THE COST OF KUHN'S ALGORITHM AND COMPLEXITY THEORY*

XU SEN-LIN (徐森林)

(China University of Science and Technology, Hefei, China)

Abstract

A comparison by Wang and Xu[6] between S. Smale's cost estimation for Newton's method and that of the author's for Kuhn's algorithm, both aiming at the zero finding of complex polynomials, showed impromements the advantage of the latter in finding zeros and approximate zeros. In this paper, important on the above work are made. Furthermore, a probabilistic estimation of the monotonicity of Kuhn's algorithm is obtained.

§ 1. Introduction

A comparison between S. Smale's cost estimation for Newton's method and that of the author's for Kuhn's algorithm, both aiming at the zero finding of complex polynomials, was presented by Wang and Xu[6]. It turns out that the latter is much better both in finding zeros and in finding approximate zeros. The ratios are respectively from n^9/μ^7 to $n^2\log(n/s)$ and from n^9/μ^7 to $n^3\log(n/\mu)$, where n is the degree of polynomials, $s>0$ is the accuracy demand for resulted zeros, and μ is the probability allowing the corresponding estimation to fail, $0<\mu<1$.

James Renegar obtained similar results[3]. His Lemma 3.1 and Proposition 5.6 in [3].

Here we improve the results of Wang and Xu[6]. A new cost estimation for Kuhn's algorithm is given by Theorem 2.12 while Theorem 3.8 answers probabilistically the problem of finding the approximate zeros of polynomials suggested by S. Smale. This is followed by a discussion in Theorem 4.15 on a probabilistic estimation of the monotonicity of Kuhn's algorithm.

§ 2. Cost of Kuhn's Algorithm

In order to be consistent, we use the same notation as used by Kuhn[2]. For simplicity, let $\bar{z}=0$ and $h=1$.

The algorithm can be sketched as follows. The half-space $C\times[-1,\infty)$ is simplicially triangulated such that every vertex is in some plane $C_d=C\times\{d\}$, $d=-1, 0, 2, \cdots$, and each plane C_d is then subdivided into isosceles right triangles with right-angle sides equal to $s(d)$, where $s(-1)=1$ and $s(d)=2^{-d}$, $d\geqslant 0$.

The labelling for vertices in C_d, $d\geqslant 0$, is that (the argument of a complex

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number is restricted to $(-\pi, \pi]$)

$$l(z)=\begin{cases}1, & \text{if } f(z)=0 \text{ or } -\pi/3\leqslant\arg f(z)\leqslant\pi/3,\\ 2, & \text{if } \pi/3<\arg f(z)\leqslant\pi,\\ 3, & \text{if } -\pi<\arg f(z)<-\pi/3,\end{cases}$$

where $f(z)=z^n+\sum_{i=0}^{n-1}a_iz^i, a_i\in C$, while the labelling for C_{-1} uses z^n instead of $f(z)$.

Let Q be the square in C_{-1} bounded by $x=\pm m$ and $y=\pm m$, where $z=x+iy$ and $m=\lceil 3(1+\sqrt{2})n/4\pi\rceil$. The symbol $\lceil a\rceil$ is the least integer not less than a.

A triangle is said to be completely labelled (c.l. triangle) if its three vertices are exactly labelled 1, 2 and 3.

Proposition 2.1. Let Σ_d be an elementary cube of the triangulation between C_d and C_{d+1}, and let B_d be a cylinder with axis $\{0\}\times[d, d+1]$. Let σ_d denote the number of tetrahedra which belong to Σ_d and are wholly contained in B_d. Then

$$\sigma_d\leqslant\begin{cases}5\cdot\mathrm{vol}(\Sigma_d\cap B_d), & \text{if } d=-1,\\ 14\cdot\mathrm{vol}(\Sigma_d\cap B_d)\cdot 2^{2d}, & \text{if } d\geqslant 0.\end{cases}$$

Proof. Let $V_d=\mathrm{vol}(\Sigma_d\cap B_d)$ for convenience. For $d=-1$, we always have $1\geqslant V_d\geqslant 0$. Obviously, if $V_d\leqslant 1/2$, then $\sigma_d=0$; if $V_d=1$, then $\Sigma_d\subset B_d$. So $\sigma_d=5$. In the case $1/2<V_d<1$, we have $\sigma_d\leqslant 1$ because every vertical edge of Σ_d touches four tetrahedra contained in Σ_d. In any of the above cases, the proposition is true.

For $d\geqslant 0$, $0\leqslant V_d\leqslant 2^{-2d}$. We discuss eight cases. Since the central point of the upper square is touched by all the fourteen tetrahedra of the cube, it is easy to obtain the corresponding results for σ_d by simple volume analysis:

(1) $V_d=2^{-2d}$, $\quad\sigma_d=14$;

(2) $\frac{7}{8}2^{-2d}<V_d<2^{-2d}$, $\quad\sigma_d\leqslant 9$;

(3) $\frac{3}{4}2^{-2d}<V_d\leqslant\frac{7}{8}2^{-2d}$, $\quad\sigma_d\leqslant 8$;

(4) $\frac{1}{2}2^{-2d}<V_d\leqslant\frac{3}{4}2^{-2d}$, $\quad\sigma_d\leqslant 7$;

(5) $\frac{3}{8}2^{-2d}<V_d\leqslant\frac{1}{2}2^{-2d}$, $\quad\sigma_d\leqslant 4$;

(6) $\frac{1}{4}2^{-2d}<V_d\leqslant\frac{3}{8}2^{-2d}$, $\quad\sigma_d\leqslant 3$;

(7) $\frac{1}{8}2^{-2d}<V_d\leqslant\frac{1}{4}2^{-2d}$, $\quad\sigma_d\leqslant 1$;

(8) $0\leqslant V_d\leqslant\frac{1}{8}2^{-2d}$, $\quad\sigma_d=0$.

In all the eight cases, $\sigma_d\leqslant 14\cdot V_d\cdot 2^{2d}$. It completes the proof.

Lemma 2.2. *Suppose that w_k are complex constants, $k=1, 2, 3$, and $-\pi/3\leqslant\arg w_1\leqslant\pi/3$; $\pi/3<\arg w_2\leqslant\pi$; $-\pi<\arg w_3<-\pi/3$. Then $\left|\arg\frac{w_2}{w_1}\right|\geqslant\frac{2\pi}{3}$, or $\left|\arg\frac{w_2}{w_3}\right|\geqslant\frac{2\pi}{3}$, or $\left|\arg\frac{w_1}{w_3}\right|\geqslant\frac{2\pi}{3}$.*

Proof. Since

$$\left|\arg\frac{w_2}{w_1}\right|<\frac{2\pi}{3}\Leftrightarrow 0<\arg w_2-\arg w_1<\frac{2\pi}{3}, \tag{2.1}$$

$$\left|\arg\frac{w_1}{w_3}\right|<\frac{2\pi}{3} \Leftrightarrow 0<\arg w_1-\arg w_3<\frac{2\pi}{3}, \tag{2.2}$$

$$\left|\arg\frac{w_2}{w_3}\right|<\frac{2\pi}{3} \Leftrightarrow \arg w_2-\arg w_3>\frac{4\pi}{3}, \tag{2.3}$$

from (2.1) and (2.3) we obtain

$$\arg w_1-\arg w_3=(\arg w_2-\arg w_3)-(\arg w_2-\arg w_1)>\frac{4\pi}{3}-\frac{2\pi}{3}=\frac{2\pi}{3}.$$

It contradicts (2.2), and this completes the proof.

