ESTIMATION FOR SOLUTIONS OF ILL-POSED CAUCHY PROBLEMS OF DIFFERENTIAL EQUATION WITH PSEUDO-DIFFERENTIAL OPERATORS*

Prat II. Case of Second Order Operators

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Abstract

The estimation for solutions of the ill-posed Cauchy problems of the differential equation

$$\frac{du(t)}{dt}=A(t)u(t)+N(t)u(t),\quad \forall t\in(0,1)$$

is discussed, where $A(t)$ is a 2-nd order p. d. o. and $N(t)$ is a uniformly bounded $H\to H$ linear operator. Two estimates of $\|u(t)\|$ are obtained.

This part is a continuation of Part I, so we use the same notation as in Part I and continue the section numbers. In section 4 we deal with diagonalizors of r-th order p. d. o.. Then in section 5 we derive the desired estimates.

§ 4. Diagonalizors

We will construct diagonalizors for r-th order p. d. o. in this section.

Lemma 4.1. *Let $P(t)\in\mathscr{L}^0_{s_1}$ have the symbol $p(t, x, \xi)\in\{S^0_1\}$ satisfying*

$$|\det p(t, x, \xi)|\geqslant \text{const}\ \gamma>0,\ \forall(t, x, \xi)\in([0, 1]\times R^m\times(|\xi|-1)).$$

Then for any integer j, there exists $P_j^{(-1)}(t)\in\mathscr{L}^0_{s_1}$ such that

$$\begin{aligned}&P(t)P_j^{(-1)}(t)-I=N_{R(-j)}(t)\in\mathscr{L}^{-j},\\&P_j^{(-1)}(t)P(t)-I=N_{L(-j)}(t)\in\mathscr{L}^{-j}.\end{aligned}\tag{4.1}$$

Proof. Let $P_j^{(-1)}(t)$ have the symbol

$$b(t, x, \xi)=\sum_{l=0}^{j-1}b_l(t, x, \xi)|\xi|^{-l}$$

with undetermined coefficients $b_l(t, x, \xi)$. We have to determine these coefficients such that $N_{R(-j)}(t)$ and $N_{L(-j)}(t)$ belong to $\mathscr{L}^{-j}$.

Applying Theorem 1.3 for $P(t)$ and $P_j^{(-1)}(t)$, we have that the difference between $P(t)P_j^{(-1)}(t)-I$ and the p. d. o. $N^a_{R(-j)}(t)$ with the symbol

$$\sum_{k=0}^{j-1}\Big(\sum_{|\alpha|=k}\frac{(-1)^k}{\alpha!}D^\alpha p(t, x, \xi)\partial^\alpha b(t, x, \xi)\Big)-I_{n\times n}$$

belongs to $\mathscr{L}^{-j}$. Calculating the coefficients of $|\xi|^{-i}$ for $i=0, 1, \cdots, j-1$ in the

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symbol of $N^a_{R(-j)}(t)$ and letting them equal to $\theta_{n\times n}$, we obtain

$$b_0(t, x, \xi)=p^{-1}(t, x, \xi)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$b_l(t, x, \xi)=-p^{-1}(t, x, \xi)\sum_{k=1}^{l}(-1)^k\sum_{|\alpha|=k}\frac{1}{\alpha!}D^\alpha p(t, x, \xi)\partial^\alpha b_{l-k}(t, x, \xi),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Since $p(t, x, \xi)\in\{S^0_1\}$ and $|\det p(t, x, \xi)|\geqslant\gamma>0$, we have $p^{-1}(t, x, \xi)\in\{S^0_1\}$ and $b(t, x, \xi)\in\{S_1\}$. Consequently the p. d. o. $P^{(-1)}_j(t)$ obtained above belongs to $\mathscr{L}^0_{s_1}$ and $N^a_{R(j)}(t)$ has the symbol whose first term with nonzero coefficient is of $|\xi|^{-j}$; hence from Theorem 1.1 $N^a_{R(j)}(t)\in\mathscr{L}^{-j}_{s_1}$. Finally we get

$$N_{R(-j)}(t)=[(P(t)P^{(-1)}_j(t)-I)-N^a_{R(-j)}(t)]+N^a_{R(-j)}(t)\in\mathscr{L}^{-j}.$$

Similarly we can construct the p. d. o. $\hat{P}^{(-1)}_j(t)\in\mathscr{L}^0_{s_1}$ such that

$$\hat{N}_{L(-j)}(t)=\hat{P}^{(-1)}_jP(t)-I\in\mathscr{L}^{-j}.$$

Multiplying the relation $P(t)P^{(-1)}_j(t)P(t)=(I+N_{R(-j)}(t))P(t)$ by $\hat{P}^{(-1)}_j(t)$ on the left we obtain

$$\begin{aligned}(I+\hat{N}_{L(-j)}(t))P^{(-1)}_j(t)P(t)&=\hat{P}^{(-1)}_j(t)(I+N_{R(-j)}(t))P(t)\\&=(I+\hat{N}_{L(-j)}(t))+\hat{P}^{(-1)}_j(t)N_{R(-j)}(t)P(t).\end{aligned}$$

Hence

$$\begin{aligned}N_{L(-j)}(t)&=P^{(-1)}_j(t)P(t)-I\\&=\hat{N}_{L(-j)}(t)+\hat{P}^{(-1)}_j(t)N_{R(-j)}(t)P(t)-\hat{N}_{L(-j)}(t)P^{(-1)}_j(t)P(t)\in\mathscr{L}^{-j}.\end{aligned}$$

The lemma is proved.

Let $a_0(t, x, \xi)$ be an $(n\times n)$ matrix belonging to $\{S^0_1\}$ and positive homogeneous of degree zero in ξ. As in Part I, $a_0(t, x, \xi)$ is said to be uniformly diagonalizable, if the $(n\times n)$ matrix consisting of the eigenvectors of $a_0(t, x, \xi)$ is uniformly nonsingular and belongs to $\{S^0_1\}$. Uniform nonsingularity means

$$|\det p(t, x, \xi)|\geqslant\text{const }\gamma>0,\ \forall(t, x, \xi)\in D=([0, 1]\times R^m\times(|\xi|-1)).$$

Obviously

$$p(t, x, \xi)a_0(t, x, \xi)=J_0(t, x, \xi)p(t, x, \xi), \tag{4.2}$$

where $J_0(t,x,\xi)\in\{DS^0_1\}$ is a diagonal matrix consisting of the eigenvalues $\lambda_j(t,x,\xi)$ of $a_0(t, x, \xi)$.

Lemma 4.2. *Let $A(t)\in\mathscr{L}^r_{s_1}$ have the symbol $a(t, x, \xi)=\sum_{l=0}^{N}a_l(t, x, \xi)|\xi|^{r-l}$, the first coefficient $a_0(t, x, \xi)$ of which is uniformly diagonalizable. If the eigenvalues $\lambda_j(t, x, \xi)$ of $a_0(t, x, \xi)$ satisfy*

$$|\lambda_l(t, x, \xi)-\lambda_j(t, x, \xi)|\geqslant\text{const }\gamma_2>0,\quad\forall(t, x, \xi)\in D\text{ and }l\neq j, \tag{4.3}$$

then there exists a p. d. o. *$D(t)\in\mathscr{L}^0_{s_1}$ such that*

$$D(t)P(t)A(t)=E(t)D(t)P(t)+N_0(t)\quad\text{in }\mathscr{L}, \tag{4.4}$$

where $P(t)\in\mathscr{L}^0_{s_1}$ has the symbol $p(t, x, \xi)$ consisting of the left eigenvectors of $a_0(t, x, \xi)$, $E(t)\in\mathscr{L}^r_{DS_1}$ has the diagonal symbol

$$J(t, x, \xi)=\sum_{l=0}^{r-1}J_l(t, x, \xi)|\xi|^{r-l}, \tag{4.5}$$

in which $J_0(t, x, \xi)\in\{DS^0_1\}$ is the diagonal matrix consisting of the eigenvalues of

$a_0(t, x, \xi)$, *and* $N_0(t)\in\mathscr{L}^0$.

Moreover, for $D(t)$ *there exists* $D_0^{-1}(t)\in\mathscr{L}(H\to H)$ *such that*

$$D_0^{-1}(t)D(t)u=u, \quad \forall u\in H. \tag{4.6}$$

The operator $D(t)P(t)$ *is called the diagonalizor of the* p. d. o. $A(t)$ *of order* r.

