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INTERPOLATION FORMULAS OF INTERMEDIATE QUOTIENTS FOR DISCRETE FUNCTIONS WITH SEVERAL INDICES*

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§ 1

The imbedding relations and the interpolation formulas of Sobolev spaces are very important and powerful tools in the study of the linear and nonlinear theory of partial differential equations and systems. The finite difference method is commonly used in pratical and theoretical study of various problems of partial differential equations and systems of different type. Many authors are attracted to establish the imbedding relations and the interpolation formulas for the discrete functions. These are undoubtedly important in pratical and theoretical use of finite difference method. As in [1] the Sobolev inequalities are given for the discrete function on number axis. In [2] the difference imbedding theorems of the Sobolev space with weight are considered. In [3—5] for the discrete functions on finite segment the interpolation formulas are established. These interpolation formulas can be used to establish the convergence and stability of the finite difference schemes for the various problems of the nonlinear partial differential systems of different types. And these can also be used to construct the generalized and weak global solutions for the nonlinear systems of partial differential equations[3—9]. Now in the present note we want to consider the interpolation formulas of intermediate quotients for the discrete functions with several indices.

§ 2

Suppose that the finite interval $[0, l]$ is divided into the small segment grids by the points $x_j=jh$ $(j=0, 1, \cdots, J)$, where $Jh=l$, J is an integer and h is the stepsize. The discrete function $u_h=\{u_j\}$ $(j=0, 1, \cdots, J)$ is defined on the grid points x_j $(j=0, 1, \cdots, J)$. Let us denote $\Delta_+u_j=u_{j+1}-u_j$, $\Delta_-u_j=u_j-u_{j-1}$. For the norms of the discrete function $u_h=\{u_j\}$ $(j=0, 1, \cdots, J)$ and its difference quotients

$$\delta^k u_h=\left\{\frac{\Delta_+^k u_j}{h^k}\right\} \quad (j=0, 1, \cdots, J-k)$$

of order $k\geqslant 0$, we take the notations

$$\begin{aligned} &\|\delta^k u_h\|_p=\left(\sum_{j=0}^{J-k}\left|\frac{\Delta_+^k u_j}{h^k}\right|^p h\right)^{\frac{1}{p}}, \quad 1\leqslant p<\infty; \\ &\|\delta^k u_h\|_\infty=\max_{j=0,1,\cdots,J-k}\left|\frac{\Delta_+^k u_j}{h^k}\right|, \quad k\geqslant 0. \end{aligned} \tag{1}$$

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The interpolation formulas of the intermediate quotients for the discrete functions defined on finite interval $[0, l]$ can be stated as follows[3-5].

Theorem 1. *For any discret function $u_h=\{u_j\}$ $(j=0, 1, \cdots, J)$ defined on the finite interval $[0, l]$, there is the interpolation formula*

$$\|\delta^k u_h\|_p \leqslant K_1 \|u_h\|_2^{1-\frac{k+\frac{1}{2}-\frac{1}{p}}{n}} \left\{\|\delta^n u_h\|_2 + \frac{\|u_h\|_2}{l^n}\right\}^{\frac{k+\frac{1}{2}-\frac{1}{p}}{n}}, \tag{2}$$

where $2\leqslant p\leqslant\infty$, $0\leqslant k<n$ and K_1 is a constant independent of h, l and u_h.

§ 3

Let us introduce some notations for discrete functions with several indices.

Take a m-dimensional rectangular domain $Q^m=\{x \mid x=(x_1, \cdots, x_m) \mid 0\leqslant x_i\leqslant l_i, i=1, 2, \cdots, m\}$ in the m-dimensional Euclidean space $\mathbf{R}^m$, where $m\geqslant 1$ and $l_i>0$, $i=1, 2, \cdots, m$. For the rectangular domain Q^m we define $l=\min\limits_{i=1,2,\cdots,m}\{l_i\}>0$. Divide the rectangular domain Q^m into small grids by the parallel hyperplanes $x_i=x_{ij_i}$ $(j_i=0, 1, \cdots, J_i;\ i=1, 2, \cdots, m)$ where $x_{ij_i}=j_ih_i$ $(j_i=0, 1, \cdots, J_i;\ i=1, 2, \cdots, m)$ and $J_ih_i=l_i$ $(i=1, 2, \cdots, m)$. The set of the grids points $(x_{1j_1}, \cdots, x_{mj_m})$ $(j_i=0, 1, \cdots, J_i;\ i=1, 2, \cdots, m)$ is denoted by $Q^m_\Delta=\{(x_{1j_1}, \cdots, x_{mj_m})\}$. We use the notation $u_\Delta=\{u_{j_1,\cdots,j_m}\}$ to denote the discrete function with m indices defined on the grid domain Q^m_Δ.