Corollary 2.3. Under the condition of Proposition 2.2, we have $|\arg w_2-\arg w_1|\geqslant\frac{2\pi}{3}$ or $|\arg w_2-\arg w_3|\geqslant\frac{2\pi}{3}$ or $|\arg w_1-\arg w_3|\geqslant\frac{2\pi}{3}$.

Proof. Similar to the proof of Lemma 2.2.

Lemma 2.4. *Let $z_1, z_2, z\in C$, $|z_1-z_2|\leqslant\delta$, $0<\alpha\leqslant\pi$, $E=\{tz_1+(1-t)z_2 \mid t\in[0, 1]\}$. If $\left|\arg\frac{z_2-z}{z_1-z}\right|\geqslant\alpha$, then*

$$E\cap B(z, \delta/\alpha)\neq\phi, \tag{2.4}$$

where $B(z, \delta/\alpha)$ is an open disc with center z and radius δ/α.

Proof. If (2.4) does not hold, then it is easily verified (by calculus) that

$$\left|\arg\frac{z_2-z}{z_1-z}\right|\leqslant 2\operatorname{arctg}\frac{\delta}{2}\Big/\frac{\delta}{\alpha}=2\operatorname{arctg}\frac{\alpha}{2}<2\cdot\frac{\alpha}{2}=\alpha,$$

contrary to the hypothesis.

Proposition 2.5. Let $\{z_1, z_2, z_3\}$ be a c.l. triangle labelled by $f(z)=\prod_{i=1}^{n}(z-\xi_i)$ and $\operatorname{diam}\{z_1, z_2, z_3\}\leqslant\delta$. Then

$$\{z_1, z_2, z_3\}\subseteq\bigcup_{i=1}^{n}B\left(\xi_i, \delta\left(1+\frac{3n}{2\pi}\right)\right). \tag{2.5}$$

Proof. Suppose $(l(z_1), l(z_2), l(z_3))=(1, 2, 3)$. According to Lemma 2.2, there is a pair z_i and z_j, such that $|\arg(f(z_i)/f(z_j))|\geqslant2\pi/3$ is valid.

Let $E=\{tz_i+(1-t)z_j \mid t\in[0, 1]\}$. If (2.5) is invalid, then for all k,

$$E\cap B(\xi_k, 3n\delta/2\pi)=\phi.$$

Set $\alpha=2\pi/3n$; then Lemma 2.4 implies

$$\left|\arg\frac{z_i-\xi_k}{z_j-\xi_k}\right|<\frac{2\pi}{3n},$$

and so
$$\left|\arg\frac{f(z_i)}{f(z_j)}\right|=\left|\arg\frac{\prod_{k=1}^{n}(z_i-\xi_k)}{\prod_{k=1}^{n}(z_j-\xi_k)}\right|\leqslant\sum_{k=1}^{n}\left|\arg\frac{z_i-\xi_k}{z_j-\xi_k}\right|<\frac{2\pi}{3},$$

a contradiction.

Proposition 2.6. The number of tetrahedra in $C\times[d, d+1]$, $d\geqslant0$, with completely labelled faces does not exceed $28n\pi\left(1+\frac{3n}{2\pi}\right)^2$.

Proof. According to Propositions 2.1 and 2.5, the number does not exceed

$$14n\pi\left(\frac{\sqrt{2}}{2^d}\right)^2\left(1+\frac{3n}{2\pi}\right)\cdot2^{2d}=28n\pi\left(1+\frac{3n}{2\pi}\right)^2.$$

Lemma 2.7. *If $|z|\geqslant\max|a_k|+1$, then $f(z)\neq0$. That is, $\max|\xi_k|<\max|a_k|+1$, where $\xi_1, \cdots, \xi_n$ are all zeros of $f(z)$.*

Proof. If $\max|a_k|=0$, then $|f(z)|=|z^n|\geqslant1$.

If $\max|a_k|>0$, then

$$|f(z)|=\left|z^n\left(1+\sum_{l=1}^{n}\frac{a_{n-l}}{z^l}\right)\right|\geqslant|z^n|\left(1-\sum_{l=1}^{n}\frac{|a_{n-l}|}{|z|^l}\right)>|z^n|\left(1-\max|a_k|\sum_{l=1}^{\infty}\frac{1}{|z|^l}\right)$$
$$=|z^n|\left(1-\frac{\max|a_k|}{|z|-1}\right)\geqslant 0.$$

Therefore, $f(z)\neq 0$.

Lemma 2.8. *Let $\{z_1, z_2, z_3\}$ be the vertices of some c.l. face in $C\times[-1, 0]$. Then,*

$$\{z_1, z_2, z_3\}\subseteq B\left(0, \max|a_k|+1+\sqrt{2}\left(1+\frac{3n}{\pi}\right)\right). \tag{2.6}$$

Proof. Lemma 2.7 shows that the zeros $\xi_1, \cdots, \xi_n$ of f lie within $B(0, \max|a_k|+1)$.

By Corollary 2.3, there is a pair of vertices, say z_1 and z_2, the arguments of whose images under the piecewise linear map induced on $C\times[-1, \infty)$ by f and $z\mapsto z^n$ differ by at least $2\pi/3$.

Suppose that (2.6) is not satisfied. If z_2 lies in C_0 and z_1 lies in C_{-1}, since $\operatorname{diam}\{z_1, z_2, z_3\}\leqslant\sqrt{2}$, then

$$E_1\cap B\left(\xi_i, \sqrt{2}\frac{3n}{\pi}\right)=\phi \quad\text{and}\quad E_2\cap B\left(z_1, \sqrt{2}\frac{3n}{\pi}\right)=\phi,$$

where $E_1=\{tz_1+(1-t)z_2\mid t\in[0, 1]\}$, $E_2=\{t\xi_i\mid t\in[0, 1]\}$. Then Lemma 2.4 implies

$$\left|\arg\frac{f(z_2)}{z_1^n}\right|=\left|\arg\frac{\prod_{i=1}^{n}(z_2-\xi_i)}{z_1^n}\right|\leqslant\sum_{i=1}^{n}\left|\arg\frac{z_2-\xi_i}{z_1}\right|$$
$$\leqslant\sum_{i=1}^{n}\left(\left|\arg\frac{z_2-\xi_i}{z_1-\xi_i}\right|+\left|\arg\frac{\xi_i-z_1}{0-z_1}\right|\right)<\sum_{i=1}^{n}\left(\frac{\pi}{3n}+\frac{\pi}{3n}\right)=\frac{2\pi}{3}.$$

If both z_1 and z_2 lie in C_0, then

$$\left|\arg\frac{f(z_2)}{f(z_1)}\right|\leqslant\sum_{i=1}^{n}\left|\arg\frac{z_2-\xi_i}{z_1-\xi_i}\right|<\sum_{i=1}^{n}\frac{\pi}{3n}=\frac{\pi}{3}.$$

If both z_1 and z_2 lie in C_{-1}, then

$$\left|\arg\frac{z_2^n}{z_1^n}\right|\leqslant\sum_{i=1}^{n}\left|\arg\frac{z_2-0}{z_1-0}\right|<\sum_{i=1}^{n}\frac{\pi}{3n}=\frac{\pi}{3}.$$

This contradicts with "at least $2\pi/3$".