Proof. We first prove the lemma for the case in which the principal part of $A(t)$ has a diagonal symbol $a_0(t, x, \xi)|\xi|^r$, i. e. $a_0(t, x, \xi)=J_0(t, x, \xi)$. In this case $p(t, x, \xi)\equiv I_{n\times n}$ and $P(t)\equiv I$. Suppose that the p. d. o. $D(t)$ has the symbol

$$d(t, x, \xi)=\sum_{i=0}^{r-1} d_i(t, x, \xi)|\xi|^{-i}$$

with undetermined coefficients $d_i(t, x, \xi)$. We have to determine these coefficients such that

$$D(t)A(t)=E(t)D(t)+N_1(t) \quad \text{in } \mathscr{L}, \tag{4.4$'$}$$

where $N_1(t)\in\mathscr{L}^0$ and $E(t)\in\mathscr{L}^r_{DS}$, has the diagonal symbol

$$J(t, x, \xi)=\sum_{i=1}^{r-1} J_i(t, x, \xi)|\xi|^{r-i}.$$

Taking $\rho=r$ and applying Theorem 3.1 for $D(t)$, $A(t)$ and $E(t)$, $D(t)$ we have

$$[D(t)A(t)-E(t)D(t)]-R(t)\in\mathscr{L}^0,$$

where $R(t)$ is a p. d. o. with the symbol

$$\sum_{k=0}^{r-1}\sum_{|\alpha|=k}\frac{(-1)^k}{\alpha!}[D^\alpha d(t, x, \xi)\partial^\alpha a(t, x, \xi)-D^\alpha J(t, x, \xi)\partial^\alpha d(t, x, \xi)].$$

In order to make (4.4)$'$ valid, it is sufficient to determine $d(t, x, \xi)$ such that $R(t)\in\mathscr{L}^0$. For that we have to calculate the coefficients of $|\xi|^{r-i}$ for $i=0, 1, \cdots, r-1$ in the symbol of $R(t)$ and make them equal to $\theta_{n\times n}$. Considering $a_0(t, x, \xi)=J_0(t, x, \xi)$ we have

$$d_0(t, x, \xi)J_0(t, x, \xi)-J_0(t, x, \xi)d_0(t, x, \xi)=\theta_{n\times n}, \tag{4.7.0}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\begin{aligned}&[d_i(t, x, \xi)J_0(t, x, \xi)-J_0(t, x, \xi)d_i(t, x, \xi)-J_i(t, x, \xi)d_0(t, x, \xi)]\\ &\quad+\Big\{d_0(t, x, \xi)a_i(t, x, \xi)+\sum_{j=1}^{i-1}[d_j(t, x, \xi)a_{i-j}(t, x, \xi)\\ &\quad-J_j(t, x, \xi)d_{i-j}(t, x, \xi)]+\sum_{k=1}^{i}\sum_{|\alpha|=k}\frac{(-1)^k}{\alpha!}\\ &\quad\times\sum_{j=0}^{i-k}[D^\alpha d_j(t, x, \xi)\partial^\alpha a_{i-k-j}(t, x, \xi)-D^\alpha J_j(t, x, \xi)\partial^\alpha d_{i-k-j}(t, x, \xi)]\Big\}=\theta_{n\times n},\end{aligned} \tag{4.7.i}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

The elements $\lambda_l(t, x, \xi)$ on the diagonal of $J_0(t, x, \xi)$ are distinct (by (4.3)). Hence, from (4.7.0) we see that the nondiagonal elements of $d_0(t, x, \xi)$ equal zero. Consequently $d_0(t, x, \xi)$ is an arbitrary diagonal matrix. For convenience we take $d_0(t, x, \xi)\equiv cI_{n\times n}\in\{DS_1\}$ with the constant $c>1$. We denote the element on the l-th row and the q-th column of matrix a by $a^{(l,q)}$ and the expression in $\{\cdots\}$ of (4.7.i) by $R_{r-i}(t, x, \xi)$ which depends only on $a_j(t, x, \xi)$, $d_j(t, x, \xi)$, $J_j(t, x, \xi)$ for $j=0, 1, \cdots, i-1$ and their derivatives. After determining $d_{i-1}(t, x, \xi)\in\{S_1\}$ and $J_{i-1}(t, x, \xi)\in\{DS_1\}$, $R_{r-i}(t, x, \xi)$ is known and belongs to $\{S_1\}$. Since $J_i(t, x, \xi)$ is diagonal, from (4.7.i) we get

$$\begin{cases} d_i^{(l,q)}(t, x, \xi)\lambda_q(t, x, \xi) - \lambda_l(t, x, \xi)d_i^{(l,q)}(t, x, \xi) \\ \quad + R_{r-i}^{(l,q)}(t, x, \xi) = 0 \quad \text{for } l \neq q, \\ J_i^{(l,l)}(t, x, \xi)d_0^{(l,l)}(t, x, \xi) = R_{r-i}^{(l,l)}(t, x, \xi). \end{cases} \tag{4.8}$$

From above we get immediately that

$$J_i^{(l,l)}(t, x, \xi) = R_{r-i}^{(l,l)}(t, x, \xi)/d_0^{(l,l)}(t, x, \xi) = R_{r-i}^{(l,l)}(t, x, \xi)/c.$$

Hence $J_i(t, x, \xi) \in \{DS_1\}$. From (4.8) the nondiagonal elements of $d_i(t, x, \xi)$ are also obtained

$$d_i^{(l,q)}(t, x, \xi) = R_{r-i}^{(l,q)}(t, x, \xi)/(\lambda_l(t, x, \xi) - \lambda_q(t, x, \xi)) \quad \text{for } l \neq q,$$

but the diagonal elements of $d_i(t, x, \xi)$ can be arbitrary.

For definiteness we take

$$d_i^{(l,l)}(t, x, \xi) \equiv \text{const } c^{(i)}.$$

Obviously, $d_i(t, x, \xi)$ thus defined belongs to $\{S_1\}$.

In this way we determine all $d_i(t, x, \xi)$ and $J_i(t, x, \xi)$ for $i=0, 1, \cdots, r-1$ successively. Consequently the p. d. o. $D(t)$ and $E(t)$ with the symbols $d(t, x, \xi)$ and $J(t, x, \xi)$ belong to $\mathscr{L}_{s_1}^0$ and $\mathscr{L}_{DS_1}^r$ respectively. For $d(t, x, \xi)$ and $J(t, x, \xi)$ thus defined, the symbol of $R(t)$ belongs to $\{S_1\}$ also, and the coefficients of $|\xi|^{r-i}$ $(i=0, 1, \cdots, r-1)$ in the symbol are $\theta_{n\times n}$. Hence from Theorem 1.1 we have $R(t) \in \mathscr{L}_{s_1}^0 \subset \mathscr{L}^0$. Consequently

$$N_1(t) = [D(t)A(t) - E(t)D(t) - R(t)] + R(t) \in \mathscr{L}^0.$$

Thus (4.4)′ is valid.

Now we consider the p. d. o. $D(t) = cI + \hat{D}(t)$, where $\hat{D}(t)$ is a p. d. o. with the symbol $\sum_{i=1}^{r-1} d_i(t, x, \xi)|\xi|^{-i}$. It is obvious from Theorem 1.1 that $\hat{D}(t) \in \mathscr{L}_{s_1}^{-1}$. Hence there exists a constant $\hat{c}$ such that

$$\|\hat{D}(t)u\| \leqslant \hat{c}\|u\|_{-1} \leqslant \hat{c}\|u\|. \tag{4.9}$$

We fix the constant $c > \hat{c}$. Therefore

$$\|\hat{D}(t)/c\|_{\mathscr{L}(H\to H)} < 1.$$

Consequently there exists the operator $(I + \hat{D}(t)/c)^{-1} = \sum_{k=0}(-\hat{D}(t)/c)^k$ in $\mathscr{L}(H\to H)$ with the norm $\leqslant c/(c-\hat{c})$. Obviously (4.6) is valid for $D_0^{(-1)}(t) = c^{-1}(I + \hat{D}(t)/c)^{-1}$. The lemma is thus proved for the case of diagonal $a_0(t, x, \xi)$.

Now we turn to the general case. Our purpose is to reduce it to the particular case already discussed.