Usually we have the abbreviations $\overset{+}{\Delta}_i u_{j_1,\cdots,j_i,\cdots,j_m}=(u_{j_1,\cdots,j_i+1,\cdots,j_m}-u_{j_1,\cdots,j_i,\cdots,j_m})/h_i$ and $\overline{\Delta}_i u_{j_1,\cdots,j_i,\cdots,j_m}=(u_{j_1,\cdots,j_i,\cdots,j_m}-u_{j_1,\cdots,j_i-1,\cdots,j_m})/h_i$. And then furthermore we adopt the notations $\delta^\alpha u_\Delta=\left\{\frac{\overset{+}{\Delta}{}^\alpha u_{j_1,\cdots,j_m}}{h^\alpha} \,\middle|\, j_i=0, 1, \cdots, J_i-\alpha_i;\ i=1, 2, \cdots, m\right\}$, where $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_m)$ is a m-index and then $\delta^\alpha=\delta_1^{\alpha_1}\cdots\delta_m^{\alpha_m}$, $\overset{+}{\Delta}{}^\alpha=\overset{+}{\Delta}{}_1^{\alpha_1}\cdots\overset{+}{\Delta}{}_m^{\alpha_m}$, $h^\alpha=h_1^{\alpha_1}\cdots h_m^{\alpha_m}$, $|\alpha|=\alpha_1+\cdots+\alpha_m\geqslant 0$ and α_i $(i=1, 2, \cdots, m)$ are integers. Similarly we have

$$\delta_s^\beta u_{j_1,\cdots,j_s,\cdot}=\left\{\frac{\overset{+}{\Delta}{}^\beta u_{j_1,\cdots,j_s,j_{s+1},\cdots,j_m}}{h^\beta} \,\middle|\, j_i=0, 1, \cdots, J_i-\beta_i;\ i=s+1, \cdots, m\right\},$$

where $\beta=(\beta_{s+1}, \cdots, \beta_m)$ and $s=0, 1, \cdots$, m.

Now we adopt the following notations of the norms for the discrete functions and their difference quotients:

$$\begin{aligned} \|\delta^\alpha u_\Delta\|_{L_p(Q^m_\Delta)} &= \left(\sum_{i=1}^m \sum_{j_i=0}^{J_i-\alpha_i} \left|\frac{\overset{+}{\Delta}{}^\alpha u_{j_1,\cdots,j_m}}{h^\alpha}\right|^p h_1h_2\cdots h_m\right)^{\frac{1}{p}}, \quad 1\leqslant p<\infty; \\ \|\delta^\alpha u_\Delta\|_{L_\infty(Q^m_\Delta)} &= \max_{\substack{j_i=0,1,\cdots,J_i-\alpha_i\\ i=1,2,\cdots,m}} \left|\frac{\overset{+}{\Delta}{}^\alpha u_{j_1,\cdots,j_m}}{h^\alpha}\right| \end{aligned} \tag{3}$$

and

$$\begin{aligned} \|\delta_s^\beta u_{j_1,\cdots,j_s,\cdot}\|_{L_p(Q^{m-s}_\Delta)} &= \left(\sum_{i=s+1}^m \sum_{j_i=0}^{J_i-\beta_i} \left|\frac{\overset{+}{\Delta}{}^\beta u_{j_1,\cdots,j_m}}{h^\beta}\right|^p h_{s+1}\cdots h_m\right)^{\frac{1}{p}}, \quad 1\leqslant p<\infty; \\ \|\delta_s^\beta u_{j_1,\cdots,j_s,\cdot}\|_{L_\infty(Q^{m-s}_\Delta)} &= \max_{\substack{j_i=0,1,\cdots,J_i-\beta_i\\ i=s+1,\cdots,m}} \left|\frac{\overset{+}{\Delta}{}^\beta u_{j_1,\cdots,j_m}}{h^\beta}\right|, \quad s=0, 1, \cdots, m, \end{aligned} \tag{4}$$