Lemma 2.9. *The computation is entirely processed inside the cylinder with radius $M=\max\left\{3\sqrt{2}(2+\pi)n/4\pi, \frac{n}{n-1}\max|a_k|+1\right\}+\sqrt{2}$ and centered at the origin of the plane.*

Proof. Because the maximal diameter of triangles is $\sqrt{2}$, if any point of a triangle is outside the cylinder of radius M, then every edge of the triangle should be outside the cylinder with same axis and radius $r=M-\sqrt{2}$. With Lemma 2.2, it suffices to show that for any edge (z', z'') of the triangulation outside the cylinder of radius r, $|\arg(F(z')/F(z''))|<2\pi/3$, where $F(z)=z^n$ for $z\in C_{-1}$ and $F(z)=f(z)$ otherwise. Since then any c.l. triangle must be wholly contained in the cylinder with radius M.

Rewrite $f(z)$ as $f(z)=z^n\left(1+\sum_{l=0}^{n-1}\frac{a_l}{z^{n-l}}\right)=z^n(1+g(z))$. If both z' and z'' are not in

C_{-1}, then

$$|g(z'')|\leqslant\sum_{l=0}^{n-1}\frac{|a_l|}{r^{n-l}}\leqslant\max|a_k|/r-1\leqslant(n-1)/n,$$

$$|g(z')-g(z'')|\leqslant|a_{n-1}|\cdot\left|\frac{1}{z'}-\frac{1}{z''}\right|+\cdots+|a_0|\cdot\left|\frac{1}{z'^n}-\frac{1}{z''^n}\right|$$

$$\leqslant\max|a_k|\cdot|z'-z''|\cdot\left(\frac{1}{r^2}+\frac{2}{r^3}+\cdots+\frac{n}{r^{n+1}}\right)<\sqrt{2}\cdot\max|a_k|/(r-1)^2$$

$$\leqslant\frac{\sqrt{2}}{r-1}\cdot\frac{n-1}{n}\leqslant\frac{n-1}{n}\cdot\sqrt{2}\Big/\left(\frac{3\sqrt{2}(2+\pi)n}{4\pi}-1\right)$$

$$<\frac{n-1}{n}\sqrt{2}\Big/\frac{3\sqrt{2}(2+\pi)(n-1)}{4\pi}=\frac{4\pi}{3(2+\pi)n},$$

$$\left|\frac{g(z')-g(z'')}{1+g(z'')}\right|\leqslant\frac{4\pi}{3(2+\pi)n}\Big/\left(1-\frac{n-1}{n}\right)=\frac{4\pi}{3(2+\pi)}<1,$$

$$|\arg F(z')/F(z'')|=|\arg f(z')/f(z'')|\leqslant n\left|\arg\frac{z'}{z''}\right|+\left|\arg\left(1+\frac{g(z')-g(z'')}{1+g(z'')}\right)\right|$$

$$\leqslant n\cdot\frac{\sqrt{2}}{3\sqrt{2}(2+\pi)\cdot n/4\pi}+\frac{\pi}{2}\cdot\left|\frac{g(z')-g(z'')}{1+g(z'')}\right|$$

$$<\frac{4\pi}{3(2+\pi)}+\frac{\pi}{2}\cdot\frac{4\pi}{3(2+\pi)}=\frac{4\pi}{3(2+\pi)}\left(1+\frac{\pi}{2}\right)=\frac{2\pi}{3}.$$

The proofs for cases $z'\in C_{-1}$ and/or $z''\in C_{-1}$ are similar.

Corollary 2.10. Assume that $\{z_1, z_2, z_3\}$ is as illustrated in Lemma 2.8. Then

$$\{z_1, z_2, z_3\}\subseteq B\left(0, \min\left\{M, \max|a_k|+1+\sqrt{2}\left(1+\frac{3n}{\pi}\right)\right\}\right).$$

Proof. According to Lemmas 2.8 and 2.9, the corollary is obvious.

Proposition 2.11. The number of steps between simplices in $C\times[-1, d]$, $d\geqslant 0$, taken by the algorithm does not exceed

$$8\lceil 3(1+\sqrt{2})n/4\pi\rceil\,(1+\lceil 3(1+\sqrt{2})n/4\pi\rceil)$$
$$+5\pi\cdot\min\left\{M, \max|a_k|+1+\sqrt{2}\left(1+\frac{3n}{\pi}\right)\right\}^2+28nd\pi\left(1+\frac{3n}{2\pi}\right)^2.$$

Proof. By Corollary 2.10, Propositions 2.1 and 2.6, the number of steps between simplices in $C\times[-1, d]$ does not exceed

$$\begin{pmatrix}\#\text{ of edges}\\ \text{in }\partial Q\end{pmatrix}+\begin{pmatrix}\#\text{ of simplices}\\ \text{in } Q\end{pmatrix}+\begin{pmatrix}\#\text{ of tetrahedra in } C\times[-1, 0]\\ \text{with a c.l. face}\end{pmatrix}$$

$$+\begin{pmatrix}\#\text{ of tetrahedra in } C\times[0, d]\\ \text{with a c.l. face}\end{pmatrix}\leqslant 8\lceil 3(1+\sqrt{2})n/4\pi\rceil\,(1+\lceil 3(1+\sqrt{2})n/4\pi\rceil)$$

$$+5\pi\cdot\min\left\{M, \max|a_k|+1+\sqrt{2}\left(1+\frac{3n}{\pi}\right)\right\}^2+28nd\pi\left(1+\frac{3n}{2\pi}\right)^2.$$

Theorem 2.12. *Given* $\varepsilon>0, f(z)=z^n+\sum_{i=0}^{n-1}a_iz^i$. *Some zeros of* f *can be approximated within a distance* ε *by Kuhn's algorithm in steps not greater than*

$$\Big[8\lceil 3(1+\sqrt{2})n/4\pi\rceil(1+\lceil 3(1+\sqrt{2})n/4\pi\rceil)$$
$$+5\pi\cdot\min\left\{M, \max|a_k|+1+\sqrt{2}\left(1+\frac{3n}{\pi}\right)\right\}^2+28nd\pi\left(1+\frac{3n}{2\pi}\right)^2\Big],$$

where
$$d=\left\lceil \log_2 \sqrt{2}\left(1+\frac{3n}{2\pi}\right)\Big/ \varepsilon\right\rceil.$$

Proof. When $d=\left\lceil \log \sqrt{2}\left(1+\frac{3n}{2\pi}\right)\Big/ \varepsilon\right\rceil$, we obtain

$$\frac{\sqrt{2}}{2^d}\left(1+\frac{3n}{2\pi}\right)\leqslant\frac{\sqrt{2}}{\sqrt{2}\left(1+\frac{3n}{2\pi}\right)\Big/\varepsilon}\cdot\left(1+\frac{3n}{2\pi}\right)=\varepsilon.$$

Proposition 2.5 shows that each vertex of a c.l. face in C_d lies within a distance ε of some zero of f.

