k\tau), \quad 0\leqslant l\leqslant n,\\ &(1+\delta l^2\pi^2)\tilde{b}_l^{(n)}(k\tau+\tau)=(1+\delta l^2\pi^2)\tilde{b}_l^{(n)}(k\tau)+\alpha\tau l\pi\tilde{a}_l^{(n)}(k\tau)+\tau\tilde{\chi}_l(x, k\tau), \quad 1\leqslant l\leqslant n,\end{aligned} \tag{9}$$

where

$$\widetilde{E}_1(x, k\tau)=\tilde{\sigma}_0(k\tau)/2+\sum_{l=1}^{\infty}(\tilde{\sigma}_l(k\tau)\cos l\pi x+\tilde{\chi}_l(k\tau)\sin l\pi x).$$

We introduce

$$(\eta(k\tau), \zeta(k\tau))=\int_0^2\eta(x, k\tau)\zeta(x, k\tau)dx,$$
$$\|\eta(k\tau)\|^2=(\eta(k\tau), \eta(k\tau)),$$
$$|\eta(k\tau)|_1^2=\|\partial\eta/\partial x(k\tau)\|^2,$$
$$\|\eta(k\tau)\|_1^2=\|\eta(k\tau)\|^2+|\eta(k\tau)|_1^2,$$
$$\|\eta(k\tau)\|_\infty=\max_x|\eta(x, k\tau)|.$$

Suppose that $\partial^2 U/\partial t^2$ is continuous. Then from (9) we obtain that

$$\|\tilde{u}(k\tau+\tau)\|^2+\delta|\tilde{u}(k\tau+\tau)|_1^2\leqslant(1+C_3\tau)(\|\tilde{u}(k\tau)\|^2+\delta|\tilde{u}(k\tau)|_1^2)+\tau\|\widetilde{E}_1(k\tau)\|^2$$

and so for all $k\tau\leqslant T$, we have

$$\|\tilde{u}(k\tau+\tau)\|^2+\delta|\tilde{u}(k\tau+\tau)|_1^2\leqslant\tau e^{C_4 T}\sum_{j=0}^{k}\|\widetilde{E}_1(j\tau)\|^2\leqslant C_5\tau^2 e^{C_4 T}.$$

By the imbedding theorem we also have

$$\|\tilde{u}(k\tau)\|_\infty \leq C_6 \tau e^{C_4 T/2}.$$

If $\alpha>0$, then we obtain

$$\begin{aligned}\|U(k\tau)-u^{(n)}(k\tau)\|_\infty &\leq \|\tilde{u}^{(n)}(k\tau)\|_\infty+\|R^{(n)}(U(x,t))\|_\infty \\ &\leq C_7\Big(\tau e^{C_4 T}+\frac{\ln n}{n^{p+\alpha}}\Big).\end{aligned}$$

Clearly, the smoother the solution $U(x, t)$, the more accurate the approximation $u^{(n)}(x, t)$ is. This is the other advantage of the scheme (5).

§ 3. The Nonlinear RLW Equation

In this section we consider the nonlinear RLW equation

$$\begin{cases}\dfrac{\partial U}{\partial t}+\alpha\dfrac{\partial U}{\partial x}+U\dfrac{\partial U}{\partial x}-\delta\dfrac{\partial^3 U}{\partial t\partial x^2}=f(x,t), & 0<x<2,\ 0<t\leq T,\\ U(0,t)=U(2,t), & 0\leq t\leq T,\\ U(x,0)=U_0(x), & 0<x<2.\end{cases} \tag{10}$$

Let

$$H_p^1=\{V/V(x)\in H^1(0,2),\ V(0)=V(2)\}.$$

The solution of (10) satisfies

$$\Big(\frac{\partial U}{\partial t},\psi\Big)+\alpha\Big(\frac{\partial U}{\partial x},\psi\Big)+\Big(U\frac{\partial U}{\partial x},\psi\Big)+\delta\Big(\frac{\partial^2 U}{\partial t\partial x},\frac{\partial\psi}{\partial x}\Big)=(f,\psi),\ \forall\psi\in H_p^1(0,2). \tag{11}$$

