A NEW VERSION OF ITERATIVE METHOD FOR SOLVING RIEMANN PROBLEM*

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Abstract

A new version of iterative method for solving Riemann problem of gas dynamics is presented. In practice the new procedure exhibited a good convergence in cases where Riemann solution involves a strong rarefaction wave or two rarefaction waves. In the other cases the new version is identical with Godunov procedure.

Introduction

Riemann solutions are the building blocks of several numerical methods for solving the equations of gas dynamics (see [1], [3], [4], [7]). The usefulness of these methods depends on the possibility of solving the Riemann problem accurately and effectively. Generally speaking, Godunov iterative procedure provides an approximating solution to the Riemann problem ([3], [5]). But as noted by Godunov, the iteration may fail to converge in the presence of strong rarefaction. To overcome this difficulty Chorin gave a modified iterative method [1]. In this paper we present a new version of iterative method. In practice we find that the new version is more effective in cases where the Riemann solution consists of two rarefactions or a rarefaction plus a shock where the rarefaction is stronger than the shock. In the other cases the new version is identical with Godunov procedure.

The New Version of Iterative Procedure

Consider the gas dynamics equations

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = 0, \\ e_t + ((e+p)u)_x = 0, \end{cases} \tag{1}$$

where ρ, u, ρu, e and p, respectively, denote the density, velocity, momentum, internal energy, and pressure of the gas, and

$$e = \rho\varepsilon + \frac{1}{2}\rho u^2$$

is the total energy of the gas. For polytropic gases ε is given by the constitutive relation

$$\varepsilon = \frac{1}{\gamma-1}\frac{p}{\rho},$$

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where γ is a constant larger than one.

The Riemann problem for (1) will have the initial data

$$(\rho(x,0),\ u(x,0),\ p(x,0))=\begin{cases} S_l=(\rho_l,\ u_l,\ p_l), & x<0,\\ S_r=(\rho_r,\ u_r,\ p_r), & x>0.\end{cases}$$

It is well-known that the solution consists of a right state S_r, a left state S_l, a middle state $S_*(p=p_*,\ u=u_*)$, separated by waves which are either rarefactions or shocks. S_* is divided by the slip line

$$\frac{dx}{dt}=u_*$$

into two parts with possibly different values of ρ_*, but equal values of u_* and p_*.

The new version of procedure is as follows.

1. In case of $u_l<u_r$, the iterative method first computes u_* in the state S_*. Define the quantity

$$M_r=(p_r-p_*)/(u_r-u_*). \tag{2}$$

The relation between S_* and S_r can be written ([2],[7]) as

$$u_*=u_r+\varphi(p_*;\ p_r,\ \rho_r), \tag{3}$$

where

$$\varphi(p_*;\ p_r,\ \rho_r)=\begin{cases} \dfrac{\sqrt{2}\,(p_*-p_r)}{\sqrt{((\gamma+1)p_*-(\gamma-1)p_r)\rho_r}}, & p_*\geqslant p_r,\\ \dfrac{2\sqrt{\gamma}}{\gamma-1}\,\dfrac{p_r^{1/2\gamma}}{\rho_r^{1/2}}\left(p_*^{(\gamma-1)/2\gamma}-p_r^{(\gamma-1)/2\gamma}\right), & p_*<p_r.\end{cases}$$

Upon solving p_* in terms of u_* from (3), we have

$$p_*=p_r+\psi(u_*-u_r;\,p_r,\ \rho_r), \tag{4}$$

where

$$\psi(u_*-u_r;\,p_r,\rho_r)=\begin{cases} \dfrac{\gamma+1}{4}(u_*-u_r)^2\rho_r\left(1+\sqrt{1+\dfrac{16\gamma p_r}{(\gamma+1)^2(u_*-u_r)^2\rho_r}}\right), & u_*\geqslant u_r,\\ p_r\left(\left(1+\dfrac{\gamma-1}{2\sqrt{\gamma}}(u_*-u_r)\sqrt{\rho_r/p_r}\right)^{\frac{2\gamma}{\gamma-1}}-1\right), & u_*<u_r.\end{cases}$$