§ 3. Approximate Zeros and Complexity Theory

Definition 3.1. *Let $z_0\in C$ and define inductively $z_k=z_{k-1}-f(z_{k-1})/f'(z_{k-1})$. z_0 is said to be an approximate zero provided the sequence z_k is well defined for all $k\geqslant 1$ and converges to z^* as $k\to\infty$ with $f(z^*)=0$, and $|f(z_k)/f(z_{k-1})|<1/2$ for all $k\geqslant 1$.*

Lemma 3.2. *Let $g(z)=\sum_{i=1}^{m} b_i z^i$, a complex polynomial with $b_1\neq 0$. Then there is a critical point $\theta\in C$ (i.e. $g'(\theta)=0$) such that for every $k=2, 3, \cdots,$*
$$|b_k/b_1|^{1/(k-1)}|f(\theta)/b_1|<K,$$
where K is a constant and $1\leqslant K\leqslant 4$.

Proof. See [5], Theorem 1, Section 1, Part II.

Lemma 3.3. *Let $c\geqslant 1$ and $\rho_f=\min\limits_{\theta, f'(\theta)=0}|f(\theta)|$ for a polynomial f. If $0<|f(z)|<\rho_f/(cK+K+1)$, then $|f(z')/f(z)|<1/c$, where $z'=z-f(z)/f'(z)$.*

Proof. See [5], Corollary A, Section 2, Part II.

Proposition 3.4. Let $c>1$ and $|f(z_0)|<\rho_f/(cK+K+1)$. Then Newton's method starting at z_0 converges to z^*, a solution of $f(z)=0$.

Proof. Let $z_k=z_{k-1}-f(z_{k-1})/f'(z_{k-1})$. By Lemma 3.3 and the definition of ρ_f, it is easy to show that z_k is well defined for all $k\geqslant 1$.

Now if for some l, $f(z_0)\neq 0$, $\cdots$, $f(z_{l-1})\neq 0$ but $f(z_l)=0$, then inductively we have $f'(z_k)\neq 0$ and $z_{k+1}=z_k$ for $k=l, l+1, \cdots$. Thus we have $\lim\limits_{k\to\infty} z_k=z^*$ with $f(z^*)=0$ and $z^*=z_l$.

If $f(z_k)\neq 0$ for any k, since
$$|f(z_k)|=|f(z_k)/f(z_{k-1})|\cdots|f(z_1)/f(z_0)|\cdot|f(z_0)|<|f(z_0)|/c^k,$$
we obtain $\lim\limits_{k\to\infty} f(z_k)=0$.

Let $L=\min\limits_{|f(z)|\leqslant\rho_f/(cK+K+1)}|f'(z)|$. Then both the compactness of set $\{z\,|\,|f(z)|\leqslant\rho_f/(cK+K+1)\}$ and $|f(z)|\leqslant\rho_f/(cK+K+1)<\rho_f$ imply $L>0$. Now for any natural numbers k, m,

$$|z_{k+m}-z_k|\leqslant\sum_{j=1}^{m}|z_{k+j}-z_{k+j-1}|=\sum_{j=1}^{m}|f(z_{k+j-1})/f'(z_{k+j-1})|\leqslant\frac{1}{L}\sum_{j=1}^{m}|f(z_{k+j-1})|$$
$$\leqslant\frac{1}{L}\sum_{j=1}^{m}\left(\frac{1}{c}\right)^{k+j-1}|f(z_0)|<|f(z_0)|\Big/L\cdot c^k\left(1-\frac{1}{c}\right).$$

Thus the sequence $\{z_k\}$ converges to some z^*, and $f(z^*)=\lim\limits_{k\to\infty} f(z_k)=0$ by the continuity of f. The theorem is then proved.

Corollary 3.5. If $|f(z)|<\rho_f/3K+1$, then z is an approximate zero of f.

Proof. In Proposition 3.4, we take $c=2$.

Let P_n be the space of the monic complex polynomials of degree n. Hence, P_n can be identified with complex Cartesian Space $C^n=\{(a_0, a_1, \cdots, a_{n-1})=a \mid a_i\in C\}$, or we can write $f\in C^n$ whenever $f(z)=z^n+\sum_{i=0}^{n-1} a_i z^i$.

We denote $P_n(R)=\{f\in P_n \mid |a_i|<R,\ i=0,\ 1,\ \cdots,\ n-1\}$. Then the volume of $P_n(R)$, or $\mathrm{vol}P_n(R)$, is $(\pi R^2)^n$, where "vol" is the Lebesque measure on $C^n=R^{2n}$ for P_n.

Let
$$P_n=\bigcup_{R>0} P_n(R),$$
$$W_n=\{f\in P_n \mid f(\theta)=0 \text{ for some } \theta \text{ with } f'(\theta)=0\},$$
$$U_\lambda(f_0)=\{f\in P_n \mid |f(0)-f_0(0)|<\lambda \text{ and } f^{(k)}(0)=f_0^{(k)}(0),\ \forall k>0\},$$
$$U_\lambda(W_n)=\bigcup_{f_0\in W_n} U_\lambda(f_0).$$

Proposition 3.6. $$\frac{\mathrm{vol}(U_\rho(W_n)\cap P_n(R))}{\mathrm{vol}\,P_n(R)}\leqslant\frac{(n-1)\rho^2}{R^2}.$$

Proof. See [5], Theorem 4A, Section 4, Part II. (Note that Smale wrote $n\rho^2/R^2$ in the theorem by a neglect.)

Lemma 3.7. *$f\in U_\rho(W_n)$ if and only if $\rho_f<\rho$.*

Proof. See [5], Lemma 2, Section 5, Part II.

Theorem 3.8. *Suppose $0<\mu<1$, and n are given. Let $\sigma=(\mu/n)^{1/2}$. Then for any $f\in P_n(R)$, using Kuhn's algorithm, with probability at least $1-\mu$, we can find n approximate zeros for zeros of f in at most s steps, where*
$$s=\Big\lceil 8\lceil 3(1+\sqrt{2})n/4\pi\rceil\,(1+\lceil 3\,(1+\sqrt{2})n/4\pi\rceil)$$
$$+5\pi\cdot\min\Big\{M,\ R+1+\sqrt{2}\Big(1+\frac{3n}{\pi}\Big)\Big\}^2+28\,nd\pi\Big(1+\frac{3n}{2\pi}\Big)^2\Big\rceil,$$
$$\tilde{M}=\max\Big\{3\sqrt{2}\,(2+\pi)n/4\pi,\ \frac{n}{n-1}R+1\Big\}+\sqrt{2},$$
$$d=\Big\lceil\ \log_2\sqrt{2}\Big(1+\frac{3n}{2\pi}\Big)\Big/\varepsilon\Big\rceil$$
$$\varepsilon=\sigma(1-N)^2/13[1-(n+1)N^n+nN^{n+1}]$$
$$=\mu^{1/2}(1-N)^2/13n^{1/2}[1-(n+1)N^n+nN^{n+1}],$$
$$N=\min\Big\{\tilde{M},\ R+1+\sqrt{2}\Big(1+\frac{3n}{\pi}\Big)\Big\}.$$