Also, we define the following difference operator

$$J(\eta(x,k\tau),\omega(x,k\tau))=\frac{1}{3}\frac{\partial}{\partial x}(\omega(x,k\tau),\eta(x,k\tau))+\frac{1}{3}\omega(x,k\tau)\frac{\partial\eta}{\partial x}(x,k\tau)$$

and let $u^{(n)}(x, k\tau)$ be the approximation of $U^{(n)}(x, k\tau)$ satisfying

$$\begin{aligned}&(u_t^{(n)}(k\tau),\psi_l)+\alpha\Big(\frac{\partial u^{(n)}(k\tau)}{\partial x},\psi_l\Big)+(J(u^{(n)}(k\tau),u^{(n)}(k\tau)),\psi_l)\\ &\quad+\delta\Big(\frac{\partial}{\partial x}u_t^{(n)}(x,k\tau),\partial\psi_l/\partial x\Big)=(f^{(n)}(k\tau),\psi_l),\quad 0\leq l\leq n,\ k\geq 0,\end{aligned} \tag{12}$$

where $\psi_l=\cos l\pi x$ or $\sin l\pi x$. Clearly the scheme (12) is explicit.

In order to estimate the error, we need the following lemmas.

Lemma 1. *If $\eta_t\in L^2(0,2)$, then* $2(\eta(k\tau),\eta_t(k\tau))=(\|\eta(k\tau)\|^2)_t-\tau\|\eta_t(k\tau)\|^2$.

Lemma 2. *If $\eta_t\in H_p^1(0,2)$, then*

$$2\Big(\frac{\partial\eta}{\partial x}(k\tau),\frac{\partial\eta_t}{\partial x}(k\tau)\Big)=(|\eta(k\tau)|_1^2)_t-\tau|\eta_t(k\tau)|_1^2.$$

Lemma 3. *If $\eta,\zeta\in H_p^1(0,2)$, then*

$$(\partial\eta/\partial x,\eta)=0$$

and

$$(J(\eta,\zeta),\eta)=0.$$

Lemma 4. *If η and ζ are as follows:*

$$\eta(x)=C_0/2+\sum_{l=1}^{n}(C_l\cos l\pi x+d_l\sin l\pi x),$$

$$\zeta(x)=C_0^1/2+\sum_{l=1}^{n}(C_l^1\cos l\pi x+d_l^1\sin l\pi x),$$

then

$$\|\eta\zeta\|^2\leqslant(2n+1)\|\eta\|^2\|\zeta\|^2.$$

Lemma 5. *If the following conditions*

(i) ρ, M_1 *and* M_2 *are non-negative constants,* $\omega(k\tau)$ *is a non-negative function,*

(ii) $\omega(k\tau)\leqslant\rho+M_1\tau\sum_{j=0}^{k-1}(\omega(j\tau)+M_2\omega^2(j\tau))$,

and

(iii) $\rho e^{2M_1T}\leqslant M_2^{-1}$, $\omega(0)\leqslant\rho$ *are satisfied, then for all* $k\tau\leqslant T$,

$$\omega(k\tau)\leqslant\rho e^{2M_1k\tau}.$$

The proofs of Lemmas 1—5 can be found in [10].

We now have

$$(U_t^{(n)}(k\tau),\ \psi_l)+\alpha\left(\frac{\partial U^{(n)}}{\partial x}(k\tau),\ \psi_l\right)+(J(U^{(n)}(k\tau),\ U^{(n)}(k\tau)),\ \psi_l)$$
$$+\delta\left(\frac{\partial U_t^{(n)}(k\tau)}{\partial x},\ \partial\psi_l/\partial x\right)=(\tilde E_3(k\tau),\ \psi_l)+\left(\tilde E_4(k\tau),\frac{\partial\psi_l}{\partial x}\right)$$
$$+(\tilde E_5(k\tau),\ \psi_l),\quad 0\leqslant l\leqslant n,\ k\geqslant 0, \tag{13}$$

