ON THE HAAR AND WALSH SYSTEMS ON A TRIANGLE*

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Abstract

In this paper we establish the Haar and Walsh systems on a triangle. These systems are complete in $L_2(\Delta)$. The uniform convergence of the Haar-Fourier series and the uniform convergence by group of the Walsh-Fourier series for any continuous function are proved.

1. Introduction

It is interesting and useful to study multivariate Haar and Walsh functions either in theory or in practice. If we investigate on a domain which can be considered a Cartesian product, then the functions are readily extended to several variables from one variable. Setting by the tensor product construct Harmuth has shown those kinds of multivariate systems in his book[5] and pointed out the applications in communication.

In this paper we attempt to focus on a triangle, or more generally on a simplex in n-dimensional space. We did not find any paper about it. Perhaps it puzzles some people temporarily.

The main contribution of this paper is to establish the Haar and Walsh system with two variables on a triangle. We prove their orthonormality and completeness in Hilbert space L_2. Moreover, the corresponding Haar-Fourier series and Walsh-Fourier series for any continuous function are uniformly convergent and convergent by group respectively.

It is easy to generalize these results to the n-dimensional simplex. For simplicity we will discuss only the two-dimensional triangle.

Now we explain some preliminaries and notations.

The Haar functions on [0, 1] are defined as follows:

$$\chi_0(t):=1, \qquad \text{for } 0\leqslant t\leqslant 1,$$

and

$$\chi_n^{(k)}(t):=\begin{cases}\sqrt{2^n}, & \text{for } \dfrac{2k-2}{2^{n+1}}\leqslant t<\dfrac{2k-1}{2^{n+1}},\\ -\sqrt{2^n}, & \text{for } \dfrac{2k-1}{2^{n+1}}<t\leqslant\dfrac{2k}{2^{n+1}},\\ 0, & \text{elsewhere in } [0,1],\end{cases} \tag{1.1}$$

$$k=1,2,3,\cdots,2^n; \qquad n=1,2,3,\cdots,\infty.$$

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The Walsh functions on $[0,1]$ consist of the following ones:

$$w_0(t):=1, \qquad \text{for } 0\leqslant t\leqslant 1,$$

$$w_1(t):=\begin{cases}1, & \text{for } 0\leqslant t<\frac{1}{2},\\ -1, & \text{for } \frac{1}{2}<t\leqslant 1,\end{cases} \tag{1.2}$$

$$w_{n+1}^{(2k-1)}(t):=\begin{cases}w_n^{(k)}(2t), & \text{for } 0\leqslant t<\frac{1}{2},\\ (-1)^{k+1}w_n^{(k)}(2t-1), & \text{for } \frac{1}{2}<t\leqslant 1,\end{cases}$$

$$w_{n+1}^{(2k)}(t):=\begin{cases}w_n^{(k)}(2t), & \text{for } 0\leqslant t<\frac{1}{2},\\ (-1)^{k}w_n^{(k)}(2t-1), & \text{for } \frac{1}{2}<t\leqslant 1,\end{cases}$$

$$k=1,2,3,\cdots,2^n; \qquad n=1,2,3,\cdots,\infty.$$

Some detailed investigation of the Haar and Walsh systems can be found in [1], [3], [5].

In order to generalize the Haar and Walsh systems to the two-dimensional case we should explain our representation in this paper. The Cartesian coordinates are not very convenient for triangular elements, and a special type of coordinate system called area coordinates should be used.

In Figure 1 it is seen that the internal point P will divide the triangle ABC into three smaller triangles, and depending on the position of the point P, the area of each of the triangles PAB, PBC, PCA can vary from zero to $|\varDelta|$, which is the area of the triangle ABC. In other words, the ratios $\frac{a}{|\varDelta|}$, $\frac{b}{|\varDelta|}$ and $\frac{c}{|\varDelta|}$ will take any value between zero and unity. Here a, b, c are the area of triangles PBC, PCA, PAB respectively.