Proof. If $\sigma R\leqslant\rho_f$ and z (a vertex of a c.l. triangle) satisfies $|z-\xi|<\varepsilon$ for some zero ξ of f, then
$$|f(z)|=|f(z)-f(\xi)|=\Big|\sum_{j=0}^{n}a_j(z^j-\xi^j)\Big|\leqslant|z-\xi|\sum_{j=1}^{n}|a_j(z^{j-1}+z^{j-2}\xi+\cdots+\xi^{j-1})|$$
$$\leqslant\varepsilon\sum_{j=1}^{n}jN^{j-1}=\varepsilon R\Big(\sum_{j=1}^{n}x^j\Big)'_N=\varepsilon R\Big(\frac{x-x^{n+1}}{1-x}\Big)'_N$$
$$=\varepsilon R\,\frac{1-(n+1)N^n+nN^{n+1}}{(1-N)^2}=\frac{\sigma R}{13}\leqslant\frac{\rho_f}{13}\leqslant\frac{\rho_f}{3K+1}.$$

By Corollary 3.5, z is an approximate zero of f.

From Theorem 2.12, we know that, for any polynomial with $\rho_f\geqslant\sigma R$, we can find n approximate zeros of f in at most s steps.

Finally, according to Proposition 3.6 and Lemma 3.7, we obtain

$$\frac{\operatorname{vol}\{f\in P_n(R)\mid \rho_f<\sigma R\}}{\operatorname{vol}P_n(R)}=\frac{\operatorname{vol}U_{\sigma R}(w_n)\cap P_n(R)}{\operatorname{vol}P_n(R)}\leqslant\frac{(n-1)(\sigma R)^2}{R^2}=(n-1)\frac{\mu}{n}<\mu.$$

§ 4. Probabilistic Estimation of Monotonicity of Kuhn's Algorithm

We proved in [7] the following results.

Assumption 4.1. Let ξ_j be a simple zero of $f(z)$, $1\leqslant j\leqslant n$, and both $\eta\geqslant\max\limits_{i\neq j}|\xi_j-\xi_i|$ and $0<\zeta\leqslant\min\limits_{i\neq j}|\xi_j-\xi_i|$ are given. Suppose $r_j=(\eta^{n-1}+\zeta^{n-1}/13)^{\frac{1}{n-1}}-\eta$. By the assumption, $r_j>0$. Moreover, let $\sigma_j=\{z\mid |z-\xi_j|<r_j\}$.

Lemma 4.2. *For any z', $z''\in\sigma_j$, we have*

$$\left|\arg\frac{f(z')}{f(z'')}\Big/\frac{z'-\xi_j}{z''-\xi_j}\right|<\frac{\pi}{12}$$

and

$$\left|\arg\frac{f(z')}{(z'-\xi_j)\prod\limits_{i\neq j}(\xi_j-\xi_i)}\right|<\frac{\pi}{24}.$$

Lemma 4.3. *There is no negative completely labelled isosceles right triangle in σ_j.*

Proposition 4.4. Let $D=\lceil\log_2(2\sqrt{2}/r_j)\rceil$. Then for any integer $d\geqslant D$, there is one and only one c.l. triangle in $\sigma_j\subset C_d$ triangulated, and it is a positive c.l. triangle.

Proposition 4.5. Suppose that ξ_j is a simple zero of $f(z)$, $1\leqslant j\leqslant n$. Let $\eta\geqslant\max\limits_{i\neq j}|\xi_j-\xi_i|$, $0<\zeta\leqslant\min\limits_{i\neq j}|\xi_j-\xi_i|$, $r_j=(\eta^{n-1}+\zeta^{n-1}/13)^{\frac{1}{n-1}}-\eta$, and $D=\lceil\log_2(2\sqrt{2}/r_j)\rceil$. Then above the plane C_D, the computation approximating ξ_j is monotonously rising.

Proposition 4.6. Let $D=\lceil\log_2(2\sqrt{2}/r_j)\rceil$. Then since C_D, with every five pivots the computation approximating ξ_j arrives at a higher level.

Proof. Suppose that the computation approximating ξ_j has arrived at a (positive) c.l. triangle in C_d, $d\geqslant D$. Suppose, without loss of generality, that the c.l. triangle is $\{A_1, B_2, C_3\}$ as shown in Fig. 4.1. Therefore, the point we are going to calculate is D.