Taking $\rho = r-1$ and applying Theorem 1.3 for $P(t)$, $A(t)$ and $E_0(t)$, $P(t)$ we have

$$[P(t)A(t) - E_0(t)P(t)] - G_1(t) = \hat{G}_1(t) \in \mathscr{L}^{r-(r-1)-1} = \mathscr{L}^0, \tag{4.10}$$

where $P(t)$, $E_0(t)$, $G_1(t)$ are p. d. o. with the symbols $p(t, x, \xi)$, $J_0(t, x, \xi)|\xi|^r$ and

$$g_1(t, x, \xi) = \sum_{k=1}^{r-1} \sum_{|\alpha|=k} \frac{(-1)^k}{\alpha!}[D^\alpha p(t, x, \xi)\partial^\alpha a(t, x, \xi) - D^\alpha(J_0(t, x, \xi)|\xi|^r)\partial^\alpha p(t, x, \xi)]$$

respectively. The first term of the last symbol is of $|\xi|^{r-1}$. Hence $G_1(t) \in \mathscr{L}_{s_1}^{r-1}$. From Lemma 4.1 there exists a p. d. o. $P_r^{(-1)}(t) \in \mathscr{L}_{s_1}^0$ with the symbol $p_r^{(-1)}(t, x, \xi)$ such that

$$N_{R(-r)}(t)=P(t)P_r^{(-1)}(t)-I\in\mathscr{L}^{-r} \text{ and } N_{L(-r)}(t)=P_r^{(-1)}(t)P(t)-I\in\mathscr{L}^{-r}. \quad (4.11)$$

From (4.10) and (4.11) we obtain

$$P(t)A(t)P_r^{(-1)}(t)-E_0(t)(I+N_{R(-r)}(t))=G_1(t)P_r^{(-1)}(t)+\hat{G}_1(t)P_r^{(-1)}(t). \quad (4.12)$$

Taking $\rho=r-2$ and applying Theorem 1.3 for $G_1(t)$ and $P_r^{(-1)}(t)$ we have

$$G_1(t)P_r^{(-1)}(t)-\hat{A}(t)\in\mathscr{L}^{(r-1)+0-(r-2)-1}=\mathscr{L}^0,$$

where $\hat{A}(t)$ is a p. d. o. of order $(r-1)$ with the symbol

$$\hat{a}(t,x,\xi)=\sum_{k=0}^{r-2}\sum_{|\alpha|=k}\frac{(-1)^k}{\alpha!}D^\alpha g_1(t,x,\xi)\partial^\alpha P_r^{(-1)}(t,x,\xi).$$

Hence

$$P(t)A(t)P_r^{(-1)}(t)=(E_0(t)+\hat{A}(t))+N_2(t), \quad (4.13)$$

where

$$N_2(t)=[G_1(t)P_r^{(-1)}(t)-\hat{A}(t)]+E_0(t)N_{R(-r)}(t)+\hat{G}_1(t)P_r^{(-1)}(t).$$

Since $E_0(t)\in\mathscr{L}^r$, $N_{R(-r)}(t)\in\mathscr{L}^{-r}$, $\hat{G}_1(t)\in\mathscr{L}^0$, $P_r^{(-1)}\in\mathscr{L}^0$, we have $N_2(t)\in\mathscr{L}^0$.

Now $E_0(t)+\hat{A}(t)\in\mathscr{L}^r_{s_1}$ and its principal part $E_0(t)$ has a diagonal symbol $J_0(t,x,\xi)|\xi|^r$. As proved above, for $E_0(t)+\hat{A}(t)$ there exists the desired p. d. o. $D(t)\in\mathscr{L}^0_{s_1}$ such that (4.4)′ is valid, i. e.

$$D(t)(E_0(t)+\hat{A}(t))=E(t)D(t)+N_1(t) \quad \text{in } \mathscr{L}.$$

From this relation and (4.13) we obtain

$$\begin{aligned}D(t)P(t)A(t)P_r^{(-1)}(t)&=D(t)\{[E_0(t)+\hat{A}(t)]+N_2(t)\}\\&=E(t)D(t)+N_1(t)+D(t)N_2(t).\end{aligned}$$

From (4.11) we have

$$\begin{aligned}D(t)P(t)A(t)P_r^{(-1)}(t)P(t)&=D(t)P(t)A(t)[I+N_{L(-r)}(t)]\\&=[E(t)D(t)+N_1(t)+D(t)N_2(t)]P(t).\end{aligned}$$

This gives the desired relation (4.4) with

$$N_0(t)=[N_1(t)+D(t)N_2(t)]P(t)-D(t)P(t)A(t)N_{L(-r)}(t).$$

$N_0(t)\in\mathscr{L}^0$, because $N_1(t)$, $D(t)$, $N_2(t)$, $P(t)$ belong to $\mathscr{L}^0$, $A(t)\in\mathscr{L}^r$ and $N_{L(-r)}(t)\in\mathscr{L}^{-r}$.

The lemma is completely proved.

§ 5. Estimation of Solutions for the Case of 2-nd Order p. d. o.

In this section we discuss the estimation for solutions of the differential Eq. (1) with second order p. d. o. $A(t)$. We first describe two lemmas.

Lemma 5.1. *Let $F(t)\in\mathscr{L}^r_{s_1}$ have the symbol $f(t)=\sum_{i=0}^{N}f_i(t,x,\xi)|\xi|^{r-i}$, whose first coefficient is diagonal and purely imaginary, i.e. $f_0(t,x,\xi)\in\{DS_1\}$ and $\operatorname{Re}f_0(t,x,\xi)=\theta_{n\times n}$. Then for any positive integer ρ there exist $\hat{F}(t)\in\mathscr{L}^{r-1}_{s_1}$ and $N_{(r-\rho)}(t)\in\mathscr{L}^{r-\rho}$, such that*

$$2\operatorname{Re}(u,F(t)u)=\operatorname{Re}(u,\hat{F}(t)u)+\operatorname{Re}(u,N_{(r-\rho)}(t)u),\ \forall u\in\varphi. \quad (5.1)$$

Proof. From Theorem 1.2 we have

$$F^{(*)}(t)-F^{(*)}_{\rho-1}(t)=N_{(r-\rho)}(t)\in\mathscr{L}^{r-\rho}.$$

Since $f_0(t,x,\xi)$ is diagonal and purely imaginary, we have $f_0(t,x,\xi)+f_0^*(t,x,\xi)$

$=\theta_{n\times n}$. Consequently the p. d. o. $\hat{F}(t)=F(t)+F^{(*)}_{\rho-1}(t)$ has a symbol whose coefficient of $|\xi|^r$ is $\theta_{n\times n}$. From Theorem 1.1 follows $\hat{F}(t)\in\mathscr{L}^{r-1}_{s_1}$. Finally we have

$$\begin{aligned}2\mathrm{Re}(u,\ F(t)u)&=(u,\ F(t)u)+(F(t)u,\ u)=(u,\ (F(t)+F^{(*)}(t))u)\\&=(u,\ (F^{(*)}(t)-F^{(*)}_{\rho-1}(t)+F^{(*)}_{\rho-1}(t)+F(t))u)\\&=\mathrm{Re}(u,\ N_{(r-\rho)}(t)u)+\mathrm{Re}(u,\ \hat{F}(t)u).\end{aligned}$$

The lemma is proved.

Lemma 5.2. *Let* $F^{(l)}(t)\in\mathscr{L}^{r_l}(r_l>0,\ l=1,\ 2)$ *have the symbols* $f^{(l)}(t,\ x,\ \xi)=\sum_{i=1}^{r_l-1}f^{(l)}_i(t,\ x,\ \xi)|\xi|^{r_l-i}$ *and* $\det|f^{(1)}_0(t,\ x,\ \xi)|\geqslant\mathrm{const}\ \gamma>0,\ \forall(t,\ x,\ \xi)\in D$. *Then there exists a* p. d. o. $E_{(f)}(t)\in\mathscr{L}^{r_2-r_1}_{s_1}$ *such that*

$$F^{(2)}(t)=E_{(f)}(t)F^{(1)}(t)+N_{(f)}(t)\quad\text{in }\mathscr{L},\tag{5.2}$$

where $N_{(f)}(t)\in\mathscr{L}^0$.