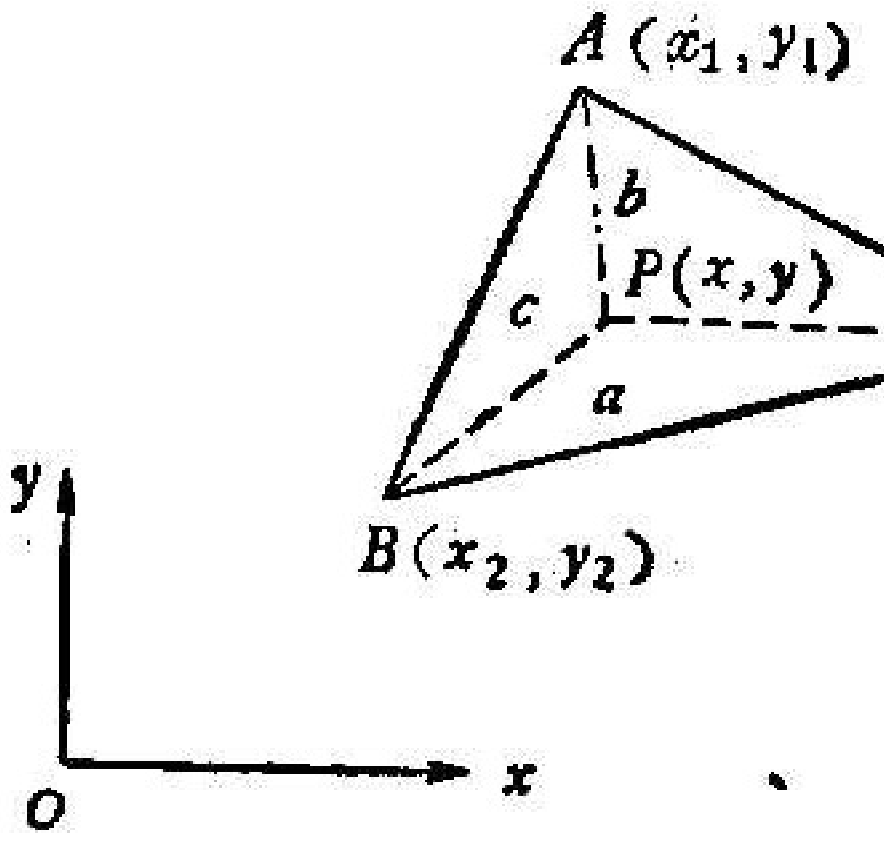


Fig. 1

These ratios are called area coordinates, defined by $l_1:=\frac{a}{|\varDelta|}$, $l_2:=\frac{b}{|\varDelta|}$, $l_3:=\frac{c}{|\varDelta|}$.

It is easy to see that

$$\begin{pmatrix}1\\ x\\ y\end{pmatrix}=\begin{pmatrix}1 & 1 & 1\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{pmatrix}\begin{pmatrix}l_1\\ l_2\\ l_3\end{pmatrix}.$$

If two points P and Q are in two similar triangles respectively, and have the same area coordinates, then we denote them by $P\sim Q$.

2. An Orthonormal Sequence χ on a Triangular Domain

Suppose $\varDelta$ (or $\varDelta_{ABC}$) is any triangle on a plane and $|\varDelta|=1$ is the area of $\varDelta_{ABC}$. If D, E, F are midpoints of AB, BC, CA respectively, connecting DE, EF, FD, we divide

Δ into four similar small triangles Δ_{ADF}, Δ_{DBE}, Δ_{FEC}, Δ_{EFD}. We call them Δ_1, Δ_2, Δ_3, Δ_4 respectively.

We define the sequence χ as follows:

$$\chi_0(P):=1, \qquad \text{for } P\in\Delta,$$

$$\chi_1^{(1)}(P):=\begin{cases}1, & \text{for } P\in\Delta_1\cup\Delta_2,\\ -1, & \text{for } P\in\Delta_3\cup\Delta_4,\end{cases}$$

$$\chi_1^{(2)}(P):=\begin{cases}\sqrt{2}, & \text{for } P\in\Delta_1,\\ -\sqrt{2}, & \text{for } P\in\Delta_2,\\ 0, & \text{for } P\in\Delta_3\cup\Delta_4,\end{cases}$$

$$\chi_1^{(3)}(P):=\begin{cases}\sqrt{2}, & \text{for } P\in\Delta_3,\\ -\sqrt{2}, & \text{for } P\in\Delta_4,\\ 0, & \text{for } P\in\Delta_1\cup\Delta_2,\end{cases}$$

....

$$\chi_n^{(3j+i)}(P):=\begin{cases}2\chi_{n-1}^{(i)}(Q), & \text{for } P\in\Delta_{j+1},\\ 0, & \text{for } P\in\Delta\backslash\Delta_{j+1},\end{cases} \tag{2.1}$$

where $Q\in\Delta$; $Q\sim P$; $j=0, 1, 2, 3$; $i=1, 2, \cdots, 3\cdot4^{n-2}$; $n=2, 3, \cdots$.

At a point of discontinuity, let the value of these functions be the average.

Now we consider the orthogonality of the sequence χ. We prove the following theorem.