where

$$\tilde E_3(x,\ k\tau)=U_t(x,\ k\tau)-\frac{\partial U}{\partial t}(x,\ k\tau),$$
$$\tilde E_4(x,\ k\tau)=\frac{\partial U_t}{\partial x}(x,\ k\tau)-\frac{\partial^2 U}{\partial t\,\partial x}(x,k\tau),$$
$$\tilde E_5(x,\ k\tau)=-J(R^{(n)}(U(x,\ k\tau)),\ U(x,\ k\tau))-J(U(x,\ k\tau),\ R^{(n)}(U(x,\ k\tau)))$$
$$+J(R^{(n)}(U(x,\ k\tau)),\ R^{(n)}(U(x,\ k\tau))).$$

Let $\tilde u^{(n)}(x,\ k\tau)=u^{(n)}(x,\ k\tau)-U^{(n)}(x,\ k\tau)$. Then from (12) and (13) we have

$$(\tilde u_t^{(n)}(k\tau),\ \psi_l)+\alpha\left(\frac{\partial\tilde u^{(n)}}{\partial x}(k\tau),\ \psi_l\right)+(J(\tilde u^{(n)}(k\tau),\ U^{(n)}(k\tau)+\tilde u^{(n)}(k\tau)),\ \psi_l)$$
$$+(J(U^{(n)}(k\tau),\ \tilde u^{(n)}(k\tau)),\ \psi_l)+\delta\left(\frac{\partial\tilde u_t^{(n)}}{\partial x},\ \frac{\partial\psi_l}{\partial x}\right)$$
$$=-(\tilde E_3(k\tau),\ \psi_l)-(\tilde E_4(k\tau),\ \partial\psi_l/\partial x)-(\tilde E_5(k\tau),\ \psi_l),\quad 0\leqslant l\leqslant n,\ k\geqslant 0. \tag{14}$$

Multiplying (14) by $\tilde a_l^{(n)}(k\tau)$ or $\tilde b_l^{(n)}(k\tau)$ according to whether $\psi_l=\cos l\pi x$ or $\sin l\pi x$ and summing up these equations, we have

$$(\tilde u_t^{(n)}(k\tau),\ \tilde u^{(n)}(k\tau))+\alpha\left(\frac{\partial\tilde u^{(n)}}{\partial x}(k\tau),\ \tilde u(k\tau)\right)$$
$$+(J(\tilde u^{(n)}(k\tau),\ U^{(n)}(k\tau)+\tilde u^{(n)}(k\tau)),\ \tilde u^{(n)}(k\tau))+(J(U^{(n)}(k\tau),\ \tilde u^{(n)}(k\tau)),\ \tilde u^{(n)}(k\tau))$$
$$+\delta\left(\frac{\partial\tilde u_t^{(n)}(k\tau)}{\partial x},\ \frac{\partial\tilde u^{(n)}(k\tau)}{\partial x}\right)$$
$$=-(\tilde E_3(k\tau),\ \tilde u^{(n)}(k\tau))-\left(\tilde E_4(k\tau),\ \frac{\partial\tilde u^{(n)}}{\partial x}(k\tau)\right)-(\tilde E_5(k\tau),\ \tilde u^{(n)}(k\tau)),\quad k\geqslant 0. \tag{15}$$

Let ε be a suitably small positive constant and A_4 be a positive constant depending on $\|U\|_\infty$ and $\|\partial U/\partial x\|_\infty$. It can be proved that (see [10])

$$|(J(U^{(n)}(k\tau),\ \tilde{u}^{(n)}(k\tau)),\ \tilde{u}^{(n)}(k\tau))|\leqslant A_1\|\tilde{u}^{(n)}(k\tau)\|^2\left(1+\left\|R^{(n)}\left(\frac{\partial U}{\partial x}\right)\right\|_\infty^2\right). \qquad (16)$$