where $\alpha=(\alpha_1, \cdots, \alpha_m)$, $\beta=(\beta_{s+1}, \cdots, \beta_m)$ $(s=0, 1, \cdots, m)$ and $|\alpha|\geqslant 0$, $|\beta|\geqslant 0$. Similarly we can define the seminorms and the norms for discrete functions in Sobolev spaces as follows:

$$\|u_\Delta\|^p_{L_p^{(k)}(Q_\Delta^m)} = \sum_{|\alpha|=k} \|\delta^\alpha u_\Delta\|^p_{L_p(Q_\Delta^m)} \tag{5}$$

and

$$\|u_\Delta\|^p_{W_p^{(k)}(Q_\Delta^m)} = \|u_\Delta\|^p_{L_p(Q_\Delta^m)} + \|u_\Delta\|^p_{L_p^{(k)}(Q_\Delta^m)}, \tag{6}$$

where $1 \le p < \infty$ and $k \ge 0$. As $p=2$, we have $W_2^{(k)}(Q_\Delta^m) = H^k(Q_\Delta^m)$.

§ 4

Now we turn to prove the interpolation formulas for the discrete functions with several indices.

Theorem 2. *For any discrete function $u_\Delta = \{u_{j_1,\dots,j_m}\}$ with m-index defined on the discrete rectangular domain $Q_\Delta^m = \{(x_{1j_1}, \dots, x_{mj_m})\}$ ($j_i = 0, 1, \dots, J_i$; $i = 1, 2, \dots, m$), there is the interpolation relation*

$$\|u_\Delta\|_{L_2^{(k)}(Q_\Delta^m)} \le K_2 \|u_\Delta\|^{1-\frac{k}{n}}_{L_2(Q_\Delta^m)} \left\{ \|u_\Delta\|_{L_2^{(n)}(Q_\Delta^m)} + \frac{\|u_\Delta\|_{L_2(Q_\Delta^m)}}{l^n} \right\}^{\frac{k}{n}}, \tag{7}$$

where $0 \le k \le n$, $m \ge 1$ and K_2 is a constant independent of the steplengths h_i ($i = 1, 2, \dots, m$), the minimum width l of Q^m and the discrete function u_Δ.

Proof. When $m=1$, the theorem reduces to the Theorem 1 for $p=2$.

Suppose that the theorem is valid for the discrete functions with not more than $(m-1)$ indices, where $m \ge 2$.

Let $\beta = (\beta_2, \dots, \beta_m)$ be a $(m-1)$-index ($m \ge 2$) and $|\beta| = \beta_2 + \dots + \beta_m = k - r \ge 0$. Then we have

$$J^2 \equiv \|\delta_1^r \delta^\beta u_\Delta\|^2_{L_2(Q_\Delta^m)} = \sum_{j_1=0}^{J_1-r} \sum_{i=2}^{m} \sum_{j_i=0}^{J_i-\beta_i} \left| \frac{\overset{+}{\Delta}{}_1^r \overset{+}{\Delta}{}^\beta u_{j_1,j_2,\dots,j_m}}{h_1^r h^\beta} \right|^2 h_1 h_2 \cdots h_m. \tag{8}$$

Regarding $\dfrac{\overset{+}{\Delta}{}_1^r \overset{+}{\Delta}{}^\beta u_{j_1,j_2,\dots,j_m}}{h_1^r h^\beta}$ as the discrete function with $(m-1)$ indices $j_2, \dots, j_m$, we have from induction assumption.