Theorem 1. *The sequence χ defined by* (2.1) *is orthonormal.*

Proof. First, it is easy to check that when $n\leqslant 2$ the sequence $\{\chi_n^{(j)}\}$ is orthonormal. We suppose that the theorem holds for $n\leqslant N$. For $2\leqslant m\leqslant N+1$; j_1, $j_2=0, 1, 2, 3$; $i_1=1, 2, \cdots, 3\cdot4^{N-1}$; $i_2=1, 2, \cdots, 3\cdot4^{m-2}$, by (2.1) and induction hypothesis, we get

$$\int_\Delta \chi_{N+1}^{(3j_1+i_1)}(P)\chi_m^{(3j_2+i_2)}(P)dP=4\delta_{j_1,j_2}\int_{\Delta_{j_1+1}}\chi_N^{(i_1)}(Q)\chi_{m-1}^{(i_2)}(Q)dP$$

$$=\delta_{j_1,j_2}\int_\Delta \chi_N^{(i_1)}(Q)\chi_{m-1}^{(i_2)}(Q)dP=\delta_{j_1,j_2}\delta_{i_1,i_2}\delta_{N,m-1}.$$

It is easy to verify that

$$\int_\Delta \chi_{N+1}^{(3j+i_1)}(P)\chi_1^{(i_2)}(P)dP=\int_\Delta \chi_{N+1}^{(3j+i_1)}(P)\chi_0(P)dP=0.$$

Therefore the theorem holds for $n=N+1$, and this finishes the induction.

3. Convergence Properties

The triangle Δ has been divided into four similar smaller triangles $\Delta_i (i=1, 2, 3, 4)$. Now set

$$\Delta_{1,i}:=\Delta_i \quad (i=1, 2, 3, 4).$$

For each $\Delta_{1,i}$, we divide it into four similar smaller triangles in the same way as we did before. We order them as $\Delta_{2,1}$, $\Delta_{2,2}$, $\cdots$, $\Delta_{2,16}$ such that

$$\Delta_{1,i}=\Delta_{2,4i}\cup\Delta_{2,4i-1}\cup\Delta_{2,4i-2}\cup\Delta_{2,4i-3}, \qquad i=1, 2, 3, 4.$$

We continue this process. For any n we get a sequence $\Delta_{n,1}$, $\Delta_{n,2}$, $\cdots$, $\Delta_{n,4^n}$ such that

$$\Delta_{n-1,i}=\Delta_{n,4i}\cup\Delta_{n,4i-1}\cup\Delta_{n,4i-2}\cup\Delta_{n,4i-3},$$

$$i=1, 2, 3, \cdots, 4^{n-1}; \quad n=1, 2, 3, \cdots, \infty.$$

$$\Delta_{0,1}:=\Delta.$$

Define a function sequence $\{f_{n,i}\}$ on the Δ:

$$f_0(P):=1, \qquad \text{for } P\in\Delta,$$

$$f_{1,i}(P):=\begin{cases}1, & \text{for } P\in\Delta_{1,i},\\ 0, & \text{for } P\in\Delta\backslash\Delta_{1,i}, \quad i=1, 2, 3, 4,\end{cases}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f_{n,i}(P):=\begin{cases}1, & \text{for } P\in\Delta_{n,i},\\ 0, & \text{for } P\in\Delta\backslash\Delta_{n,i},\end{cases} \tag{3.1}$$

$$i=1, 2, 3, \cdots, 4^n; \quad n=1, 2, 3, \cdots, \infty.$$

It is obvious that the sequence $\{f_{n,i}\}$ is orthogonal.

Let

$$M_n:=\operatorname{span}(f_{n,1}, f_{n,2}, \cdots, f_{n,4^n}), \quad n\geqslant 0. \tag{3.2}$$

Thus

$$\dim M_n=4^n.$$

For convenience, sometimes we use notation

$$\chi_1:=\chi_0,$$

$$\chi_{4^{n-1}+i}:=\chi_n^{(i)}, \qquad n\geqslant 1;\ i=1, 2, \cdots, 3\cdot 4^{n-1}. \tag{3.3}$$

Set

$$H_n:=\operatorname{span}(\chi_1, \chi_2, \cdots, \chi_n).$$

It is clear that

$$H_{4^n}=M_n, \tag{3.4}$$

since $H_{4^n}\subset M_n$ and $\dim H_{4^n}=\dim M_n=4^n$.

We define

$$L_2(\Delta):=\left\{f\,\middle|\,\int_\Delta f^2\,d\sigma<\infty\right\}$$

and

$$\|f\|_2^2:=\int_\Delta f^2\,d\sigma.$$

Then the Fourier series of a given function $F\in L_2(\Delta)$ in terms of the function sequence $\{\chi_n\}$ is

$$F\sim\sum_{i=1}^{\infty}\alpha_i\chi_i \tag{3.5}$$

with

$$\alpha_i:=\int_\Delta F(P)\chi_i(P)dP.$$

Let

$$\mathscr{P}_nF:=\sum_{i=1}^{n}\alpha_i\chi_i(P) \tag{3.6}$$

be the n-th partial sum of the series (3.5).