We suppose that $\frac{\partial^p U(x,\ t)}{\partial x^p}\in A_\alpha$ where $p\geqslant 1$, $\alpha>0$; then from (16) and Lemmas 1—3 we obtain

$$(\|\tilde{u}(k\tau)\|^2+\delta|\tilde{u}(k\tau)|_1^2)_t-\tau(\|\tilde{u}_t(k\tau)\|^2+\delta|\tilde{u}_t(k\tau)|_1^2)$$
$$\leqslant A_2\left(\|\tilde{u}^{(n)}(k\tau)\|_1^2+\sum_{i=3}^{5}\|\tilde{E}_i(k\tau)\|^2\right). \qquad (17)$$

Let m be a positive constant. Multiplying (14) by $m\tau[\tilde{a}_l^{(n)}(k\tau)]_t$ or $m\tau[\tilde{b}_l^{(n)}(k\tau)]_t$ according to whether $\psi_l=\cos l\pi x$ or $\sin l\pi x$ and summing up these equations we obtain

$$\tau(m-\varepsilon)(\|\tilde{u}_t(k\tau)\|^2+\delta|\tilde{u}_t(k\tau)|_1^2)+F_1(k\tau)+F_2(k\tau)$$
$$\leqslant\frac{C_9\tau m^2}{\varepsilon}\|\tilde{u}^{(n)}(k\tau)\|_1^2+\frac{C_{10}\tau m^2}{\varepsilon}\left(\sum_{i=3}^{5}\|\tilde{E}_i(k\tau)\|^2\right), \qquad (18)$$

where

$$F_1(k\tau)=m\tau(J(\tilde{u}^{(n)}(k\tau),\ U^{(n)}(k\tau)),\ \tilde{u}_t^{(n)}(k\tau))$$
$$+m\tau(J(U^{(n)}(k\tau),\ \tilde{u}^{(n)}(k\tau)),\tilde{u}_t^{(n)}(k\tau))$$

and

$$F_2(k\tau)=m\tau(J(\tilde{u}^{(n)}(k\tau),\ \tilde{u}^{(n)}(k\tau)),\ \tilde{u}_t^{(n)}(k\tau)).$$

Clearly

$$|F_1(k\tau)|\leqslant\varepsilon\tau\|\tilde{u}_t^{(n)}(k\tau)\|^2+\frac{A_3\tau}{\varepsilon}m^2\|\tilde{u}^{(n)}(k\tau)\|_1^2.$$

Lemma 4 gives

$$|F_2(k\tau)|\leqslant\varepsilon\tau\|\tilde{u}_t^{(n)}(k\tau)\|^2+\frac{A_4\tau nm^2}{\varepsilon}\|\tilde{u}^{(n)}(k\tau)\|^2\|\tilde{u}^{(n)}(k\tau)\|_1^2.$$

Thus from (18) it follows that

$$\tau(m-3\varepsilon)(\|\tilde{u}_t(k\tau)\|^2+\delta|\tilde{u}_t(k\tau)|_1^2)$$
$$\leqslant\frac{A_5\tau m^2}{\varepsilon}\left\{\|\tilde{u}^{(n)}(k\tau)\|_1^2+n\|\tilde{u}^{(n)}(k\tau)\|_1^4+\sum_{i=3}^{5}\|\tilde{E}_i(k\tau)\|^2\right\}. \qquad (19)$$

Combining (17) with (19) and taking $m=1+3\varepsilon$, we have

$$(\|\tilde{u}^{(n)}(k\tau)\|^2+\delta\|\tilde{u}^{(n)}(k\tau)\|_1^2)_t\leqslant A_6(\|\tilde{u}^{(n)}(k\tau)\|_1^2+\tau n\|\tilde{u}^{(n)}(k\tau)\|_1^4+\tilde{\varepsilon}^2(k\tau)),$$

where

$$\tilde{\varepsilon}^2(k\tau)=\sum_{i=3}^{5}\|\tilde{E}_i(k\tau)\|^2;$$

putting

$$\rho^{(n)}(k\tau)=C_{11}\tau\sum_{j=0}^{k-1}\tilde{\varepsilon}^2(j\tau),$$

we obtain

$$\|\tilde{u}^{(n)}(k\tau)\|_1^2\leqslant\rho^{(n)}(k\tau)+A_7\tau\sum_{j=0}^{l-1}(\|\tilde{u}^{(n)}(j\tau)\|_1^2+\tau n\|\tilde{u}^{(n)}(j\tau)\|_1^4).$$