$$\sum_{i=2}^{m} \sum_{j_i=0}^{J_i-\beta_i} \left| \frac{\overset{+}{\Delta}{}_1^r \overset{+}{\Delta}{}^\beta u_{j_1,j_2,\dots,j_m}}{h_1^r h^\beta} \right|^2 h_2 \cdots h_m = \left\| \frac{\overset{+}{\Delta}{}_1^r}{h_1^r} \delta^\beta u_{j_1,\cdot} \right\|^2_{L_2(Q_\Delta^{m-1})}$$

$$\le K_2^2 \left\| \frac{\overset{+}{\Delta}{}_1^r u_{j_1,\cdot}}{h_1^r} \right\|^{2\left(1-\frac{k-r}{n-r}\right)}_{L_2(Q_\Delta^{m-1})} \left\{ \left\| \frac{\overset{+}{\Delta}{}_1^r u_{j_1,\cdot}}{h_1^r} \right\|_{L_2^{(n-r)}(Q_\Delta^{m-1})} + \frac{1}{\bar{l}^{n-r}} \left\| \frac{\overset{+}{\Delta}{}_1^r u_{j_1,\cdot}}{h_1^r} \right\|_{L_2(Q_\Delta^{m-1})} \right\}^{2\frac{k-r}{n-r}},$$

where $\bar{l} = \min\limits_{i=2,3,\dots,m} \{l_i\}$. Then (8) becomes

$$J^2 \le 4K_2^2 \sum_{j_1=0}^{J_1-r} \left\| \frac{\overset{+}{\Delta}{}_1^r u_{j_1,\cdot}}{h_1^r} \right\|^{2\left(1-\frac{k-r}{n-r}\right)}_{L_2(Q_\Delta^{m-1})} \left\| \frac{\overset{+}{\Delta}{}_1^r u_{j_1,\cdot}}{h_1^r} \right\|^{2\frac{k-r}{n-r}}_{L_2^{(n-r)}(Q_\Delta^{m-1})} h_1 + \frac{4K_2^2}{\bar{l}^{2(k-r)}} \sum_{j_1=0}^{J_1-r} \left\| \frac{\overset{+}{\Delta}{}_1^r u_{j_1,\cdot}}{h_1^r} \right\|^2_{L_2(Q_\Delta^{m-1})} h_1$$

$$\le 4K_2^2 \left\| \delta_1^r u_\Delta \right\|^{2\left(1-\frac{k-r}{n-k}\right)}_{L_2(Q_\Delta^m)} \left\| \delta_1^r u_\Delta \right\|^{2\frac{k-r}{n-r}}_{L_2^{(n-r)}(Q_\Delta^m)} + \frac{4K_2^2}{\bar{l}^{2(k-r)}} \left\| \delta_1^r u_\Delta \right\|^2_{L_2(Q_\Delta^m)}. \tag{9}$$

Here we have

$$\|\delta_1^r u_\Delta\|^2_{L_2^{(n-r)}(Q_\Delta^m)} \le \|u_\Delta\|^2_{L_2^{(n)}(Q_\Delta^m)}. \tag{10}$$

And also from Theorem 1, we have

$$\|\delta_1^r u_\Delta\|^2_{L_2(Q_\Delta^m)} = \sum_{i=2}^{m} \sum_{j_i=0}^{J_i} \|\delta_1^r u_{\cdot,j_2,\dots,j_m}\|^2_{L_2(Q_\Delta^1)} h_2 \cdots h_m$$

$$\le 4K_1^2 \sum_{i=2}^{m} \sum_{j_i=0}^{J_i} \|u_{\cdot,j_2,\dots,j_m}\|^{2\left(1-\frac{r}{n}\right)}_{L_2(Q_\Delta^1)} \|\delta_1^n u_{\cdot,j_2,\dots,j_m}\|^{\frac{2r}{n}}_{L_2(Q_\Delta^1)} h_2 \cdots h_m$$

$$+\frac{4K_1^2}{l_1^{2r}}\sum_{i=2}^{m}\sum_{j_i=0}^{J_i}\|u_{\cdot,j_2,\cdots,j_m}\|^2_{L_2(Q_\Delta^1)}h_2\cdots h_m$$

$$\leqslant 4K_1^2\left\|u_\Delta\right\|^{2(1-\frac{r}{n})}_{L_2(Q_\Delta^m)}\left\|u_\Delta\right\|^{\frac{2r}{n}}_{L_2^{(n)}(Q_\Delta^m)}+\frac{4K_1^2}{l_1^{2r}}\|u_\Delta\|^2_{L_2(Q_\Delta^m)}. \tag{11}$$