From the orthogonality of sequence $\{\chi_n\}$, $\mathscr{P}_n F$ is the best L_2-approximation to F from H_n. Hence it is convergent to F if F is in $L_2(\Delta)$, since H_n is dense in $L_2(\Delta)$. Thus we get the following theorem.

Theorem 2. *If $F \in L_2(\Delta)$, then*

$$\lim_{n\to\infty} \| F - \mathscr{P}_n F \|_2 = 0.$$

In order to study the uniform convergence we let

$$C(\Delta) := \{ f \mid f \text{ is continuous on } \Delta \}$$

and

$$\| f \|_\infty := \max_{P \in \Delta} | f(P) |.$$

For $F \in C(\Delta)$ we define

$$\mathscr{P}_n^{(j)} F := \int_\Delta F \chi_0 \, d\sigma \cdot \chi_0 + \int_\Delta F \chi_1^{(1)} \, d\sigma \cdot \chi_1^{(1)} + \cdots + \int_\Delta F \chi_n^{(j)} \, d\sigma \cdot \chi_n^{(j)}. \tag{3.7}$$

Set

$$K_0(P, Q) := \chi_0(P) \chi_0(Q), \ (\text{for } P, Q \in \Delta)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$K_n^{(j)}(P, Q) := \chi_0(P)\chi_0(Q) + \chi_1^{(1)}(P)\chi_1^{(1)}(Q) + \cdots + \chi_n^{(j)}(P)\chi_n^{(j)}(Q), \tag{3.8}$$
$$j = 1, 2, \cdots, 3\cdot 4^{n-1}; \quad n = 1, 2, \cdots.$$

Thus

$$\mathscr{P}_n^{(j)} F(P) = \int_\Delta K_n^{(j)}(P, Q) F(Q) dQ. \tag{3.9}$$

Let $A := (a_{ij})$ $(i, j = 1, 2, 3, \cdots, 4^n)$ be any $4^n \times 4^n (n = 1, 2, \cdots)$ matrix and $G(P, Q)$ be any function defined on $\Delta \times \Delta$.

The notation $G(P, Q) \leftrightarrow A$ means that the value of $G(P, Q)$ is a_{ij} when $P \in \Delta_{n,j}$, $Q \in \Delta_{n,i}$. It leads to the following relationship:

$$\chi_0(P)\chi_0(Q) \leftrightarrow \sigma_0 := \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\chi_1^{(1)}(P)\chi_1^{(1)}(Q) \leftrightarrow \sigma_1 := \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix},$$

$$\chi_1^{(2)}(P)\chi_1^{(2)}(Q) \leftrightarrow \sigma_2 := \begin{bmatrix} 2 & -2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\chi_1^{(3)}(P)\chi_1^{(3)}(Q) \leftrightarrow \sigma_3 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}. \tag{3.10}$$

To write those more shortly, we use the notation

$$\text{diag block}(A_1, A_2, \cdots, A_m) := \begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_m \end{bmatrix}, \tag{3.11}$$

where A_i is a square submatrix.

Using (3.11) we get

$$K_1^{(1)}(P, Q) \leftrightarrow \sigma_0+\sigma_1 = \text{diag block}\left[\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\right],$$

$$K_1^{(2)}(P, Q) \leftrightarrow \sigma_0+\sigma_1+\sigma_2 = \text{diag block}\left[4, 4, \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\right],$$

$$K_1^{(3)}(P, Q) \leftrightarrow \sigma_0+\sigma_1+\sigma_2+\sigma_3 = \text{diag block}(4, 4, 4, 4),$$

where diag block$(4, 4, 4, 4) = 4I_4$, I_n is $n\times n$ identity matrix. We denote the $m\times m$ zero-element matrix by 0_m below.