Finally we use Lemma 5 with $\omega(k\tau)=\|\tilde{u}^{(n)}(k\tau)\|_1^2$ and if

$$\rho^{(n)}(T)e^{2A_7T}\leqslant\frac{1}{\tau n}, \qquad (20)$$

then for all $k\tau\leqslant T$,

$$\|\tilde{u}^{(n)}(k\tau)\|_1^2\leqslant\rho^{(n)}(k\tau)e^{2A_7k\tau}.$$

Furthermore,

$$\|\tilde{u}^{(n)}(k\tau)\|_\infty^2 \leqslant C_{12}\rho^{(n)}(k\tau)e^{2A_1 k\tau}$$

and

$$\|U(k\tau)-u^{(n)}(k\tau)\|_\infty^2 \leqslant C_{12}\rho^{(n)}(k\tau)e^{2A_1 k\tau}+O\left(\frac{(\ln n)^2}{n^{2p+2\alpha}}\right).$$

Remark. If $\partial^2 U/\partial t^2$ is continuous, then

$$\rho^{(n)}(k\tau)=O(\tau^2)+O\left(\frac{(\ln n)^2}{n^{2p+2\alpha-2}}\right).$$

If $\tau=O\left(\frac{1}{n^q}\right)$, $q>1/3$, $p\geqslant 1$, $\alpha>1/3$, then (20) is satisfied and so for all $k\tau\leqslant T$,

$$\|U(k\tau)-u^{(n)}(k\tau)\|_\infty \leqslant A_8/n^{1/3}.$$

If $\tau=O\left(\frac{1}{n^{p+\alpha-1}}\right)$, then we have

$$\|U(k\tau)-u^{(n)}(k\tau)\|_\infty = O\left(\frac{\ln n}{n^{p+\alpha-1}}\right).$$

Again it is obvious that the smoother the solution $U(x, t)$, the more accurate the approximation $u^{(n)}(x, k\tau)$ will be.

§ 4. Numerical Results

For the numerical experiments, the corresponding function $f(x, t)$ in equations (1) and (10) are constructed in such a way that the solutions of the problems are of the form

$$U(x, t)=A\exp(B\sin\pi x+\omega t), \tag{21}$$

where A, B and ω are constants, and in this section by "error" (E) we refer to

$$\max_{x}|U(x, k\tau)-u^{(n)}(x,k\tau)|/|U(x, k\tau)|. \tag{22}$$

Linear problem.

When the spectral scheme (6) is used to solve the linear RLW equation (1), we find that for large δ, the error obtained using the time step $\tau=0.01$ is roughly ten times smaller than the error due to $\tau=0.1$ (Table 1). This pattern is also observed for small δ (Table 2). However, provided both A and B have moderately small values, for both small and large δ values, only ten terms in the spectral expansion are enough to get reasonable solutions (Table 3).

Nonlinear problem.

The finite difference and finite element schemes used in solving the nonlinear RLW equation (10) to do a comparative study with the spectral method (12) are the following:

Scheme 1:

$$\left(\frac{u_m^{k+1}-u_m^k}{\tau}\right)+(1+u_m^k)\left(\frac{u_{m+1}^k-u_{m-1}^k}{2h}\right)-\frac{\delta}{\tau}\left(\frac{u_{m+1}^{k+1}-2u_m^{k+1}+u_{m-1}^{k+1}}{h^2}\right)$$
$$+\frac{\delta}{\tau}\left(\frac{u_{m+1}^k-2u_m^k+u_{m-1}^k}{h^2}\right)=f_m^n. \tag{23}$$