By substituting p_* of (4) into (2), one gets

$$M_r=(p_r\rho_r)^{1/2}\Psi((u_*-u_r)(\rho_r/p_r)^{1/2}). \tag{5}$$

where

$$\Psi(x)=\begin{cases} \dfrac{\gamma+1}{4}\left(x+\left(x^2+\dfrac{16\gamma}{(\gamma+1)^2}\right)^{1/2}\right), & x\geqslant 0,\\ \dfrac{1}{x}\left(\left(1+\dfrac{\gamma-1}{2\sqrt{\gamma}}x\right)^{\frac{2\gamma}{\gamma-1}}-1\right), & x<0.\end{cases}$$

Similarly, M_l is defined by

$$M_l=-(p_l-p_*)/(u_l-u_*) \tag{6}$$

The relation between S_* and S_l is

$$u_*=u_l-\varphi(p_*;\ p_l,\ \rho_l), \tag{7}$$

or

$$p_*=p_l+\psi(u_l-u_*;\ p_l,\ \rho_l). \tag{8}$$

From (6) and (7), one gets

$$M_l=(p_l\rho_l)^{1/2}\Psi((u_l-u_*)(\rho_l/p_l)^{1/2}). \tag{9}$$

Upon elimination of p_* from (2) and (6), we have

$$u_* = \frac{(p_l - p_r + M_r u_r + M_l u_l)}{(M_r + M_l)}. \tag{10}$$

There are three equations (5), (9) and (10) with three unknowns (u_*, M_l, M_r). For calculating the unknowns we construct by induction a sequence $(u_*^\nu, M_l^\nu, M_r^\nu, \nu = 0, 1, \cdots)$ as follows:

$$M_r^{\nu+1} = (p_r\rho_r)^{1/2}\Psi((u_*^\nu - u_r)(\rho_r/p_r)^{1/2}),$$
$$M_l^{\nu+1} = (p_l\rho_l)^{1/2}\Psi((u_l - u_*^\nu)(\rho_l/p_l)^{1/2}),$$
$$u_*^{\nu+1} = (p_l - p_r + M_r^{\nu+1}u_r + M_l^{\nu+1}u_l)/(M_r^{\nu+1} + M_l^{\nu+1}).$$

This iteration is stopped when

$$\max(|M_r^{\nu+1} - M_r^\nu|,\ |M_l^{\nu+1} - M_l^\nu|) \leqslant \varepsilon,$$

where $\varepsilon > 0$ is a given small number; One then sets $M_r = M_r^{\nu+1}$, $M_l = M_l^{\nu+1}$ and $u_* = u_*^{\nu+1}$. This procedure is started by setting

$$u_*^0 = (u_l + u_r)/2,\ M_r^0 = M_l^0 = 100.$$

Once u_*, M_r, M_l are found, then we compute

$$p_* = (u_l - u_r + p_r/M_r + p_l/M_l)/(1/M_r + 1/M_l) \tag{11}$$

from the definition of M_l and M_r.

2. In case of $u_l > u_r$, the new version is the same as the Godunov procedure([1], [5], [6]). The procedure calculates p_* first. Substituting (3), (7) into (2), (6) respectively, one gets

$$M_r = (p_r\rho_r)^{1/2}\Phi(p_*/p_r), \tag{12}$$
$$M_l = (p_l\rho_l)^{1/2}\Phi(p_*/p_l), \tag{13}$$

where

$$\Phi(x) = \begin{cases} \left(\dfrac{\gamma+1}{2}x + \dfrac{\gamma-1}{2}\right)^{1/2}, & x \geqslant 1, \\ \dfrac{\gamma-1}{2\sqrt{\gamma}}\ \dfrac{1-x}{1-x^{(\gamma-1)/2\gamma}}, & x < 1. \end{cases}$$