Proof. Suppose that $E_{(f)}(t)$ has the symbol

$$e_{(f)}(t,\ x,\ \xi)=\sum_{i=0}^{r_2-1}e_{(f)i}(t,\ x,\ \xi)|\xi|^{r_2-r_1-i}$$

with the undetermined coefficients $e_{(f)i}(t,x,\xi)$. We have to determine these coefficients such that (5.2) is valid. Taking $p=r_2-1$ and applying Theorem 1.3 for $E_{(f)}(t)$ and $F^{(1)}(t)$ we get

$$E_{(f)}(t)F^{(1)}(t)-R_{(f)}(t)\in\mathscr{L}^0,$$

where $R_f(t)$ is a p. d. o. with the symbol

$$\sum_{k=0}^{r_2-1}\sum_{|\alpha|=k}\frac{(-1)^k}{\alpha!}D^\alpha e_{(f)}(t,\ x,\ \xi)\partial^\alpha f^{(1)}(t,\ x,\ \xi).$$

Now equalize the coefficients of $|\xi|^{r_2-i}$ for $i=0,\ 1,\ \cdots,\ r_2-1$ in the symbols of $R_{(f)}(t)$ and $F^{(2)}(t)$, so that $F^{(2)}(t)-R_f(t)\in\mathscr{L}^0$. Thus we obtain

$$f^{(2)}_0(t,\ x,\ \xi)-e_{(f)0}(t,\ x,\ \xi)f^{(1)}_0(t,\ x,\ \xi)=\theta_{n\times n},$$

....

$$\begin{aligned}f^{(2)}_i(t,\ x,\ \xi)-e_{(f)i}(t,\ x,\ \xi)f^{(1)}_0(t,\ x,\ \xi)-\sum_{j=0}^{i-1}e_{(f)j}(t,\ x,\ \xi)f^{(1)}_{i-j}(t,\ x,\ \xi)\\-\sum_{k=1}^{i}\sum_{|\alpha|=k}\frac{(-1)^k}{\alpha!}\sum_{j=0}^{i-k}D^\alpha e_{(f)j}(t,\ x,\ \xi)\partial^\alpha f^{(1)}_{i-k-j}(t,\ x,\ \xi)=\theta_{n\times n},\end{aligned}$$

....

Since $|\det f^{(1)}_0(t,\ x,\ \xi)|\geqslant\gamma>0$, from above we can determine successively all $e_{(f)i}(t,\ x,\ \xi)$, $i=0,\ 1,\ \cdots,\ r_2-1$. Obviously $e_{(f)}(t,\ x,\ \xi)$ thus defined belongs to $\{S_1\}$. Hence $E_{(f)}(t)\in\mathscr{L}^{r_2-r_1}_{s_1}$ and

$$\begin{aligned}N_{(f)}(t)&=F^{(2)}(t)-E_{(f)}(t)F^{(1)}(t)\\&=[F^{(2)}(t)-R_f(t)]-[E_{(f)}(t)F^{(1)}(t)-R_f(t)]\in\mathscr{L}^0.\end{aligned}$$

The proof of the lemma is completed.

Now we turn to the estimate of solutions for the differential equation

$$\frac{du(t)}{dt}=A(t)u(t)+N(t)u(t),\quad\forall t\in(0,\ 1),$$

where $u(t)=u(t;\ x)$ is an n-dimensional vector function, $A(t)\in\mathscr{L}^2_{s_1}$ and $N(t)\in\mathscr{L}(H\to H)$.

We denote by $\{U_{II}\}$ the set of all solutions which belong to $C([0, 1]; H)\cap C^1([0, 1]; H)\cap C([0, 1]; H_2)$ and make the following hypothesis.

Hypothesis II:

(1) $A(t)$ has the symbol $a(t, x, \xi)=a_0(t, x, \xi)|\xi|^2+a_1(t, x, \xi)|\xi|\in\{S_1\}$.

(2) The diagonal matrix $J_0(t, x, \xi)$ consisting of the eigenvalues $\lambda_l(t, x, \xi)$ of $a_0(t, x, \xi)$ belongs to $\{DS_1^0\}$, and $\lambda_l(t, x, \xi)$ satisfy

$$|\lambda_l(t, x, \xi)-\lambda_q(t, x, \xi)|\geqslant\text{const }\gamma_2>0, \quad \forall(t, x, \xi)\in D \text{ and } l\neq q.$$

(3) All eigenvalues $\lambda_l(t, x, \xi)$ are real, i.e. $\operatorname{Im}\lambda_l(t, x, \xi)\equiv 0$, and $|\lambda_l(t, x, \xi)|\geqslant\text{const }\gamma_1>0$, $\forall l$ and $(t, x, \xi)\in D$.

(4) In the interval $(0, 1)$, $N'(t)=\lim\limits_{\Delta t\to 0}\dfrac{N(t+\Delta t)-N(t)}{\Delta t}\in\mathscr{L}(H\to H)$ exists.

As remarked in section 3, condition (1) does not restrict the generality. Condition (3) means that some characteristic combinations corresponding to $\lambda_l(t, x, \xi)\geqslant\gamma_1$ have parabolic character, and others corresponding to $\lambda_l(t, x, \xi)\leqslant-\gamma_1$ have backward parabolic character. Concerning condition (2), we remark that under this condition $a_0(t, x, \xi)$ is uniformly diagonalizable. This assertion will be proved in the Appendix.

Theorem 2. *Suppose that Hypothesis* II *holds. Then*

$$\|u(t)\|\leqslant M_1^{(2)}\max_{i=1,2}\{\|u(0)\|^{1-\delta_i(t)}\|u(1)\|^{\delta_i(t)}\}, \quad \forall u(t)\in\{U_{II}\}. \tag{5.3}$$

Moreover, if $u(1)\in H_2$, *then*

$$\|u(t)\|\leqslant M_2^{(2)}\|u(0)\|\exp\{M_3^{(2)}[\|u(1)\|_2/\|u(1)\|]t\}, \tag{5.4}$$

where $\delta_i(t)$ *are the functions defined in Lemma* 2.1, $M_1^{(2)}$, $M_2^{(2)}$, $M_3^{(2)}$ *and* M *in the expression of* $\delta_i(t)$ *are all constants independent of* $u(t)$.

Proof. As in the proof of Theorem 1, we first diagonalize the differential equation and then put it in the case discussed in Lemma 2.2.

As mentioned above, under condition (2) in Hypothesis II, $a_0(t, x, \xi)$ is uniformly diagonalizable. Hence there exists a uniformly nonsingular matrix $p(t, x, \xi)\in\{S_1^0\}$ such that

$$|\det p(t, x, \xi)|\geqslant\text{const }\gamma>0, \quad \forall(t, x, \xi)\in D \tag{5.5}$$

and

$$p(t, x, \xi)a_0(t, x, \xi)=J_0(t, x, \xi)p(t, x, \xi). \tag{5.6}$$

From Lemma 4.2, there exist $D(t)\in\mathscr{L}^0_{s_1}$ and $D_0^{-1}(t)\in\mathscr{L}(H\to H)$ such that

$$D(t)P(t)A(t)=E(t)D(t)P(t)+N_0(t) \quad \text{in } \mathscr{L} \tag{5.7}$$

and

$$D_0^{-1}(t)D(t)u=u, \quad \forall u\in H, \tag{5.8}$$

where $N_0(t)\in\mathscr{L}^0$, and $P(t)$, $E(t)$ are p. d. o. with the symbols $p(t, x, \xi)$ and diagonal $J(t, x, \xi)=J_0(t, x, \xi)|\xi|^2+J_1(t, x, \xi)|\xi|$ respectively. We decompose $E(t)=E_R(t)+E_I(t)$. Here $E_R(t)\in\mathscr{L}^2_{DS_1}$ and $E_I(t)\in\mathscr{L}^1_{DS_1}$ are the p. d. o. with the symbols $\operatorname{Re}J(t, x, \xi)=J_0(t, x, \xi)|\xi|^2+\operatorname{Re}J_1(t, x, \xi)|\xi|$ and $\sqrt{-1}\operatorname{Im}J(t, x, \xi)=\sqrt{-1}\operatorname{Im}J_1(t, x, \xi)|\xi|$ respectively (from II (3), $J_0(t, x, \xi)$ is real). From (5.5) and Lemma 2.4 we see that there exist $P^{(-1)}(t)\in\mathscr{L}^0_{s_1}$, $D_{(k)}(t)\in\mathscr{L}(H\to H)$ and $L_{(-1)}(t)\in\mathscr{L}^{-1}$ such that

$$D_{(k)}(t)P^{(-1)}(t)P(t)u=u+D_{(k)}(t)L_{(-1)}(t)T(k)u, \quad \forall u\in H. \tag{5.9}$$

Now let $u(t)\in\{U_{II}\}$ and set

$$D(t)P(t)u(t)=v(t), \quad T(k)u(t)=w(t). \tag{5.10}$$

Then from (5.10), (5.7) and (5.9), (5.8) we have

$$\frac{dv(t)}{dt}=D(t)P(t)A(t)u(t)+[D(t)P(t)N(t)+D'(t)P(t)+D(t)P'(t)]u(t)$$
$$=E(t)D(t)P(t)u(t)+[N_0(t)+D(t)P(t)N(t)+D'(t)P(t)+D(t)P'(t)]$$
$$\cdot\,[D_{(k)}(t)P^{(-1)}(t)D_0^{-1}(t)D(t)P(t)-D_{(k)}(t)L_{(-1)}(t)T(k)]u(t),$$
$$\frac{dw(t)}{dt}=T(k)[A(t)+N(t)][D_{(k)}(t)P^{(-1)}(t)D_0^{-1}(t)D(t)P(t)$$
$$-D_{(k)}(t)L_{(-1)}(t)T(k)]u(t).$$