Substituting (10) and (11) into (9), we obtain by direct calculation

$$J\leqslant 4K_1K_2\left\|u_\Delta\right\|^{1-\frac{k}{n}}_{L_2(Q_\Delta^m)}\Bigg\{\|u_\Delta^1\|_{L_2^{(n)}(Q_\Delta^m)}+\left(\frac{\|u_\Delta\|_{L_2(Q_\Delta^m)}}{l_1^n}\right)^{\frac{r(n-k)}{k(n-r)}}\left\|u_\Delta\right\|^{\frac{n(k-r)}{k(n-r)}}_{L_2^{(n)}(Q_\Delta^m)}$$

$$+\left(\frac{\|u_\Delta\|_{L_2(Q_\Delta^m)}}{l^n}\right)^{1-\frac{r}{k}}\|u_\Delta\|^{\frac{r}{k}}_{L_2^{(n)}(Q_\Delta^m)}+\frac{\|u_\Delta\|_{L_2(Q_\Delta^m)}}{l^n\left(\frac{l_1}{l}\right)^{\frac{rn}{k}}}\Bigg\}^{\frac{k}{n}}.$$

Then we get the estimate

$$J\leqslant C_1\left\|u_\Delta\right\|^{1-\frac{k}{n}}_{L_2(Q_\Delta^m)}\left\{\|u_\Delta\|_{L_2^{(n)}(Q_\Delta^m)}+\frac{\|u_\Delta\|_{L_2(Q_\Delta^m)}}{l^n}\right\}^{\frac{k}{n}},$$

where $l=\min\{l_1, l\}$ and C_1 is a constant. This proves the interpolation formula (7).

§ 5

Theorem 3. *For any discrete function $u_\Delta=\{u_{j_1,\cdots,j_m}\}$ with m indices defined on $Q_\Delta^m=\{(x_{1j_1},\cdots x_{mj_m})\}$ $(j_i=0, 1, \cdots, J_i;\ i=1, 2, \cdots, m)$, there is the interpolation relation*

$$\|u_\Delta\|_{L_\infty^{(k)}(Q_\Delta^m)}\leqslant K_3\left\|u_\Delta\right\|^{1-\frac{k+\frac{m}{2}}{n}}_{L_2(Q_\Delta^m)}\left\{\|u_\Delta\|_{L_2^{(n)}(Q_\Delta^m)}+\frac{\|u_\Delta\|_{L_2(Q_\Delta^m)}}{l^n}\right\}^{\frac{k+\frac{m}{2}}{n}}, \tag{12}$$

where $n\geqslant k+m$, $m\geqslant 1$ and K_3 is a constant independent of h_i $(i=1, 2, \cdots, m)$, l and u_Δ.

Proof. When $m=1$, (12) becomes (2) as $p=\infty$.

Suppose that (12) is valid for the discrete functions with $(m-1)$ indices, where $m\geqslant 2$. Similarly denote $\beta=(\beta_2, \beta_3,\cdots, \beta_m)$ and $|\beta|=\beta_2+\cdots+\beta_m=k-r\geqslant 0$.

Let us consider

$$J=\|\delta_1^r\hat\delta_1^\beta u_\Delta\|_{L_\infty(Q_\Delta^m)}=\max_{j_1=0,1,\cdots,J_1-r}\left\|\frac{\overset{+}{\Delta}{}_1^r}{h_1^r}\hat\delta_1^\beta u_{j_1,\cdot}\right\|_{L_\infty(Q_\Delta^{m-1})}.$$