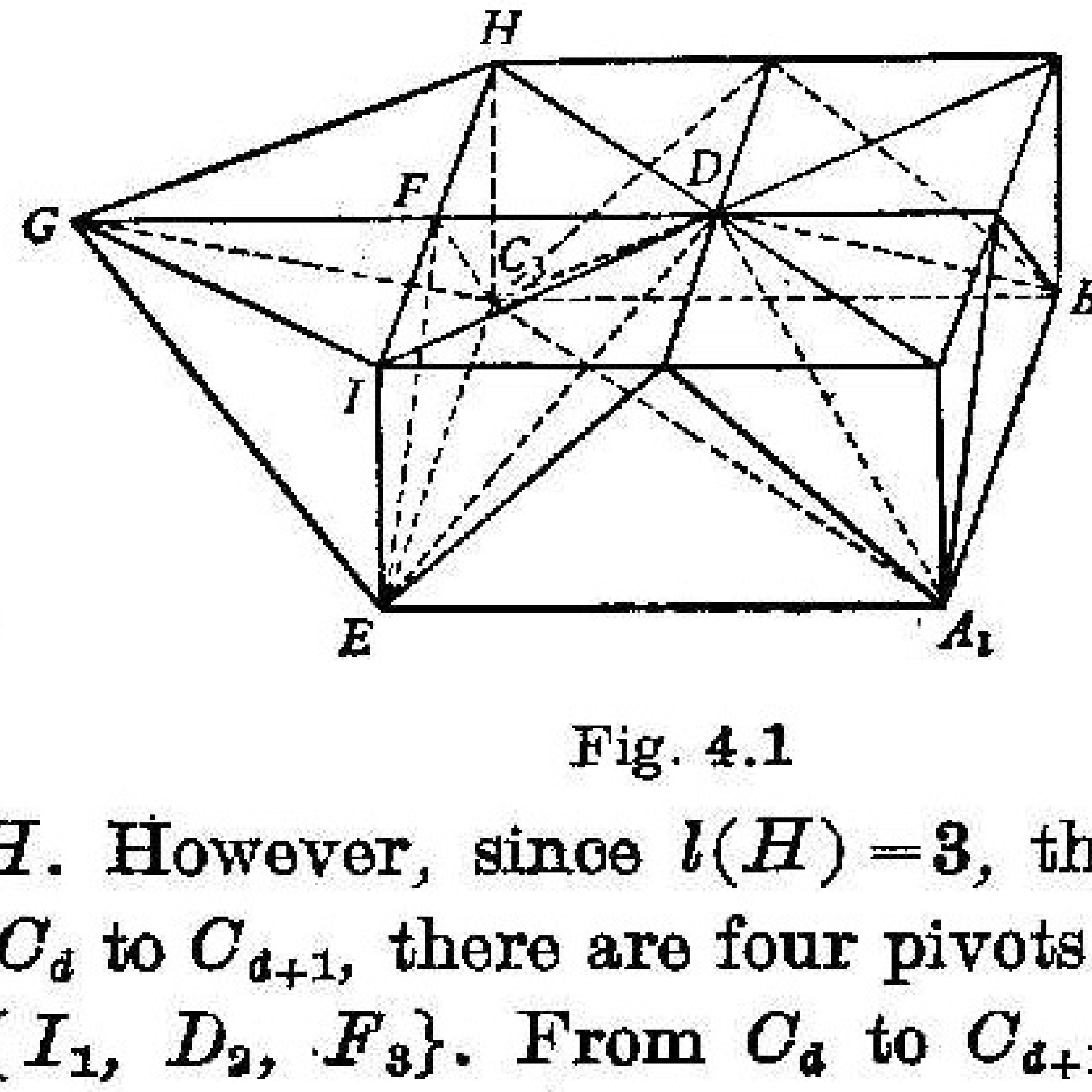

Fig. 4.1

If $l(D)=2$, we should consider E next. With Lemma 3.3, $l(E)\neq 2$. By the symmetry, suppose $l(E)=1$. Then $\{E_1, D_2, C_3\}$ determines F.

If $l(F)=1$, then $\{F_1, D_2, C_3\}$ determines H. However, since $l(H)=3$, the computation gets into C_{d+1} by $\{F_1, D_2, H_3\}$. From C_d to C_{d+1}, there are four pivots.

If $l(F)=3$, the computation gets into C_{d+1} by $\{I_1, D_2, F_3\}$. From C_d to C_{d+1}, there are still four pivots.

If $l(F)=2$, then $\{E_1, F_2, C_3\}$ determines G. With Lemma 3.3, $l(G)\neq 2$; otherwise $\{I_1, H_3, G_2\}$ is a negative o.l. isosceles right triangle. If $l(G)=1$, we have $\{G_1, F_2, H_3\}\subset C_{d+1}$; if $l(G)=3$, we have $\{I_1, F_2, G_3\}\subset C_{d+1}$. In both cases there are five pivots from C_d to C_{d+1}.

Therefore, if $l(D)=2$, with at most five pivots, the computation approximating ξ_i must get at a higher level.

The proofs for the cases $l(D)=1$ and $l(D)=3$ are similar and thus omitted.

Proposition 4.7. (Bierberbach–Koebe Theorem) Let $g:D\to C$ be an analytic function on the unit disc, $1-1$ for $|z|<1$, and $g(0)=0$, $g'(0)=1$. Then

$$\left\{w \,\middle|\, |w|<\frac{1}{4}\right\}\subseteq g(D).$$

Proof. See [1].

Proposition 4.8. Let $g:B(z, r)\to C$ be analytic, $1-1$, and $g'(z)\neq 0$. Then $\{w \mid |w-g(z)|<r|g'(z)|/4\}\subseteq g(B(z, r))$.

Proof. For $x\in D$, let $g(x)=\dfrac{g(z+rx)-g(z)}{rg'(z)}$. Obviously, $g(0)=0$, $g'(0)=1$. Applying the Bierberbach–Koebe Theorem to g, we obtain the proposition.

Lemma 4.9. *Suppose $f\in P_n(R)$, $f\overline{\in} U_\lambda(w_n)$, $f(\xi_i)=0$, $|\xi_i|\leqslant 1$. Then there is a branch f_i^{-1} of f^{-1} defined on $B(0, \lambda)$ with $f_i^{-1}(0)=\xi_i$. Furthermore, for $0<\alpha\leqslant 1$,*

$$B(\xi_i, \alpha\lambda/2n(2+(n+1)R))\subseteq f_i^{-1}(B(0, \alpha\lambda)).$$

Proof. Note that $f\overline{\in} U_\lambda(w_n)$ implies $|f(\theta)|>\lambda$ if $f'(\theta)=0$. Thus there is a branch f_i^{-1} of f defined on $B(0, \alpha\lambda)$, $0<\alpha\leqslant 1$, with $f_i^{-1}(0)=\xi_i$. Applying Proposition 4.8 to f_i^{-1}, we obtain

$$B(\xi_i, \alpha\lambda/4|f'(\xi_i)|)\subseteq f_i^{-1}(B(0, \alpha\lambda)).$$

Since $f\in P_n(R)$, $|\xi_i|\leqslant 1$, we have

$$|f'(\xi_i)|=\left|n+\sum_{j=0}^{n-1} ja_j\xi_i^{j-1}\right|\leqslant n+R\sum_{j=0}^{n-1} j=n\left(1+(n+1)\frac{R}{2}\right)$$

and

$$B(\xi_i, \alpha\lambda/2n(2+(n+1)R))\subseteq B(\xi_i, \alpha\lambda/4|f'(\xi_i)|)\subseteq f_i^{-1}(B(0, \alpha\lambda)).$$

Proposition 4.10. Let $f(z)=\prod_{j=1}^{n}(z-\xi_j)\in P_n(R)$, $f'(z)=\prod_{j=1}^{n-1}(z-\theta_j)$. If $f\overline{\in} U_\lambda(W_n)$, $|\xi_i|\leqslant 1$, then

$$\min_{i\neq j}|\xi_i-\xi_j|>\lambda/2n(2+(n+1)R),$$
$$\min_{j}|\xi_i-\theta_j|>\lambda/2n(2+(n+1)R).$$

Proof. This follows immediately from Lemma 4.9 by setting $\alpha=1$, since the image of f_i^{-1} contains only the zero ξ_i of f and no critical point of f[4].

Lemma 4.11. *Let $f\in P_n(R)$, $n\geqslant 2$, $f(\xi_1)=0$, $f(\xi_2)=0$, $|\xi_1|\geqslant 1$, $|\xi_2|\geqslant 1$, $\xi_1\neq\xi_2$. If $f(z)=z^n+\sum_{i=0}^{n-1}a_iz^i$, then there is a polynomial $g(z)=z^n+\sum_{i=0}^{n-1}b_iz^i$ with $g(\xi_1)=0$, $g'(\xi_1)=0$ and*

$$|a_i-b_i|<|\xi_1-\xi_2|(R(n-1)(n-2)^2+R+1)\quad if\quad i=n-2, n-1,$$
$$a_i=b_i\quad if\quad i\neq n-2, n-1.$$

Proof. Let $f(z)=(z-\xi_1)(z-\xi_2)h(z)$ where $h(z)=z^{n-2}+\sum_{i=0}^{n-3}c_iz^i$, $\beta=(\xi_1-\xi_2)\times h(\xi_1)/\xi_1^{n-2}$ and $g(z)=f(z)-\beta(z-\xi_1)z^{n-2}=(z-\xi_1)[(z-\xi_2)h(z)-\beta z^{n-2}]=z^n+\sum_{i=0}^{n-1}b_iz^i$.