Scheme 2:

$$\left(\frac{u_m^{k+1}-u_m^k}{\tau}\right)+\left(1+\frac{1}{3}u_m^k\right)\left(\frac{u_{m+1}^k-u_{m-1}^k}{2h}\right)+\frac{1}{3}\frac{(u_{m+1}^k)^2-(u_{m-1}^k)^2}{2h}$$
$$-\frac{\delta}{\tau}\left(\frac{u_{m+1}^{k+1}-2u_m^{k+1}+u_{m-1}^{k+1}}{h^2}\right)+\frac{\delta}{\tau}\left(\frac{u_{m+1}^k-2u_m^k+u_{m-1}^k}{h^2}\right)=f_m^n. \quad (24)$$

Scheme 3:

$$\left[\frac{1}{6}\left(\frac{u_{m+1}^{k+1}-u_{m+1}^k}{\tau}\right)+\frac{4}{6}\left(\frac{u_m^{k+1}-u_m^k}{\tau}\right)+\frac{1}{6}\left(\frac{u_{m-1}^{k+1}-u_{m-1}^k}{\tau}\right)\right]+\left(1+\frac{1}{3}u_m^k\right)\left(\frac{u_{m+1}^k-u_{m-1}^k}{2h}\right)$$
$$+\frac{1}{3}\frac{(u_{m+1}^k)^2-(u_{m-1}^k)^2}{2h}-\frac{\delta}{\tau}\left(\frac{u_{m+1}^{k+1}-2u_m^{k+1}+u_{m-1}^{k+1}}{h^2}\right)+\frac{\delta}{\tau}\left(\frac{u_{m+1}^k-2u_m^k+u_{m-1}^k}{h^2}\right)=f_m^k, \quad (25)$$

where h is the mesh width in space.

It is interesting to note that, although the behaviour of the above schemes are almost the same as (or a little better than) the spectral method for large N (the number of terms in the spectral expansion) (Table 4), for N small, the spectral method is a lot better irrespective of the value of δ (Tables 5—7). However, the computational cost of the spectral scheme is comparable with those of (23), (24) and (25) for small n.

Therefore, provided we expect a smooth solution of the RLW equation, because of its simplicity and the explicit nature it is reasonable to use a spectral scheme with small n rather than a finite-difference or finite element scheme.

Table 1

$A=B=2.0$, $\omega=0.5$, $\delta=5.0$, $n=10$

Time	E(τ=0.1)	E(τ=0.01)
0.5	5.925400-3	6.088618-4
1.0	1.147120-2	1.177780-3
1.5	1.655400-2	1.699260-3
2.0	2.113946-2	2.168700-3
2.5	2.521867-2	2.584740-3
3.0	2.880981-2	2.950861-3
3.5	3.194160-2	3.269687-3
4.0	3.465124-2	3.544835-3
4.5	3.697974-2	3.781567-3
5.0	3.896877-2	3.983741-3

Table 2

$A=B=1.0$, $\omega=0.5$, $\delta=0.01$, $n=10$

Time	E(τ=0.1)	E(τ=0.01)
0.5	9.495588-3	1.109793-3
1.0	3.009789-2	3.188463-3
1.5	5.527154-2	5.102806-3
2.0	6.421713-2	5.634552-3
2.5	5.003357-2	5.584738-3
3.0	5.948730-2	6.309864-3
3.5	7.834576-2	7.007519-3
4.0	7.151223-2	7.386882-3
4.5	8.023500-2	7.493547-3
5.0	8.808314-2	7.536437-3

Table 3

$A=B=1.0,\ \omega=0.5,\ \delta=0.01,\ \tau=0.01$			
Time	E($n=5$)	E($n=10$)	E($n=20$)
0.5	1.170026-3	1.109793-3	1.109740-3
1.0	3.109356-3	3.188463-3	3.188524-3
1.5	5.071532-3	5.102806-3	5.102797-3
2.0	5.748613-3	5.634552-3	5.634508-3
2.5	5.698943-3	5.584738-3	5.584657-3
3.0	6.336858-3	6.309864-3	6.309830-3
3.5	7.121520-3	7.007519-3	7.007533-3
4.0	7.501003-3	7.386882-3	7.386915-3
4.5	7.607647-3	7.493547-3	7.493547-3
5.0	7.650481-3	7.536437-3	7.536423-3