By induction assumption we get

$$J\leqslant C_2\max_{j=0,1,\cdots,J_1-r}\left\|\frac{\overset{+}{\Delta}{}_1^r u_{j_1,\cdot}}{h_1^r}\right\|^{1-\frac{k-r+\frac{m-1}{2}}{n-r-1}}_{L_2(Q_\Delta^{m-1})}\left\|\frac{\overset{+}{\Delta}{}_1^r u_{j_1,\cdot}}{h_1^r}\right\|^{\frac{k-r+\frac{m-1}{2}}{n}}_{L_2^{(n-r-1)}(Q_\Delta^{m-1})}$$

$$+\frac{C_3}{l^{(k-r+\frac{m-1}{2})}}\max_{j_1=0,1,\cdots,J_1-r}\left\|\frac{\overset{+}{\Delta}{}_1^r u_{j_1,\cdot}}{h_1^r}\right\|_{L_2(Q_\Delta^{m-1})}. \tag{13}$$

Here we have

$$\max_{j_1=0,1,\cdots,J_1-r}\left\|\frac{\overset{+}{\Delta}{}_1^r u_{j_1,\cdot}}{h_1^r}\right\|^2_{L_\infty(Q_\Delta^{m-1})}\leqslant\sum_{i=2}^{m}\sum_{j_i=0}^{J_i}\|\delta_1^r u_{\cdot,j_2,\cdots,j_m}\|^2_{L_\infty(Q_\Delta^1)}h_2\cdots h_m$$

$$\leqslant C_4\sum_{i=2}^{m}\sum_{j_i=0}^{J_i}\Bigg\{\|u_{\cdot,j_2,\cdots,j_m}\|^{2\left(1-\frac{r+\frac{1}{2}}{n}\right)}_{L_2(Q_\Delta^1)}\|\delta_1^n u_{\cdot,j_2,\cdots,j_m}\|^{\frac{2r+1}{n}}_{L_2(Q_\Delta^1)}$$

$$+\frac{1}{l_1^{2r+1}}\|u_{\cdot,j_2,\cdots,j_m}\|^2_{L_2(Q_\Delta^1)}\Bigg\}h_2\cdots h_m$$

$$\leqslant C_5\|u_\Delta\|^{2\left(1-\frac{r+\frac{1}{2}}{n}\right)}_{L_2(Q_\Delta^m)}\|\delta_1^n u_\Delta\|^{\frac{2r+1}{n}}_{L_2(Q_\Delta^m)}+[illegible]\|u_\Delta\|^2_{L_2(Q_\Delta^m)} \quad [illegible]$$

On the other hand we also have

$$\begin{aligned}
\max_{j_1=0,1,\cdots,J_1-r}\left\|\frac{\stackrel{+}{\Delta}{}_1^r u_{j_1,\cdot}}{h_1^r}\right\|^2_{L^{(n-r-1)}(Q_1^{T-1})}
&\leqslant \sum_{|\beta|=n-r-1}\sum_{i=2}^{m}\sum_{j_i=0}^{J_i-\beta_i}\left\|\frac{\stackrel{+}{\Delta}{}^{\beta}}{h^{\beta}}\,\delta_1^r u_{\cdot,j_2,\cdots,j_m}\right\|^2_{L_2(Q_1^1)} h_2h_3\cdots h_m\\
&\leqslant C_4\sum_{|\beta|=n-r-1}\sum_{i=2}^{m}\sum_{j_i=0}^{J_i-\beta_i}\Bigg\{\left\|\frac{\stackrel{+}{\Delta}{}^{\beta}u_{\cdot,j_2,\cdots,j_m}}{h^{\beta}}\right\|^{\frac{1}{r+1}}_{L_2(Q_1^1)}\left\|\delta_1^{r+1}\frac{\stackrel{+}{\Delta}{}^{\beta}}{h^{\beta}}u_{\cdot,j_2,\cdots,j_m}\right\|^{\frac{2r+1}{r+1}}_{L_2(Q_1^1)}\\
&\qquad+\frac{1}{l_1^{2r+1}}\left\|\frac{\stackrel{+}{\Delta}{}^{\beta}}{h^{\beta}}u_{\cdot,j_2,\cdots,j_m}\right\|^2_{L_2(Q_1^1)}\Bigg\}h_2\cdots h_m\\
&\leqslant C_4\|u_\Delta\|^{\frac{1}{r+1}}_{L_2^{(n-r-1)}(Q^T)}\|u_\Delta\|^{\frac{2r+1}{r+1}}_{L_2^{(n)}(Q^T)}+\frac{C_4}{l_1^{2r+1}}\|u_\Delta\|_{L_2^{(n-r-1)}(Q^T)}.
\end{aligned}$$