Factorizing $(z-\xi_1)$ from g yields $(z-\xi_2)h(z)-\beta z^{n-2}$ which has ξ_1 as a zero. Thus $g(\xi_1)=0$, $g'(\xi_1)=0$. Note that $|b_{n-1}-a_{n-1}|=|\beta|$, $|b_{n-2}-a_{n-2}|=|\beta\xi_1|$, $a_i=b_i$ if $i\neq n-2, n-1$. Since $|\xi_1|\geqslant 1$, $|\beta|\leqslant|\beta\xi_1|$, it suffices to show that $|\beta\xi_1|\leqslant |\xi_1-\xi_2|\,(R(n-1)(n-2)^2+R+1)$.

Let $(z-\xi_2)\,h(z)=z^{n-1}+\sum_{i=0}^{n-2}\alpha_i z^i$. Then $z^n+\sum_{i=0}^{n-1}a_i z^i=(z-\xi_1)(z-\xi_2)\,h(z)=z^n+(\alpha_{n-2}-\xi_1)z^{n-1}+\sum_{i=1}^{n-2}(\alpha_{i-1}-\xi_1\alpha_i)z^i-\xi_1\alpha_0$, $a_0=-\xi_1\alpha_0$, $|\alpha_0|=|a_0/\xi_1|$, and $a_i=\alpha_{i-1}-\xi_1\alpha_i$, $1\leqslant i\leqslant n-1\,(\alpha_{n-1}=1)$, $|\alpha_i|=\left|\frac{\alpha_{i-1}-a_i}{\xi_1}\right|\leqslant\left|\frac{a_i}{\xi_1}\right|+\left|\frac{\alpha_{i-1}}{\xi_1}\right|\leqslant\cdots\leqslant\sum_{j=0}^{i}|a_j/\xi_1^{i+1-j}|$.

In exactly the same manner, $|c_i|\leqslant\sum_{j=0}^{i}|a_j/\xi_2^{i+1-j}|$. Thus,

$$
\begin{aligned}
|\beta\xi_1| &= |(\xi_1-\xi_2)h(\xi_1)/\xi_1^{n-3}|\\
&\leqslant\frac{|\xi_1-\xi_2|}{|\xi_1|^{n-3}}\Big(|\xi_1|^{n-2}+\sum_{i=0}^{n-3}|\xi_1|\sum_{j=0}^{i}|\xi_2|^{j-(i+1)}\sum_{k=0}^{j}|a_k|\cdot|\xi_1|^{k-(j+1)}\\
&\leqslant|\xi_1-\xi_2|\Big[|\xi_1|+R\Big(\sum_{i=0}^{n-3}\sum_{j=0}^{i}|\xi_2|^{j-(i+1)}\sum_{k=0}^{j}|\xi_1|^{i+k-(j+1)-(n-3)}\Big)\Big]\\
&\leqslant|\xi_1-\xi_2|\Big[|\xi_1|+R\sum_{i=0}^{n-3}\sum_{j=0}^{i}\sum_{k=0}^{j}1\Big]=|\xi_1-\xi_2|\Big[|\xi_1|+R\sum_{i=0}^{n-3}\sum_{j=0}^{i}(j+1)\Big]\\
&=|\xi_1-\xi_2|\Big[|\xi_1|+R\sum_{i=0}^{n-3}\frac{(i+1)(i+2)}{2}\Big]=|\xi_1-\xi_2|\Big[|\xi_1|+R\,\frac{(n-2)(n-1)n}{6}\Big]\\
&\leqslant|\xi_1-\xi_2|\,(|\xi_1|+R(n-1)(n-2)^2)
\end{aligned}
$$

(Note $|\xi_1|\geqslant 1$, $|\xi_2|\geqslant 1$, $j-(i+1)<0$, $i+k-(j+1)-(n-3)<0$). We have noted that the zeros of polynomials in $P_n(R)$ lie in $B(0, R+1)$. Hence $|\xi_1|<R+1$, and we conclude the proof.

Let $L_\lambda(W_n)=\bigcup_{f_0\in W_n}L_\lambda(f_0)$, where $L_\lambda(f_0)=\Big\{f\in P_n\,\Big|\,\Big|\frac{f^{(k)}(0)}{k!}-\frac{f_0^{(k)}(0)}{k!}\Big|\leqslant\lambda$ if $k=n-2,\ n-1$ and $f^{(k)}(0)=f_0^{(k)}(0)$ if $k<n-2\Big\}$.

Throughout the rest of this section let

$$\rho_1=\rho(2n(2+(n+1)R)),$$
$$\rho_2=\rho(R(n-1)(n-2)^2+R+1).$$

Proposition 4.12. If $f(z)=\prod_{i=1}^{n}(z-\xi_i)\in P_n(R)$, $f\bar{\in} U_{\rho_1}(W_n)\cup\bar{L}_{\rho_2}(W_n)$. Then $|\xi_i-\xi_j|>\rho$ for $i\neq j$.

Proof. If $|\xi_i|\leqslant 1$, this follows from Proposition 4.10. If $|\xi_i|\geqslant 1$, $|\xi_j|\geqslant 1$, the result follows from Lemma 4.11 and the definition of $L_{\rho_2}(W_2)$. In fact, if $|\xi_i-\xi_j|\leqslant\rho$, then there is $g\in W_n$ such that

$$
\begin{aligned}
|a_l-b_l|=\left|\frac{f^{(l)}(0)}{l!}-\frac{g^{(l)}(0)}{l!}\right|&<|\xi_i-\xi_j|\,(R(n-1)(n-2)^2+R+1)\\
&\leqslant\rho\,(R(n-1)(n-2)^2+R+1)=\rho_2
\end{aligned}
$$

if $l=n-2,\ n-1$; $a_l=b_l$ if $l\neq n-2,\ n-1$. So, $f\in L_{\rho_2}(W_n)$. This contradicts the hypothesis.

With P_n as C^n under the mapping $z^n+\sum_{i=0}^{n-1}a_i j^i\to(a_0, a_1, \cdots, a_{n-1})$, W_n is the zero set of $R(f, f')$, the resultant of f[8]. $R(f, f')$ is a polynomial of degree $n+1$ in $(a_{n-2},$

$q_{n-1})$, Similar to S. Smale's Theorem 40, Section 4, Part II[5], we obtain

Proposition 4.13. $\dfrac{\operatorname{vol}[L_{\rho_2}(W_n)\cap P_n(R)]}{\operatorname{vol} P_n(R)}\leqslant 32\left(\dfrac{\rho_2}{R}\right)^2(n+1)$.