Since

$$\chi_2^{(i)}(P)\chi_2^{(i)}(Q) \leftrightarrow \text{diag}(4\sigma_i, 0_4, 0_4, 0_4), \ (i=1, 2, 3)$$

we get

$$K_2^{(i)}(P, Q) \leftrightarrow \text{ diag block}\left(4\sum_{j=0}^{i}\sigma_j, 4I_4, 4I_4, 4I_4\right),$$

$$K_2^{(3+i)}(P, Q) \leftrightarrow \text{ diag block}\left(4^2I_4, 4\sum_{j=0}^{i}\sigma_j, 4I_4, 4I_4\right),$$

$$K_2^{(6+i)}(P, Q) \leftrightarrow \text{ diag block}\left(4^2I_4, 4^2I_4, 4\sum_{j=0}^{i}\sigma_j, 4I_4\right),$$

and

$$K_2^{(9+i)}(P, Q) \leftrightarrow \text{ diag block}\left(4^2I_4, 4^2I_4, 4^2I_4, 4\sum_{j=0}^{i}\sigma_j\right), \ (i=1, 2, 3)$$

especially

$$K_2^{(12)}(P, Q) \leftrightarrow 4^2I_{16} = \text{diag block}(4^2I_4, 4^2I_4, 4^2I_4, 4^2I_4).$$

Suppose in the general case that

$$K_n^{(3\cdot 4^{n-1})}(P, Q) \leftrightarrow 4^nI_{4^n}. \tag{3.12}$$

By definitions of (2.1) and (3.8),

$$K_{n+1}^{(1)}(P, Q) \leftrightarrow \text{diag block}(K_{1,1}, K_{2,2}, \cdots, K_{4^n, 4^n})$$

where each $K_{i,i}$ is a 4×4 matrix. More precisely

$$\begin{aligned} &K_{n+1}^{(i)}(P, Q) \leftrightarrow \text{diag block}\left(4^n\sum_{t=0}^{i}\sigma_t, 4^nI_4, \cdots, 4^nI_4\right), \ (i=1, 2, 3), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &K_{n+1}^{(3j+i)}(P, Q) \leftrightarrow \text{ diag block}\left(4^{n+1}I_4, \cdots, 4^n\sum_{t=0}^{i}\sigma_t, \cdots, 4^nI_4\right), \end{aligned} \tag{3.13}$$

where the term $4^n\sum_{t=0}^{i}\sigma_t$ is the $(j+1)$-th block $(j=1, 2, \cdots, 4^n-1)$; in particular

$$K_{n+1}^{(3\cdot 4^n)}(P, Q) \leftrightarrow 4^{n+1}I_{4^{n+1}}.$$

Therefore for any $n=1, 2, \cdots$ (3.12) holds.

Suppose

$$a\in\Delta_{n,\,4(l-1)+i}\subset\Delta_{n-1,\,l},\qquad i=1,\,2,\,3,\,4.$$

By (3.9),

$$\mathscr{P}_n^{(j)}F(a)=\int_\Delta K_n^{(j)}(a,\,Q)F(Q)dQ. \tag{3.14}$$

By (3.13), (3.14) we obtain

$$\mathscr{P}_n^{(j)}F(a)=4^{n-1}\int_{\Delta_{n-1,l}}F(Q)dQ=\frac{1}{|\Delta_{n-1,\,l}|}\int_{\Delta_{n-1,l}}F(Q)dQ \tag{3.15}$$

for $j\leqslant 3(l-1)$ and

$$\mathscr{P}_n^{(j)}F(a)=4^n\int_{\Delta_{n,l}}F(Q)dQ=\frac{1}{|\Delta_{n,\,l}|}\int_{\Delta_{n,l}}F(Q)dQ \tag{3.16}$$

for $j\geqslant 3l$.

For $j=3l-2,\ 3l-1$ we have

$$\mathscr{P}_n^{(3l-2)}F(a)=\frac{2}{|\Delta_{n-1,\,l}|}\left(\int_{\Delta_{n,4(l-1)+i}}F(Q)dQ+\int_{\Delta_{n,4(l-1)+i+1}}F(Q)dQ\right) \tag{3.17}$$

($i=1$ or 3) and

$$\mathscr{P}_n^{(3l-1)}F(a)=\begin{cases}\dfrac{1}{|\Delta_{n,\,l}|}\displaystyle\int_{\Delta_{n,4(l-1)}}F(Q)dQ, & a\in\Delta_{n,\,4(l-1)+i},\ i=1,\,2,\\[2ex] \dfrac{2}{|\Delta_{n-1,\,l}|}\left(\displaystyle\int_{\Delta_{n,4(l-1)+3}}F(Q)dQ+\int_{\Delta_{n,4(l-1)+4}}F(Q)dQ\right), & i=3,\,4.\end{cases} \tag{3.18}$$

In any case, from (3.15) to (3.18) we conclude

$$\lim_{n\to\infty}\mathscr{P}_n^{(j)}F(a)=\frac{1}{|\Delta_a|}\int_{\Delta_a}F(Q)dQ=F(a), \tag{3.19}$$

where $\Delta_a\in\{\Delta_{m,\,i}\}$ and $a\in\Delta_a$, $|\Delta_a|\to 0$ when $n\to\infty$.