Therefore

$$\begin{cases}\dfrac{dv(t)}{dt}=[E_R(t)+E_I(t)]v(t)+N_{1,1}^{(2)}(t)v(t)+N_{1,2}^{(2)}w(t), & (5.11)\\[2mm] \dfrac{dw(t)}{dt}=N_{2,1}^{(2)}(t)v(t)+N_{2,2}^{(2)}(t)w(t), & (5.12)\end{cases}$$

where

$$N_{1,1}^{(2)}(t)=[N_0(t)+D(t)P(t)N(t)+D'(t)P(t)+D(t)P'(t)]D_{(k)}(t)P^{(-1)}(t)D_0^{-1}(t),$$
$$N_{1,2}^{(2)}(t)=-[\cdots\cdots]D_{(k)}(t)L_{(-1)}(t),$$
$$N_{2,1}^{(2)}(t)=T(k)[A(t)+N(t)]D_{(k)}(t)P^{(-1)}(t)D_0^{-1}(t),$$
$$N_{2,2}^{(2)}(t)=-T(k)[A(t)+N(t)]D_{(k)}(t)L_{(-1)}(t).$$

From Lemma 2.3 we have $N_{2,i}^{(2)}(t)\in\mathscr{L}(H\to H)$.

Since $N_0(t)$, $D(t)$, $P(t)$, $D'(t)$, $P'(t)$, $P^{(-1)}(t)$ belong to $\mathscr{L}^0\subset\mathscr{L}(H\to H)$ and $N(t)$, $D_{(k)}(t)$, $D_0^{-1}(t)$ belong to $\mathscr{L}(H\to H)$, it is easily verified that $N_{1,i}^{(2)}(t)\in\mathscr{L}(H\to H)$. Hence

$$N_{i,j}^{(2)}(t)\in\mathscr{L}(H\to H). \tag{5.13}$$

The system (5.11), (5.12) is already diagonalized. We put it in the case of Lemma 2.2 by setting

$$y(t)=\begin{pmatrix}v(t)\\ w(t)\end{pmatrix},\quad \mathscr{B}(t)=\begin{pmatrix}E_R(t) & \theta_{n\times n}\\ \theta_{n\times n} & \theta_{n\times n}\end{pmatrix}. \tag{5.14}$$

In this case $\mathscr{H}=H\times H$, $\mathscr{D}(\mathscr{B})=\mathscr{H}_2=H_2\times H_2$.

Now we have to verify conditions (2.3) and (2.4).

Set

$$N_I(t)=E_I^{(*)}(t)+E_I(t). \tag{5.15}$$

Taking $\rho=0$ and applying Theorem 1.2 for $E_I(t)$, we have

$$E_I^{(*)}(t)-E_{I\rho}^{(*)}(t)\in\mathscr{L}^{1-0-1}=\mathscr{L}^0.$$

In addition $E_{I\rho}^{(*)}(t)$ has the symbol $(\sqrt{-1}\,\mathrm{Im}\,J_1(t,x,\xi)|\xi|)^*=-\sqrt{-1}\,\mathrm{Im}\,J_1(t,x,\xi)|\xi|$. Hence $E_{I\rho}^{(*)}=-E_I(t)$.

Consequently

$$N_I(t)=[E_I^{(*)}(t)-E_{I\rho}^{(*)}(t)]+[E_{I\rho}^{(*)}(t)+E_I(t)]\in\mathscr{L}^0.$$

From above and (5.11), (5.12), (5.13) we have

$$2\mathrm{Re}\Big(y(t),\ \frac{dy(t)}{dt}-\mathscr{B}(t)y(t)\Big)_{\mathscr{H}}=2\mathrm{Re}\{(v(t),\ E_I(t)v(t))+(v(t),\ N^{(2)}_{1,1}(t)v(t)$$
$$+N^{(2)}_{1,2}(t)w(t))+(w(t),\ N^{(2)}_{2,1}(t)v(t)+N^{(2)}_{2,2}(t)w(t))\}$$
$$=2\mathrm{Re}\Big\{\frac{1}{2}(v(t),\ N_I(t)v(t))+(v(t),\ N^{(2)}_{1,1}(t)v(t)+N^{(2)}_{1,2}(t)w(t))$$
$$+(w(t),\ N^{(2)}_{2,1}(t)v(t)+N^{(2)}_{2,2}(t)w(t))\Big\}=\widetilde{M}_4(t)\,\|y(t)\|^2_{\mathscr{H}}.$$

Since $N_I(t)$ and $N^{(2)}_{i,j}(t)$ belong to $\mathscr{L}(H\to H)$, we have

$$|\widetilde{M}_4(t)|\leqslant\text{const}\ M^{(2)}_4.$$

Thus condition (2.3) is valid.

Set

$$N_R(t)=E^{(*)}_R(t)-[E_R(t)+E_L(t)E_R(t)],\tag{5.16}$$

where $E_L(t)\in\mathscr{L}^{-1}_{DS_1}$ has the symbol

$$e_L(t,\ x,\ \xi)=[\sum_{|\alpha|=1}(-1)D^\alpha\partial^\alpha(J_0(t,\ x,\ \xi))\,|\xi|^2][J_0(t,\ x,\ \xi)\,|\xi|^2]^{-1}.$$

Taking $\rho=1$ and applying Theorem 1.2 we have

$$E^{(*)}_R(t)-E^{(*)}_{R\rho}(t)\in\mathscr{L}^{2-1-1}=\mathscr{L}^0.$$

Taking $\rho=1$ and applying Theorem 1.3 for $E_L(t)$ and $E_R(t)$ we can verify that $E^{(*)}_{R\rho}(t)-[E_R(t)+E^*_L(t)E_R(t)]\in\mathscr{L}^0$. Hence $N_R(t)\in\mathscr{L}^0$.

Since $J_0(t,\ x,\ \xi)$ is a real matrix, $D_j=-\sqrt{-1}\dfrac{\partial}{\partial x_j}$, we see that $e_L(t,\ x,\ \xi)$ is purely imaginary, i. e.

$$\mathrm{Re}\ e_L(t,\ x,\ \xi)=\theta_{n\times n}.\tag{5.17}$$

From Lemma 5.1 we have

$$2\mathrm{Re}(v,\ E_L(t)v)=\mathrm{Re}(v,\ \hat{E}_L(t)v)+\mathrm{Re}(v,\ N_{(-3)}(t)v),\quad\forall v\in\varphi,\tag{5.18}$$

where $\hat{E}_L(t)\in\mathscr{L}^{-1-1}_{s_1}=\mathscr{L}^{-2}_{s_1},\ N_{(-3)}(t)\in\mathscr{L}^{-3}.$

By applying Theorem 1.3 it is easily verified that $-E_I(t)E_R(t)+E_R(t)E_I(t)$ and $E_L(t)E_R(t)E_I(t)$ are p. d. o. of order 2 (up to an operator of order zero). And $E'_R(t)\in\mathscr{L}^2_{DS}$, $E_I(t)\in\mathscr{L}^1_{DS_1}$. Hence according to Lemma 5.2 there exist the p. d. o. $E_{\sigma''}(t)\in\mathscr{L}^0_{DS_1}$, $E_{\sigma'}(t)\in\mathscr{L}^0_{DS_1}$, $E_D(t)\in\mathscr{L}^0_{DS}$, $E_{I'}(t)\in\mathscr{L}^{-1}_{DS}$ and $N_{\sigma''}(t)$, $N_{\sigma'}(t)$, $N_D(t)$, $N_{I'}(t)$ in $\mathscr{L}^0$ such that

$$-E_I(t)E_R(t)+E_R(t)E_I(t)=E_{\sigma''}(t)E_R(t)+N_{\sigma''}(t),\tag{5.19}$$
$$E_L(t)E_R(t)E_I(t)=E_{\sigma'}(t)E_R(t)+N_{\sigma'}(t),\tag{5.20}$$
$$E'_R(t)=E_D(t)E_R(t)+N_D(t),\tag{5.21}$$
$$E_I(t)=E_{I'}(t)E_R(t)+N_{I'}(t).\tag{5.22}$$

Consequently, from the above relations and (5.15), (5.16) we get

$$E^{(*)}_I(t)E_R(t)+E^{(*)}_R(t)E_I(t)$$
$$=-E_I(t)E_R(t)+E_R(t)E_I(t)+N_I(t)E_R(t)+E_L(t)E_R(t)E_I(t)+N_R(t)E_I(t)$$
$$=(E_{\sigma''}(t)+N_I(t)+E_{\sigma'}(t)+N_R(t)E_{I'}(t))E_R(t)$$
$$+(N_{\sigma''}(t)+N_{\sigma'}(t)+N_R(t)N_{I'}(t)=E_0(t)E_R(t)+N_0(t),\tag{5.23}$$

where $$E_c(t)=E_{c''}(t)+N_I(t)+E_{c'}(t)+N_R(t)E_{I'}(t)$$
and $$N_c(t)=N_{c''}(t)+N_{c'}(t)+N_R(t)N_{I'}(t).$$

It is obvious that both $E_c(t)$ and $N_c(t)$ belong to $\mathscr{L}(H\to H)$.