Instead of having the factor 32, S. Smale had 4. This is because his Lemmas 4 and 5 should read

$$N_\rho(\gamma)\cap P_n(R)\subset V_{\sqrt{2}\rho}\cup T_{\sqrt{2}\rho}(\gamma\cap P_n(R+\sqrt{2}\rho))\cup\{\text{set of measure zero}\}$$

and

$$\operatorname{vol} T_\rho(\gamma)\leqslant\pi\left[\frac{\rho^2}{2}\operatorname{area}\gamma+\frac{\rho^4}{12}\int_\gamma K\right].$$

Proposition 4.14. Let $0<\mu<1$, $Y_{\mu,n}(R)=P_n(R)\cap(U_{\rho_1}(W_n)\cup L_{\rho_2}(W_n))$, where $\rho_1=\rho(2n(2+(n+1)R)$, $\rho_2=\rho(R(n-1)(n-2)^2+R+1)$, $\rho=\mu^{1/2}/F(R,n)$, and $F(R,n)=\dfrac{1}{R}[32(n+2)(R(n-1)(n-2)^2+R+1)^2+(n-1)(2n(2+(n+1)R))^2]^{1/2}$.

Thus we have

$$\frac{\operatorname{vol} Y_{\mu,n}(R)}{\operatorname{vol} P_n(R)}\leqslant\mu$$

such that if $f(z)=\prod_{i=1}^n(z-\xi_i)\in P_n(R)$, $f(z)\overline{\in} Y_{\mu,n}(R)$, then

$$\min_{i\neq j}|\xi_i-\xi_j|>\rho.$$

Proof. According to Propositions 3.6 *and* 4.13, *we obtain*

$$\frac{\operatorname{vol} Y_{\mu,n}(R)}{\operatorname{vol} P_n(R)}\leqslant\frac{\operatorname{vol}[U_{\rho_1}(W_n)\cap P_n(R)]}{\operatorname{vol} P_n(R)}+\frac{\operatorname{vol}[L_{\rho_2}(W_n)\cap P_n(R)]}{\operatorname{vol} P_n(R)}$$

$$\leqslant\frac{(n-1)\rho_1^2}{R^2}+32\left(\frac{\rho_2^2}{R}\right)(n+1)$$

$$=\frac{\rho^2}{R^2}[32(n+1)(R(n-1)(n-2)^2+R+1)^2+(n-1)(2n(2+(n+1)R))^2]$$

$$=(\mu^{1/2}/F(R,n))^2\cdot F^2(R,n)=\mu.$$

Finally, if $f\in P_n(R)$, $f\overline{\in} Y_{\mu,n}(R)$, then Proposition 4.12 implies $\min_{i\neq j}|\xi_i-\xi_j|>\rho$.

Theorem 4.15. *Suppose* $f(z)=\prod_{i=1}^n(z-\xi_i)\in P_n(R)$, $0<\mu<1$,

$$D=\left\lceil\log_2\frac{2\sqrt{2}}{[(2R+2)^{n-1}+\rho^{n-1}/13]^{1/(n-1)}-(2R+2)}\right\rceil,$$

$$K=\Big\lceil 8\lceil 3(1+\sqrt{2})n/4\pi\rceil(1+\lceil 3(1+\sqrt{2})n/4\pi\rceil)$$

$$+5\pi\cdot\min\left\{\widetilde{M},\ R+1+\sqrt{2}\left(1+\frac{3n}{\pi}\right)\right\}^2+28nD\pi\left(1+\frac{3n}{2\pi}\right)^2\Big\rceil,$$

$$L=\left(\frac{1}{2}\right)^{D-1/2}\left(1+\frac{3n}{2\pi}\right),$$

with probability of failure no greater than μ. *There exist positive integers* $K_1,\cdots,K_n$, *where* $\sum_{i=1}^n K_i\leqslant K$. *Such that if* $k\geqslant K_i$, *then* z_{ik} *are monotonic and*

(1) $|z_{ik}-\xi_i|<\left(\dfrac{1}{2}\right)^{\frac{k-K_i}{5}}L$;

(2) $|f(z_{ik})|<\left(\frac{1}{2}\right)^{\frac{k-K_i}{5}}L(2R+L+2)^{n-1}$.

Proof. Using the result of Proposition 4.14, for $f\in P_n(R)$ and $f\notin Y_{\mu,n}(R)$, $\operatorname{vol} Y_{\mu,n}(R)/\operatorname{vol} P_n(R)\leqslant\mu$ is valid. Moreover, in Proposition 4.5, take $\eta=2R+2$, $\zeta=\rho$. Then $\eta=2R+2>\max_{i\neq j}|\xi_i-\xi_j|$, $0<\zeta=\rho<\min_{i\neq j}|\xi_i-\xi_j|$. Thus, we obtain

$$D=\left\lceil\log_2\frac{2\sqrt{2}}{(2R+2)^{n-1}+\rho^{n-1}/13^{1/(n-1)}-(2R+2)}\right\rceil.$$

By Proposition 2.11, there are positive integers $K_1, \cdots, K_n$, where $\sum_{i=1}^{n}K_i\leqslant K$, such that if $k\geqslant K_i$, then z_{ik} are monotonic. Proposition 4.6 shows that, for the sequence $\{(z_{ik}, d_k)\}$, $k-K_i\leqslant 5(d_k-D)$, i.e. $d_k\geqslant\frac{k-K_i}{5}+D$. Therefore, according to Proposition 2.5 we obtain

(1) $$|z_{ik}-\xi_i|\leqslant\frac{\sqrt{2}}{2^{d_k}}\left(1+\frac{3n}{2\pi}\right)\leqslant\frac{\sqrt{2}\left(1+\frac{3n}{2\pi}\right)}{2^{\frac{k-K_i}{5}+D}}=\left(\frac{1}{2}\right)^{\frac{k-K_i}{5}}L.$$

(2) Let $\hat{f}(z)=(z-\xi_i)\prod_{j\neq i}(z-\xi_j)$.

If $k\geqslant K_i$, then we have

$$|f(z_{ik})|=|z_{ik}-\xi_i|\prod_{j\neq i}|z_{ik}-\xi_j|\leqslant\left(\frac{1}{2}\right)^{\frac{k-K_i}{5}}L\cdot\prod_{j\neq i}(|z_{ik}-\xi_i|+|\xi_i|+|\xi_j|)$$
$$\leqslant\left(\frac{1}{2}\right)^{\frac{k-K_i}{5}}L\prod_{j\neq i}(L+2R+2)=\left(\frac{1}{2}\right)^{\frac{k-K_i}{5}}L(2R+L+2)^{n-1}.$$

References

[1] E. Hille, *Analytic Function Theory*, II, Ginn, Boston 1962.
[2] H. W. Kuhn, Finding roots of polynomials by pivoting, in *Fixed Points: Algorithms and Application*, Academic Press, 1977, 11—40.
[3] J. Renegar, On the cost of approximating all roots of a complex polynomial, to be published, *Math. Programming*.
[4] M. Shub, S. Smale, Computational complexity: On the geometry of polynomials and a theory of cost: Part I, Preprint.
[5] S. Smale, The fundamental theorem of algebra and complexity theory, *Bulletin AMS*, **4** (1981), 1—36.
[6] 王则柯，徐森林，逼近零点与计算复杂性理论，中国科学，1, 1984, 8—15.
[7] Z. K. Wang, S. L. Xu, On the monotonicity of Kuhn's algorithm approximating a simple root of polynomials, to be published.
[8] B. Vander Waerden, Modern Algebram, Vol. I, Ungar, New York, 1953.