It is easy to check that

$$\int_\Delta|K_n^{(j)}(P,\,Q)|dQ=1. \tag{3.20}$$

Now (3.19), (3.20) imply the following theorem.

Theorem 3. *For* $F\in C(\Delta)$, $\lim\limits_{n\to\infty}\|\mathscr{P}_n^{(j)}F-F\|_\infty=0$ $(j=1,\,2,\,\cdots,\,3\cdot 4^{n-1})$.

4. On the Walsh System

Naturally there exist some different forms of definition which are equivalent. We use area coordinates to define the two-variable Walsh function. Some notations follow the case of the Haar functions.

$$\begin{aligned}
W_0(P)&:=1, && \text{for } P\in\Delta;\\
W_{n+1}^{(i)}(P)&:=W_n^{(i)}(Q), && \text{for } P\in\Delta,\ i=1,\,2,\,\cdots,\,4^n,\\
W_{n+1}^{(4^n+i)}(P)&:=\begin{cases}\lambda,\\-\lambda,\end{cases} && \begin{matrix}\text{for } P\in\Delta_1\cup\Delta_3,\\ \text{for } P\in\Delta_2\cup\Delta_4,\end{matrix}\\
W_{n+1}^{(2\cdot 4^n+i)}(P)&:=\begin{cases}\lambda,\\-\lambda,\end{cases} && \begin{matrix}\text{for } P\in\Delta_1\cup\Delta_2,\\ \text{for } P\in\Delta_3\cup\Delta_4,\end{matrix}\\
W_{n+1}^{(3\cdot 4^n+i)}(P)&:=\begin{cases}\lambda,\\-\lambda,\end{cases} && \begin{matrix}\text{for } P\in\Delta_1\cup\Delta_4,\\ \text{for } P\in\Delta_2\cup\Delta_3,\end{matrix}
\end{aligned} \tag{4.1}$$

where

$$\lambda := W_n^{(i)}(Q), \qquad Q\in\Delta;\ Q\sim P;\ i=1,\ 2,\ 3,\ \cdots,\ 4^n;\ n=0,\ 1,\ 2,\ \cdots,$$

$$W_0^{(1)}(P) := W_0(P) = 1, \qquad \text{for } P\in\Delta.$$

This finishes the definition of the sequence W.

Sometimes we prefer $W_l(l=1,\ 2,\ 3,\ \cdots,\ 4^{m+1})$ to $W_{m+1}^{(j\cdot 4^m+i)}$ with $l=j\cdot 4^m+i$ $(i=1,\ 2,\ \cdots,\ 4^m;\ j=0,\ 1,\ 2,\ 3)$.

At a point of discontinuity, the values of these functions are taken as the average.

Figures 2 and 3 show the Walsh sequence when $n=0,\ 1$.

Before the discussion of the orthogonality of the Walsh system we introduce the Hadamard matrix([2], [4], p. 207).

The Hadamard matrix is a square array whose elements consist only of $+1$ and -1 and whose rows (and columns) are orthogonal to one another. Obviously the lowest order nontrivial Hadamard matrix is of order two, viz.

$$H_2 := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{4.2}$$

Higher order matrices whose orders are powers of two can be obtained from the recurrent relationship

$$H_n = H_{n/2}\otimes H_2, \tag{4.3}$$

where $\otimes$ denotes the direct or Kronecker product and n is a power of two. The direct product means replacing each element in the matrix by H_2. With the help of the Hadamard matrix the one-dimensional Walsh function can be defined[1]. In the two-dimensional case we should use the 4×4 matrix

$$H_4 = H_2\otimes H_2$$

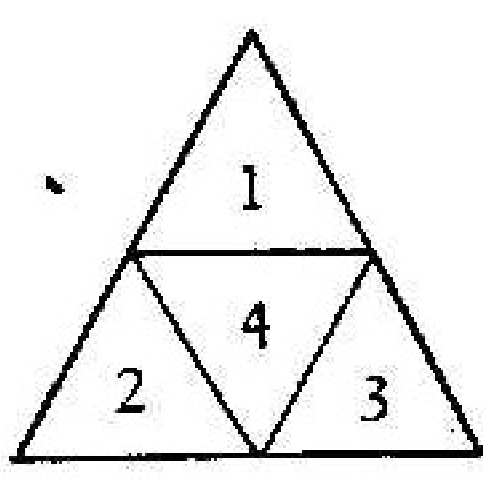
and get the recurrent relationship

$$H_N = H_{N/4}\otimes H_4. \tag{4.4}$$

The Hadamard matrix (4.4) corresponds to the Walsh sequence $\{W_i\}$ $(i=1,\ 2,\ \cdots,\ 4^N)$ for a given N.