From (5.11), (5.16), (5.21), (5.23), (5.18), (5.22) we have

$$\begin{aligned}
&\frac{d}{dt}\operatorname{Re}(v(t),\ E_R(t)v(t))-\operatorname{Re}[(N_{1,1}^{(2)}(t)v(t)+N_{1,2}^{(2)}w(t),\ E_R(t)v(t))\\
&\quad+(E_R^{(*)}(t)v(t),\ N_{1,1}^{(2)}(t)v(t)+N_{1,2}^{(2)}(t)w(t))]\\
&=\operatorname{Re}[(E(t)v(t),\ E_R(t)v(t))+(E_R^{(*)}(t)v(t),\ E(t)v(t))+(v(t),\ E'_R(t)v(t))]\\
&=\operatorname{Re}[2(E_R(t)v(t),\ E_R(t)v(t))+(v(t),\ (E_I^{(*)}(t)E_R(t)+E_R^{(*)}(t)E_I(t))v(t))\\
&\quad+((E_L(t)E_R(t)+N_R(t))v(t),\ E(t)v(t))+(v(t),\ (E_D(t)E_R(t)+N_D(t))v(t))]\\
&=\operatorname{Re}[2(E_R(t)v(t),\ E_R(t)v(t))+(v(t),\ (E_O(t)E_R(t)\\
&\quad+N_O(t))v(t))+(v(t),\ (E_D(t)E_R(t)+N_D(t))v(t))\\
&\quad+((\hat{E}_L(t)+N_{(-3)}(t))E_R(t)v(t),\ E_R(t)v(t))+(E_R(t)v(t),\ E_L^{(*)}(t)E_I(t)v(t))\\
&\quad+(N_R(t)v(t),\ (I+E_{I'}(t))E_R(t)v(t)+N_{I'}(t)v(t))]\\
&=\operatorname{Re}[2(E_R(t)v(t),\ E_R(t)v(t))+((E_O^*(t)+E_D^*(t)\\
&\quad+(\hat{E}_L(t)+N_{(-3)}(t))E_R(t)+E_L^{(*)}(t)E_I(t)\\
&\quad+(I+E_{I'}(t))^*N_{I'}(t))v(t),\ E_R(t)v(t))+(v(t),\ (N_O(t)\\
&\quad+N_D^*(t))v(t))+(N_R(t)v(t),\ N_{I'}(t)v(t))].
\end{aligned}$$

Since $E_L(t)$ (also $E_L^{(*)}(t))\in\mathscr{L}^{-1}$, $E_I(t)\in\mathscr{L}^1$, and $\hat{E}_L(t)\in\mathscr{L}^{-2}$, $N_{(-3)}(t)\in\mathscr{L}^{-3}$, $E_R(t)\in\mathscr{L}^2$, we have that $E_L^{(*)}(t)E_I(t)$ and $(\hat{E}_L(t)+N_{(-3)}(t))E_R(t)$ belong to $\mathscr{L}^0$. Moreover $E_O(t)$ (also $E_O^*)\in\mathscr{L}(H\to H)$, $E_D(t)$ (also $E_D^*(t))\in\mathscr{L}^0$, $E_{I'}(t)\in\mathscr{L}^{-1}$, $N_{I'}(t)\in\mathscr{L}^0$, $N_O(t)\in\mathscr{L}(H\to H)$, $N_D(t)\in\mathscr{L}^0$, $N_R(t)\in\mathscr{L}^0$. So we have that

$$(E_O^*(t)+E_D^*(t)+(\hat{E}_L(t)+N_{(-3)}(t))E_R(t)+E_L^{(*)}(t)E_I(t)+(I+E_{I'}(t))^*N_{I'}(t)$$

and $(N_O(t)+N_D(t))$ belong to $\mathscr{L}(H\to H)$.

From above and (5.13) we get

$$\begin{aligned}
&\frac{d}{dt}\operatorname{Re}(y(t),\ \mathscr{B}(t)y(t))_{\mathscr{H}}-2\|\mathscr{B}(t)y(t)\|_{\mathscr{H}}^2\\
&\quad=\frac{d}{dt}\operatorname{Re}(v(t),\ E_R(t)v(t))-2\|E_R(t)v(t)\|^2\\
&\quad=\widetilde{M}_5^{(2)}(t)\|E_R(t)v(t)\|(\|v(t)\|^2+\|w(t)\|^2)^{1/2}+\widetilde{M}_6^{(2)}(t)(\|v(t)\|^2+\|w(t)\|^2)\\
&\quad=\widetilde{M}_5^{(2)}(t)\|\mathscr{B}(t)y(t)\|_{\mathscr{H}}\|y(t)\|_{\mathscr{H}}+\widetilde{M}_6^{(2)}(t)\|y(t)\|_{\mathscr{H}}^2,
\end{aligned}$$

where $$|\widetilde{M}_5^{(2)}(t)|\leqslant M_5^{(2)},\quad |\widetilde{M}_6^{(2)}(t)|\leqslant M_6^{(2)}.$$

The constants $M_5^{(2)}$ and $M_6^{(2)}$ depend only on $A(t)$ and $N(t)$.

Thus condition (2.4) is also valid.

Applying Lemma 2.2 we get

$$\begin{aligned}
(\|v(t)\|^2+\|w(t)\|^2)^{\frac12}=\|y(t)\|_{\mathscr{H}}&\leqslant M_7^{(2)}\max_{i=1,2}\{\|y(0)\|_{\mathscr{H}}^{1-\delta_i(t)}\|y(1)\|_{\mathscr{H}}^{\delta_i(t)}\}\\
&=M_7^{(2)}\max_{i=1,2}\{[(\|v(0)\|^2+\|w(0)\|^2)^{\frac12}]^{1-\delta_i(t)}[(\|v(1)\|^2+\|w(1)\|^2)^{\frac12}]^{\delta_i(t)}\} \qquad (5.24)
\end{aligned}$$

and
$$\begin{aligned}
(\|v(t)\|^2+\|w(t)\|^2)^{\frac12}&\leqslant M_8^{(2)}(\|v(0)\|^2+\|w(0)\|^2)^{\frac12}\\
&\quad\cdot\exp\{M_9^{(2)}[\|E_R(1)v(1)\|/(\|v(1)\|^2+\|w(1)\|^2)^{\frac12}]t\}. \qquad (5.25)
\end{aligned}$$

Moreover from (5.9), (5.8), (5.10) we have

$$\|u(t)\| \leqslant \|D_{(k)}(t)P^{(-1)}(t)D_0^{-1}(t)v(t)\| + \|D_{(k)}(t)L_{(-1)}(t)w(t)\|$$

$$\leqslant M_{10}^{(2)}(\|v(t)\|^2 + \|w(t)\|^2)^{\frac{1}{2}}$$

$$= M_{10}^{(2)}(\|D(t)P(t)u(t)\|^2 + \|T(k)u(t)\|^2)^{\frac{1}{2}} \leqslant M_{11}^{(2)}\|u(t)\|. \tag{5.26}$$

In addition,

$$\|E_R(1)v(1)\| = \|E_R(1)D(1)P(1)U(1)\| \leqslant M_{12}^{(2)}\|U(1)\|_2. \tag{5.27}$$

From (5.24), (5.25), (5.26), (5.27) we obtain the desired estimates (5.3), (5.4) with the constants $M_1^{(2)} = M_7^{(2)}M_{10}^{(2)}M_{11}^{(2)}$, $M_2^{(2)} = M_8^{(2)}M_{10}^{(2)}M_{11}^{(2)}$ and $M_3^{(2)} = M_9^{(2)}M_{10}^{(2)}M_{12}^{(2)}$. These constants and constant M in expression of $\delta_i(t)$ are all independent of $u(t)$. The theorem is completely proved.