From Theorem 2, there is

$$\|u_\Delta\|_{L_2^{(n-r-1)}(Q^T)}\leqslant C_5\|u_\Delta\|^{\frac{r+1}{n}}_{L_2(Q^T)}\|u_\Delta\|^{\frac{n-r-1}{n}}_{L_2^{(n)}(Q^T)}+\frac{C_5}{l^{n-r-1}}\|u_\Delta\|_{L_2(Q^T)}.$$

Substituting the last relation into the previous inequality, we obtain

$$\max_{j_1=0,1,\cdots,J_1-r}\left\|\frac{\stackrel{+}{\Delta}{}_1^r u_{j_1,\cdot}}{h_1^r}\right\|_{L_2^{(n-r-1)}(Q_1^{T-1})}\leqslant C_6\|u_\Delta\|^{\frac{1}{2n}}_{L_2(Q^T)}\left\{\|u_\Delta\|_{L_2^{(n)}(Q^T)}+\frac{\|u_\Delta\|_{L_2(Q^T)}}{l^n}\right\}^{1-\frac{1}{2n}}, \tag{15}$$

where C_6 is a constant independent of u_Δ.

Substituting (14) and (15) into the right hand part of (13), we have the expression

$$\begin{aligned}
J\leqslant C_7\|u_\Delta\|^{1-\frac{k+\frac{m}{2}}{n}}_{L_2(Q^T)}\Bigg\{&\|u_\Delta\|_{L_2^{(n)}(Q^T)}\\
&+\left(\frac{\|u_\Delta\|_{L_2(Q^T)}}{l^n}\right)^{\frac{(n-\frac{1}{2})(k-r+\frac{m-1}{2})}{(k+\frac{m}{2})(n-r-1)}}\|u_\Delta\|^{\frac{(r+\frac{1}{2})(n-k-\frac{m+1}{2})}{(k+\frac{m}{2})(n-r-1)}}_{L_2^{(n)}(Q^T)}\\
&+\left(\frac{\|u_\Delta\|_{L_2(Q^T)}}{l_1^n}\right)^{\frac{(r+\frac{1}{2})(n-k-\frac{m+1}{2})}{(k+\frac{m}{2})(n-r-1)}}\|u_\Delta\|^{\frac{(n-\frac{1}{2})(k-r+\frac{m-1}{2})}{(k+\frac{m}{2})(n-r-1)}}_{L_2^{(n)}(Q^T)}\\
&+\left(\frac{\|u_\Delta\|_{L_2(Q^T)}}{l^n}\right)^{\frac{k-r+\frac{m-1}{2}}{n}}\|u_\Delta\|^{\frac{r+\frac{1}{2}}{n}}_{L_2^{(n)}(Q^T)}\\
&+\frac{\|u_\Delta\|_{L_2(Q^T)}}{\left(\frac{l_1}{l}\right)^{\frac{n(r+\frac{1}{2})(n-\frac{1}{2})}{(k+\frac{m}{2})(n-r-1)}}\left(\frac{l^{n-\frac{1}{2}}}{l_1^{r+\frac{1}{2}}}\right)^{\frac{n}{n-r-1}}}+\frac{\|u_\Delta\|_{L_2(Q^T)}}{l^n\left(\frac{l}{l}\right)^{\frac{n(r+\frac{1}{2})}{k+\frac{m}{2}}}}\Bigg\}^{\frac{k+\frac{m}{2}}{n}}.
\end{aligned}$$

Replacing l and l_1 by the minimum width l of rectangular domain Q^m and using the inequality $A^{1-\alpha}B^{\alpha}\leqslant(1-\alpha)A+\alpha B$ $(0<\alpha<1,\ A>0,\ B>0)$, we obtain

$$J\leqslant C_8\|u_\Delta\|^{1-\frac{k+\frac{m}{2}}{n}}_{L_2(Q^T)}\left\{\|u_\Delta\|_{L_2^{(n)}(Q^T)}+\frac{\|u\|_{L_2(Q^T)}}{l^n}\right\}^{\frac{k+\frac{m}{2}}{n}},$$

where C_8 is a constant independent of h_i $(i=1,2,\cdots,m)$, l and u_Δ. This implices (12) and then the theorem is proved.