Figures 2 and 3 show the Walsh sequence associated with the Hadamard matrix.

In Figures 2 and 3 black areas represent $+1$, and white areas -1. The following triangle shows a certain order.



$$n=0 \quad \begin{array}{c} W_1^{(1)}\leftrightarrow \\ W_1^{(2)}\leftrightarrow \\ W_1^{(3)}\leftrightarrow \\ W_1^{(4)}\leftrightarrow \end{array} \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix} =: H_4$$

(where we omit 1 in these elements of the Hadamard matrix)

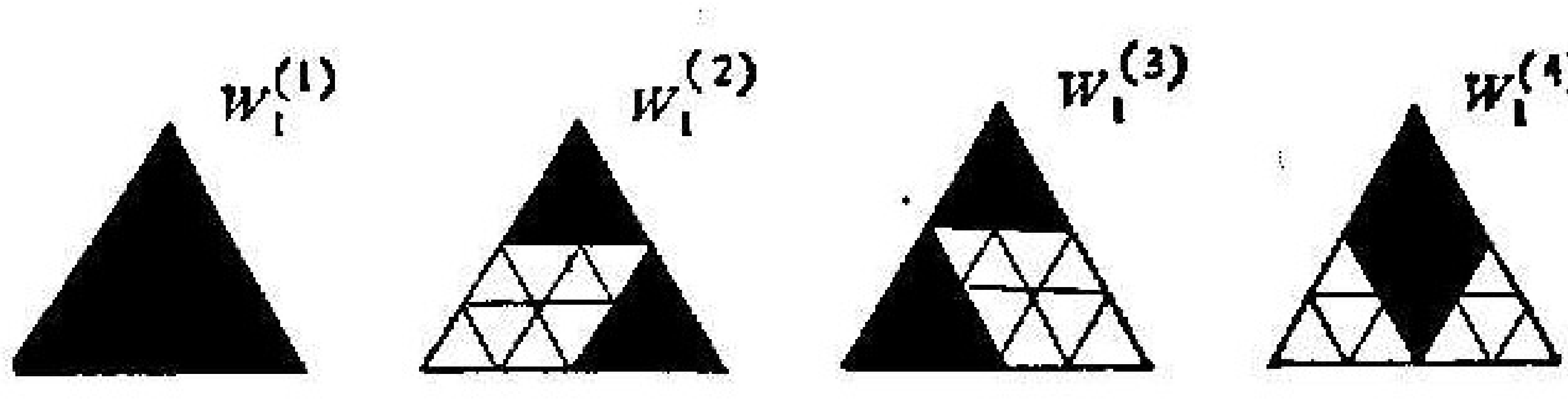
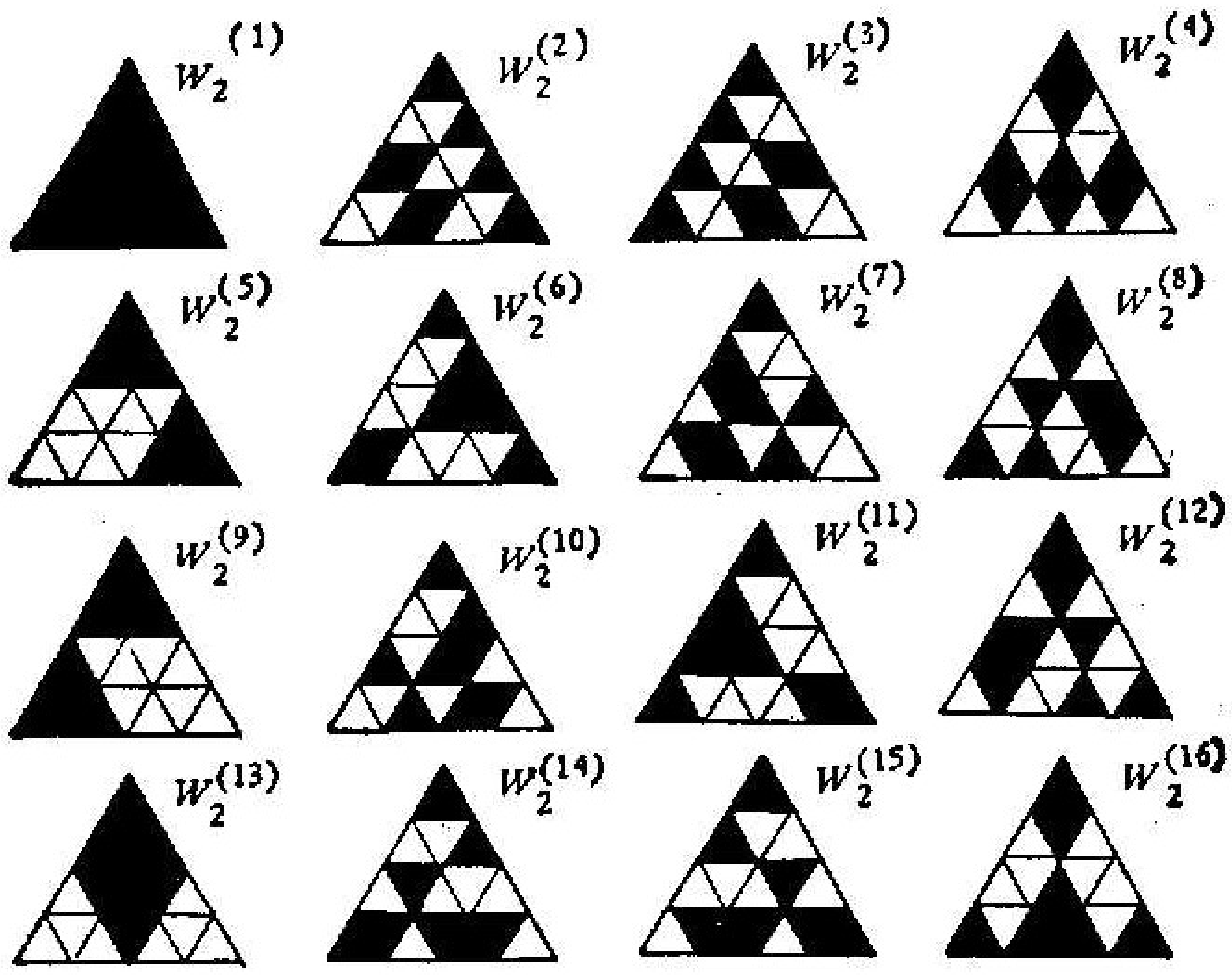