As an example we consider the ill-posed Cauchy problem of the backward parabolic equation

$$\begin{cases} \dfrac{\partial u(t, x)}{\partial t} = -\left\{\sum_{i,j=1}^{m} \dfrac{\partial}{\partial x_i}\left(A_{ij}(t, x)\dfrac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{m} B_i(t, x)\dfrac{\partial u}{\partial x_i} + C(t, x)u\right\}, & (5.28) \\ u(t, x)|_{t=0} = u_0(x) \in H(R^m), & (5.29) \end{cases}$$

where $x = (x_1, \cdots, x_m)$, $u(t, x)$ is an n-dimensional vector valued function. We suppose that $C(t, x)$ is bounded in D and $A_{ij}(t, x)$, $B_i(t, x)$ belong to $\{S_1^0\}$ (in reality, it is sufficient that A_{ij}, B_i are sufficiently smooth and satisfy the uniform convergence property (1.2) when $|x| \to \infty$).

If $n = 1$, $A_{11}(t, x) \geqslant \gamma_1 > 0$, (5.28) is the backward heat equation. The estimation of its solutions is discussed in [3] (Ref. of Part I), when A_{11}, C are independent of t and $B_1 \equiv \theta_{n\times n}$. If A_{ij}, B_i are symmetric and there exists a constant γ_1 such that the matrix

$$a_0(t, x, \xi) - \gamma_1 I_{n\times n} = \sum_{i,j=1}^{m} A_{ij}(t, x)\xi_i\xi_j/|\xi|^2 - \gamma_1 I_{n\times n}$$

is positive definite for all $(t, x, \xi) \in D$, then the problem (5.28), (5.29) is discussed in [6] (Ref. of Part I). If $a_0(t, x, \xi)$ is uniformly diagonalizable, its eigenvalues are all real, positive ($\lambda_i(t, x, \xi) > \gamma_1 > 0$) and distinct ($|\lambda_i(t, x, \xi) - \lambda_j(t, x, \xi)| > \gamma_2 > 0$ for $i \neq j$), then Theorem 2 can be applied to the problem (5.28), (5.29) without requiring the symmetry of A_{ij} and B_i.

Appendix

We prove the following proposition.

Proposition. Let $a_0(t, x, \xi) \in \{S_1^0\}$ be positive homogeneous of degree zero in ξ, and let the diagonal matrix $J_0(t, x, \xi)$ consisting of the eigenvalues $\lambda_l(t, x, \xi)$ of $a_0(t, x, \xi)$ belong to $\{DS_1^0\}$ and $\lambda_l(t, x, \xi)$ satisfy

$$|\lambda_l(t, x, \xi) - \lambda_q(t, x, \xi)| \geqslant \text{const}\ \gamma_2 > 0, \ \forall (t, x, \xi) \in D \text{ and } l \neq q.$$

Then $a_0(t, x, \xi)$ is uniformly diagonalizable. This means that there exists a uniformly nonsingular $(n\times n)$ matrix $p(t, x, \xi) \in \{S_1^0\}$ consisting of the eigenvectors of $a_0(t, x, \xi)$, i. e.

$$p(t, x, \xi)a_0(t, x, \xi) = J_0(t, x, \xi)p(t, x, \xi), \tag{6.1}$$

$$|\det p(t, x, \xi)| \geqslant \text{const}\ \gamma > 0, \ \forall (t, x, \xi) \in D. \tag{6.2}$$

Proof. Since $a_0(t, x, \xi)\in\{S_1^0\}$, we can extend the definition of $a_0(t, x, \xi)$ up to $t=1$ continuously, and discuss our proposition in the closed interval $[0, 1]$ of t.

For simplicity we denote the point (t, x, ξ) and the set $\{[0, 1]\times R^m\times(|\xi|=1)\}$ by z and $\{z\}$. First, for every internal point $z_0=(t_0, x_0, \xi_0)$ we construct the eigenvector $\hat{p}_{z_0}^{(j)}(t, x, \xi)$ (corresponding to the eigenvalue $\lambda_j(t, x, \xi)$) which is differentiable in t and infinitely differentiable in x and ξ in some neighbourhood of z_0. Then by connecting $\hat{p}_{z_0}^{(j)}(t, x, \xi)$ we construct the single valued vector function $p^{(j)}(t, x, \xi)$ which is differentiable in t and infinitely differentiable in x and ξ in the whole $\{z\}$. Finally we prove that the matrix consisting of these eigenvectors $p^{(j)}(t, x, \xi)$ belongs to $\{S_1^0\}$ and is uniformly nonsingular.

Since $\lambda_j(t, x, \xi)$ are distinct, the rank of the $(n\times n)$ matrix $a_0(z_0)-\lambda_j(z_0)I$ is equal to $(n-1)$. Hence the product of a certain constant $\hat{c}$ and all the minors of the elements in a certain column of this matrix form the left eigenvector (an n-dimensional row vector) of $a_0(z_0)$, corresponding to $\lambda_j(z_0)$. We choose the constant $\hat{c}$ such that the norm of this vector at point z_0 is equal to 1 and denote this vector by $\hat{p}_{z_0}^{(j)}(t, x, \xi)$. Obviously $|\hat{p}_{z_0}^{(j)}(z_0)|=1$. Since $a_0(t, x, \xi)\in\{S_1^0\}$ and $J_0(t, x, \xi)\in\{DS_1^0\}$, we have $a_0(t, x, \xi)-\lambda_j(t, x, \xi)I\in\{S_1\}$. Hence it is differentiable in t and infinitely differentiable in x, ξ and satisfies the uniform convergence property (1.2). Consequently there exists a number $\varepsilon_1>0$ independent of z_0 such that

$$|\hat{p}_{z_0}^{(j)}(z)|>\frac{1}{2},\quad \forall z\in O(z_0, \varepsilon_1), \tag{6.3}$$

where $O(z_0, \varepsilon_1)$ is an open neighbourhood of z_0, defined by

$$O(z_0, \varepsilon_1)=\begin{cases}\{z:\ |z-z_0|^2=(t-t_0)^2+|x-x_0|^2+|\xi-\xi_0|^2<\varepsilon_1^2\}\ \text{for}\ |x_0|\leqslant\dfrac{1}{\varepsilon_1},\\ \left\{z=(t, x, \xi):\ (t-t_0)^2+|\xi-\xi_0|^2<\varepsilon_1\ \text{and}\ |x|>\dfrac{1}{\varepsilon_1}\right\}\ \text{for}\ |x_0|>\dfrac{1}{\varepsilon_1}.\end{cases} \tag{6.4}$$

Let S_ε be an arbitrary set of z, the collection Q of $O(z_0, \varepsilon_1)$ constitutes a covering of S_ε, if

$$S_\varepsilon\subset\bigcup_{O(z_0,\varepsilon_1)\in Q}O(z_0, \varepsilon_1).$$

Let $\{z\}_1$ denote the open set $\left\{z\in\{z\}: |x|>\dfrac{1}{\varepsilon_1}\right\}$. Then $\left([0, 1]\times\left(|x|\leqslant\dfrac{1}{\varepsilon_1}\right)\times(|\xi|=1)\right)=\{z\}\setminus\{z\}_1$ is a closed set of z. Hence from the covering $\{O(z_0, \varepsilon_1): z_0\in\{z\}\setminus\{z\}_1\}$ we can select a finite open covering $\{O(z_0, \varepsilon_1)\}_1$ of $\{z\}\setminus\{z\}_1$. In addition $([0, 1]\times(|\xi|=1))$ is a closed set of (t, ξ). Hence from the covering $\{O(z_0, \varepsilon_1): z_0\in\{z\}_1\}$ we can also select a finite open covering $\{O(z_0, \varepsilon_1)\}_2$ of $\{z\}_1$. Then the sum of $\{O(z_0, \varepsilon_1)\}_1$ and $\{O(z_0, \varepsilon_1)\}_2$ constitutes a finite open covering $\{O(z_l, \varepsilon_1)\}_{l=1}^N$ of $\{z\}$.