Table 4

$A=B=0.1,\ \omega=0.5,\ \delta=5.0,\ \tau=0.01,\ n=10$				
Time	E(Spectral)	E(23)	E(24)	E(25)
0.5	7.085890-4	8.641859-4	8.636913-4	8.515090-4
1.0	1.302863-3	1.526191-3	1.525135-3	1.526031-3
1.5	1.820826-3	2.035302-3	2.033628-3	2.059457-3
2.0	2.295187-3	2.429719-3	2.427315-3	2.480313-3
2.5	2.755676-3	2.739320-2	2.736010-3	2.811442-3
3.0	3.231578-3	2.987613-3	2.983143-3	3.070840-3
3.5	3.752505-3	3.193632-3	3.187652-3	3.272714-3
4.0	4.350367-3	3.372975-3	3.365124-3	3.428482-3
4.5	5.060934-3	3.539232-3	3.528860-3	3.546930-3
5.0	5.925296-3	3.704908-3	3.691308-3	3.635273-3

Table 5

$A=B=0.1,\ \omega=0.5,\ \delta=5.0,\ \tau=0.01,\ n=5$				
Time	E(Spectral)	E(23)	E(24)	E(25)
0.5	7.085745-4	3.083941-3	3.080641-3	3.955317-3
1.0	1.302852-3	5.407932-3	5.400887-3	7.073691-3
1.5	1.820817-3	7.214305-3	7.217663-3	9.528217-3
2.0	2.295187-3	8.636566-3	8.641381-3	1.145607-2
2.5	2.755676-3	9.755973-3	9.762593-3	1.296527-2
3.0	3.231578-3	1.065067-2	1.065942-2	1.414120-2
3.5	3.752505-3	1.138328-2	1.139480-2	1.505064-2
4.0	4.350367-3	1.200566-2	1.202065-2	1.574587-2
4.5	5.060934-3	1.256206-2	1.258137-2	1.626772-2
5.0	5.925296-3	1.313289-2	1.311714-1	1.664770-2

Table 6

$A=B=0.1,\ \omega=0.5,\ \delta=0.01,\ \tau=0.01,\ n=5$				
Time	E(Spectral)	E(23)	E(24)	E(25)
0.5	6.240481-3	3.292321-2	3.268875-2	6.749889-2
1.0	9.069753-3	5.362737-2	5.305456-2	8.842583-2
1.5	7.451680-3	4.764098-2	4.665613-2	6.105477-2
2.0	9.189239-3	4.589194-2	4.590062-2	6.604430-2
2.5	1.325873-2	3.730251-2	3.723072-2	7.646288-2
3.0	1.321978-2	3.454743-2	3.782271-2	8.901591-2
3.5	1.574614-2	4.317997-2	4.652779-2	6.740339-2
4.0	1.789208-2	4.472429-2	4.883848-2	5.752699-2
4.5	1.925411-2	3.833180-2	4.425357-2	5.752699-2
5.0	2.094770-2	3.705758-2	4.394875-2	4.091342-2

Table 7

$A=B=0.1,\quad \omega=0.5,\quad \delta=0.001,\quad \tau=0.01,\quad n=5$

Time	E(Spectral)	E(23)	E(24)	E(25)
0.5	6.719408-3	3.582542-2	3.555305-2	8.931380-2
1.0	8.832252-3	5.613129-2	5.546584-2	6.114991-2
1.5	6.675292-3	4.475714-2	4.479550-2	7.574176-2
2.0	1.026703-2	4.599629-2	4.597437-2	7.791302-2
2.5	1.292265-2	3.395545-2	3.421262-2	9.982187-2
3.0	1.266211-2	4.112869-2	4.427290-2	6.546807-2
3.5	1.655488-2	4.728465-2	5.064517-2	7.065172-2
4.0	1.688559-2	4.131371-2	4.628788-2	6.696868-2
4.5	2.005767-2	3.625120-2	4.271839-2	4.431025-2
5.0	2.062896-2	4.084922-2	4.688197-2	6.025408-2

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