Fig. 2



$$
\begin{array}{ll cccc cccc cccc cccc l}
W_2^{(1)} & \leftrightarrow & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & \\
W_2^{(2)} & \leftrightarrow & + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & \\
W_2^{(3)} & \leftrightarrow & + & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - & \\
W_2^{(4)} & \cdot & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + & \\
W_2^{(5)} & \cdot & + & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - & \\
W_2^{(6)} & \cdot & + & - & + & - & - & + & - & + & + & - & + & - & - & + & - & + & \\
W_2^{(7)} & & + & + & - & - & - & - & + & + & + & + & - & - & - & - & + & + & \\
W_2^{(8)} & & + & - & - & + & - & + & + & - & + & - & - & + & - & + & + & - & = H_4 \otimes H_4 \\
W_2^{(9)} & & + & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - & \\
W_2^{(10)} & & + & - & + & - & + & - & + & - & - & + & - & + & - & + & - & + & \\
W_2^{(11)} & & + & + & - & - & + & + & - & - & - & - & + & + & - & - & + & + & \\
W_2^{(12)} & & + & - & - & + & + & - & - & + & - & + & + & - & - & + & + & - & \\
W_2^{(13)} & & + & + & + & + & - & - & - & - & - & - & - & - & + & + & + & + & \\
W_2^{(14)} & \cdot & + & - & + & - & - & + & - & + & - & + & - & + & + & - & + & - & \\
W_2^{(15)} & \cdot & + & + & - & - & - & - & + & + & - & - & + & + & + & + & - & - & \\
W_2^{(16)} & \cdot & + & - & - & + & - & + & + & - & - & + & + & - & + & - & - & + &
\end{array}
$$

Fig. 3

Now we notice that

$$\mathscr{P}_n F := \sum_{i=1}^{n} \alpha_i W_i$$

is the best L_2–approximation to a given function F from

$$\hat{M}_n := \operatorname{span}(W_i)_1^n,$$

where

$$\alpha_i := \int_\Delta F(P) W_i(P) dP.$$

Hence it is convergent to F if F is in L_2, since $\hat{M}_n$ is dense in L_2, i. e.

Theorem 4. *If $F \in L_2(\Delta)$, then $\lim_{n\to\infty} \|F - \mathscr{P}_n F\|_2 = 0$.*

Since $\hat{M}_{4^n} = M_n$ and from Theorem 3 we get the following theorem.

Theorem 5. *Let $F \in C(\Delta)$, $\mathscr{P}_{4^n}$ be L_2–projector onto M_{4^n} on Δ; then*

$$\lim_{n\to\infty}\|F-\mathscr{P}_{4^n}F\|_\infty=0.$$

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