Now we construct a c^∞-partition of unity $\{\psi_l(z)\}_{l=1}^N$ subordinate to the covering $\{O(z_l, \varepsilon_1)\}_{l=1}^N$. First we select a number $\varepsilon_2<\varepsilon_1$ such that $\{O(z_l, \varepsilon_2)\}_{l=1}^N$ is still a finite open covering of $\{z\}$. Now for $z_l\in\{z\}\setminus\{z\}_1$ we take $\hat{\psi}_l(z)$ equal to a nonnegative $c_0^\infty(O(z_l, \varepsilon_1))$ function satisfying $\hat{\psi}_l(z)>0$ for $z\in O(z_l, \varepsilon_2)$. And for $z_l\in\{z\}_1$, we take $\hat{\psi}_l(z)=\hat{\psi}_l^{(1)}(t, \xi)\hat{\psi}_l^{(2)}(x)$, where $\hat{\psi}_l^{(1)}$ is a nonnegative $c_0^\infty(|t-t_l|^2+|\xi-\xi_l|^2<\varepsilon_1^2)$ function satisfying $\hat{\psi}_l^{(1)}(t, \xi)>0$ for $(t, \xi)\in(|t-t_l|^2+|\xi-\xi_l|^2<\varepsilon_2^2)$ and $\hat{\psi}_l^{(2)}(x)$ is a nonnegative $c^\infty(R^m)$ function satisfying $\hat{\psi}_l^{(2)}(x)>0$ for $|x|>\dfrac{1}{\varepsilon_2}$, $\hat{\psi}_l^{(2)}(x)\equiv1$ for $|x|>\dfrac{2}{\varepsilon_2}$ and

$\hat{\psi}_l^{(2)}(x)\equiv 0$ for $|x|\leqslant\frac{1}{s_1}$. Now we take $\psi_l(z)=\hat{\psi}_l(z)\Big/\sum_{j=1}^{N}\hat{\psi}_j(z)$. It is easily verified that all $\psi_l(t, x, \xi)$ satisfy the uniform convergence property similar to (1, 2) and functions $\psi_l(z)$ thus defined form a c^∞-partition of unity $\{\psi_l(z)\}_{l=1}^N$ subordinate to $\{O(z_l, \varepsilon_1)\}_{l=1}^N$. Obviously

$$\sum_{l=1}^{N}\psi_l(z)\equiv 1. \tag{6.5}$$

Now we connect $\hat{p}_{z_l}^{(j)}(z)$ by using $\psi_l(z)$. Since

$$|p_{z_l}^{(j)}(z)|>\frac{1}{2}\quad\text{in}\quad O(z_l, \varepsilon_1), \tag{6.6}$$

the normalized vector $p_{z_l}^{(j)}(z)=\hat{p}_{z_l}^{(j)}(z)/|\hat{p}_{z_l}^{(j)}(z)|$ is also differentiable in t and infinitely differentiable in x and ξ in $O(z_l, \varepsilon_1)$. If $z\in O(z_l, \varepsilon_1)\cap O(z_k, \varepsilon_1)$, then the vectors $p_{z_l}^{(j)}(z)$ and $p_{z_k}^{(j)}$ are both eigenvectors of $a_0(z)$, corresponding to the same eigenvalue $\lambda_j(z)$ and having the same norm equal to 1. Hence these two vectors are equal up to a complex scalar factor having modulus 1 (since λ_j are distinct), i. e.

$$p_{z_k}^{(j)}(z)=\exp(\sqrt{-1}\,\delta_{l,k}^{(j)}(z))\,p_{z_l}^{(j)}(z), \tag{6.7}$$

where $\delta_{l,k}^{(j)}(z)$ is a real valued function.

If we denote the i-th component of vector $p^{(j)}$ by $p^{(j,i)}$, then $\delta_{l,k}^{(j)}(z)$ is equal to the argument of the complex value $p_{z_k}^{(j,i)}(z)/p_{z_l}^{(j,i)}(z)$ for any i for which $p_{z_l}^{(j,i)}\neq 0$, i. e.

$$\delta_{l,k}^{(j)}(z)=-\sqrt{-1}\ \ln\,[p_{z_k}^{(j,i)}(z)/p_{z_l}^{(j,i)}(z)]. \tag{6.8}$$

Since $|p_z^{(j)}(z)|=1$ we can always find the index i such that the absolute value of $p_{z_l}^{(j,i)}(z)$ is not smaller than $1/\sqrt{n}$. Hence $\delta_{l,k}^{(j)}(z)$ is also differentiable in t and infinitely differentiable in x, ξ as $p_{z_l}^{(j)}(z)$.

Set

$$p^{(j)}(t, x, \xi)=p^{(j)}(z)=p_{z_l}^{(j)}(z)\exp\Big[\sqrt{-1}\sum_{q=1}^{N}\psi_q(z)\delta_{l,q}^{(j)}(z)\Big]\quad\text{for } z\in O(z_l, \varepsilon_1). \tag{6.9}$$

It is obvious that $p^{(j)}(t, x, \xi)$ is the eigenvector of $a_0(t, x, \xi)$ corresponding to $\lambda_j(t, x, \xi)$, and $p^{(j)}(t, x, \xi)$ is differentiable in t and infinitely differentiable in x, ξ. We have to verify that $p^{(j)}(t, x, \xi)$ is single valued, i. e. for $z\in O(z_l, \varepsilon_1)\cap O(z_k, \varepsilon_1)$,

$$p_{z_l}^{(j)}(z)\exp\Big[\sqrt{-1}\sum_{q=1}^{N}\psi_q(z)\delta_{l,q}^{(j)}(z)\Big]=p_{z_k}^{(j)}(z)\exp\Big[\sqrt{-1}\sum_{q=1}^{N}\psi_q(z)\delta_{k,q}^{(j)}(z)\Big]. \tag{6.10}$$

(6.10) can be obtained from (6.7), (6.5), (6.8) as follows

$$\begin{aligned}
p_{z_k}^{(j)}(z)\exp\Big[\sqrt{-1}\sum_{q=1}^{N}\psi_q(z)\delta_{k,q}^{(j)}(z)\Big]&=p_{z_l}^{(j)}(z)\exp\Big[\sqrt{-1}\Big(\sum_{q=1}^{N}\psi_q(z)\delta_{k,q}^{(j)}(z)+\delta_{l,k}^{(j)}(z)\Big)\Big]\\
&=p_{z_l}^{(j)}(z)\exp\Big[\sqrt{-1}\sum_{q=1}^{N}\psi_q(z)(\delta_{k,q}^{(j)}(z)+\delta_{l,k}^{(j)}(z))\Big]\\
&=p_{z_l}^{(j)}(z)\exp\Big[\sum_{q=1}^{N}\psi_q(z)\ln(p_{z_q}^{(j,i)}(z)/p_{z_l}^{(j,i)}(z))\Big]=p_{z_l}^{(j)}(z)\exp\Big[\sqrt{-1}\sum_{q=1}^{N}\psi_q(z)\delta_{l,q}^{(j)}(z)\Big].
\end{aligned}$$

Now we denote the matrix consisting of these eigenvectors $p^{(j)}(t, x, \xi)$ by $p(t, x, \xi)$. It is obvious that (6.1) is valid. Since $p(t, x, \xi)$ is differentiable in t and infinitely differentiable in x, ξ as $p^{(j)}(t, x, \xi)$, and $\psi_l(t, x, \xi)\in c^\infty(O(z_l, \varepsilon_1))$ satisfy the uniform convergence property similar to (1.2), it is easily verified that $\psi_l(t, x, \xi)\,p(t, x, \xi)\in\{S_1^0\}$. Consequently

$$p(t, x, \xi)=\sum_{l=1}^{N}\psi_l(t, x, \xi)p(t, x, \xi)\in\{S_1^0\}.$$

$p^{(j)}(t, x, \xi)(j=1, 2, \cdots, n)$ are eigenvectors corresponding to distinct eigenvalues λ_j. Hence $p^{(j)}(t, x, \xi)$ are linearly independent. Consequently

$$\det p(t, x, \xi)\neq 0, \quad \forall(t, x, \xi)\in\{z\}. \tag{6.11}$$

Since $p(t, x, \xi)\in\{S_1^0\}$, it is easily proved that $\lim\limits_{|x|\to\infty}\lambda_j(t, x, \xi)=\lambda_j(t, \infty, \xi)$ and $\lim\limits_{|x|\to\infty}p(t, x, \xi)=p(t, \infty, \xi)$ exist. And $p(t, \infty, \xi)$ is a matrix consisting of the eigenvectors of $a_0(t, \infty, \xi)$, corresponding to distinct eigenvalues $\lambda_j(t, \infty, \xi)$. Hence there exists

$$\lim_{|x|\to\infty}\det p(t, x, \xi)=\det p(t, \infty, \xi)\neq 0, \quad \forall(t, \xi)\in([0, 1]\times(|\xi|=1)). \tag{6.12}$$

From (6.11), (6.12) we obtain immediately that there exists a constant γ such that

$$|\det p(t, x, \xi)|\geqslant \text{const}\ \gamma>0, \quad \forall(t, x, \xi)\in\{z\}.$$

(6.2) is valid. Thus we have proved the proposition.