§ 6

From the theorems obtained in the previous sections, we can verify immediately the following theorems.

Theorem 4. *For any discrete function* $u_\Delta=\{u_{j_1,j_2,\cdots,j_m}\}$ *defined on* $\mathbb{R}_\Delta^m=\{(x_{1j_1},\cdots,x_{mj_m})\}$, $(j_1,\cdots,j_m=0,\pm1,\pm2,\cdots)$ *for* $m\geqslant1$, *there are interpolation formulas*

$$\|u_\Delta\|_{L_2^{(k)}(\mathbb{R}_\Delta^m)}\leqslant K_4\|u_\Delta\|_{L_2(\mathbb{R}_\Delta^m)}^{1-\frac{k}{n}}\|u_\Delta\|_{L_2^{(n)}(\mathbb{R}_\Delta^m)}^{\frac{k}{n}},\quad 0\leqslant k\leqslant n, \tag{16}$$

and

$$\|u_\Delta\|_{L_\infty^{(k)}(\mathbb{R}_\Delta^m)}\leqslant K_5\|u_\Delta\|_{L_2(\mathbb{R}_\Delta^m)}^{1-\frac{k+\frac{m}{2}}{n}}\|u_\Delta\|_{L_2^{(n)}(\mathbb{R}_\Delta^m)}^{\frac{k+\frac{m}{2}}{n}},\quad 0\leqslant k\leqslant n-m, \tag{17}$$

where K_4 *and* K_5 *are constants independent of* h_i $(i=1,\cdots,m)$ *and* u_Δ.

Theorem 5. *For any discrete function* $u_\Delta=\{u_{j_1,j_2,\cdots,j_m}\}$ *defined on* Q_Δ^m (*or* $\mathbb{R}_\Delta^m$), *then for any* $\varepsilon>0$, *there is a constant* $K(\varepsilon)$ *such that*

$$\|u_\Delta\|_{L_2^{(k)}(Q_T)}\leqslant\varepsilon\|u_\Delta\|_{L_2^{(n)}(Q_T)}+K(\varepsilon)\|u_\Delta\|_{L_2(Q_T)},\quad 0\leqslant k<n \tag{18}$$

and

$$\|u_\Delta\|_{L_\infty^{(k)}(Q_T)}\leqslant\varepsilon\|u_\Delta\|_{L_2^{(n)}(Q_T)}+K(\varepsilon)\|u_\Delta\|_{L_2(Q_T)},\quad 0\leqslant k\leqslant n-m, \tag{19}$$

where $K(\varepsilon)$ *is independent of* h_i $(i=1,2,\cdots,m)$ *and* u_Δ.

Theorem 6. *For any discrete function* $u_\Delta=\{u_{j_1,j_2,\cdots,j_m}\}$ *defined on* Q_Δ^m (*or* $\mathbb{R}_\Delta^m$) *for* $m\geqslant1$, *there is interpolation relation*

$$\|u_\Delta\|_{L_p^{(k)}(Q_T)}\leqslant K_6\|u_\Delta\|_{L_2(Q_T)}^{1-\frac{k+\frac{m}{2}-\frac{m}{p}}{n}}\left\{\|u_\Delta\|_{L_2^{(n)}(Q_T)}+\frac{\|u_\Delta\|_{L_2(Q_T)}}{l^n}\right\}^{\frac{k+\frac{m}{2}-\frac{m}{p}}{n}}, \tag{20}$$

where $0\leqslant k\leqslant n-m$, $2\leqslant p\leqslant\infty$ *and* K_6 *is independent of* h_i $(i=1,2,\cdots,m)$, l *and* u_Δ.

In fact we have

$$\|u_\Delta\|_{L_p^{(k)}(Q_T)}^p\leqslant\|u_\Delta\|_{L_\infty^{(k)}(Q_T)}^{p-2}\|u_\Delta\|_{L_2^{(k)}(Q_T)}^2.$$

Then (20) follows immediately by substituting (7) and (12) into the right hand part of the above inequality.

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