For solving the unknowns (p_*, M_l, M_r) from the equations (11), (12) and (13), the Godunov method uses the following procedure; Pick starting values

$$p_*^0 = (p_l + p_r)/2,\quad M_r^0 = M_l^0 = 100$$

and then compute M_r^ν, M_l^ν, $p_*^\nu (\nu = 0, 1, \cdots)$ using

$$M_r^{\nu+1} = (p_r\rho_r)^{1/2}\Phi(p_*^\nu/p_r),$$
$$M_l^{\nu+1} = (p_l\rho_l)^{1/2}\Phi(p_*^\nu/p_l),$$
$$p_*^{\nu+1} = (u_l - u_r + p_r/M_r^{\nu+1} + p_l/M_l^{\nu+1})/(1/M_r^{\nu+1} + 1/M_l^{\nu+1}).$$

This iteration is stopped when

$$\max(|M_r^{\nu+1} - M_r^\nu|,\ |M_l^{\nu+1} - M_l^\nu|) \leqslant \varepsilon;$$

One then sets $M_r = M_r^{\nu+1}$, $M_l = M_l^{\nu+1}$ and $p_* = p_*^{\nu+1}$. Once p_*, M_r, M_l are known, we have

$$u_* = (p_l - p_r + M_l u_l + M_r u_r)/(M_l + M_r)$$

from (2) and (4).

After p_*, u_*, M_l and M_r are determined one can get complete Riemann solution from the jump conditions of shock and from the isentropic law and the constancy of Riemann invariants of rarefaction. For detial see [6].

Note that in case of $u_l < u_r$, the Riemann solution contains at least one rarefaction, and the rarefaction is stronger than the shock if a shock exists in the solution. In fact according to the value of u_* the pattern of solution has three possibilities:

1. When $u_l < u_* < u_r$, the solution consists of two rarefactions.

2. When $u_* < u_l < u_r$, the solution consists of one right rarefaction and one left shock. If we measure the strength of waves by the difference of velocities on the two sides of the wave, the strength of rarefaction ($|u_r - u_*|$) is larger than the strength of shock ($|u_l - u_*|$).

3. When $u_l < u_r < u_*$, the solution is similar to the one in case of 2.

This new procedure was applied to reacting gas flow problems [7] and to several test problems where the Godunov procedure does not converge or converges slowly, such as a strong rarefaction or two rarefactions involved in the solutions. The new algorithm exhibited a good convergence in the most cases.

Remark. We may go along the line described in [8] with our new version to get a more effective but more complicated algorithm. We omit the details here.

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References

[1] A. J. Chorin, Random choice solution of hyperbolic systems, *J. Comput. Phys.*, 22, 1976, 517—533.

[2] R. Courant and K. O. Friedrichs, Supersonic Flow and Shock Waves, Interscience, New York, 1948.

[3] S. K. Godunov, A finite difference method for the numerical computation and discontinuous solutions of the equations of fluid dynamics, *Mat. Sb.*, 47, 1959, 271—360.

[4] B. Van Leer, Towards the ultimate difference scheme, *J. Comput. Phys.*, 32, 1979, 101—136.

[5] R. D. Richtmyer and K. W. Morton, Finite Difference Methods for Initial Value Problems, Interscience, New York, 1967.

[6] G. A. Sod, A survey of several finite difference methods for systems of nonlinear hyperbolic conservation laws, *J. Comput. Phys.*, 27, 1978, 1—31.

[7] Teng Zhen-huan, A. J. Chorin and Liu Tai-ping, Riemann problems for reacting gas, with applications to transition, *SIAM J. Appl. Math.*, 42, 1982, 964—981.

[8] S. K. Godunov, Numerical methods of multidimension problems in gas dynamics, Moscow, 1976.