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MAXIMUM NORM ESTIMATE, EXTRAPOLATION AND OPTIMAL POINTS OF STRESSES FOR THE FINITE ELEMENT METHODS ON THE STRONGLY REGULAR TRIANGULATION*

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Abstract

Under the condition that the triangulation of the given domain is strongly regular, the maximum norm estimate with accuracy $O(h^2)$ of the linear finite element approximation is obtained, the optimal points of stresses at the midpoints of common sides for all adjacent elements are shown, and the estimate with higher accuracy for the extrapolation approximation based on mesh refinement and extrapolation is given.

§1. Introduction

The L^∞-error estimates of the finite element approximations for second order linear elliptic boundary value problems have been established by Frehse, Nitsche, Rannacher, Scott, et al. Fried has published an example which indicates that the pointwise estimate

$$\|u-u^h\|_{0,\infty}\leqslant ch^2\ln\frac{1}{h}\|u\|_{2,\infty} \tag{1}$$

may be of optimal order in the usual case. However, if some restrictive assumptions are imposed, then the convergence order can be improved. As an example, when the triangulation of the given domain is strongly regular (see [5]—[7] or the next section), and $u\in H^3(\Omega)\cap W^2_\infty(\Omega)$, the following result is obtained for the linear finite element approximation in [7]:

$$\|u-u^h\|_{0,\infty,D}\leqslant ch^2\left(\ln\frac{1}{h}\right)^{1/2}[\|u\|_{2,\infty,D}+\|u\|_{3,\Omega}], \tag{2}$$

where $D\subseteq\Omega$.

In the present paper, we shall prove in section 2 the following

Theorem 1. *If the triangulation Π_h of the given domain Ω is strongly regular and $u\in W^3_\infty(\Omega)\cap H^1_0(\Omega)$, then the pointwise accuracy of the linear finite element approximation u^h will be*

$$\max_{p\in\Omega}|u(p)-u^h(p)|\leqslant ch^2\|u\|_{3,\infty}. \tag{3}$$

One of the new developments in finite element analysis is the investigation of the phenomena of superconvergence. Obviously, it is of interest to improve the accuracy of stresses by using the optimal points of stresses. The superconvergence

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estimate for the gradient with accuracy $O(h^2)$ has been obtained in [5, 6] by using the means of gradients for two adjacent elements as the approximations to the gradient at the midpoints of common sides for some elements. However, it was not proved that there exist the optimal points of stresses for all elements and it was not stated where the elements which have the optimal points of stresses are located. In [8], the above result was improved and the inner superconvergence estimates of gradient for the optimal point of stresses was obtained. In section 3, we shall prove the following.

Theorem 2. *If the triangulation Π_h is strongly regular and $u\in C^3(\Omega)\cap H_0^1(\Omega)$, then the midpoints of common sides for all adjacent elements are the entirely optimal points of stresses with accuracy $O\left(h^2\ln\frac{1}{h}\right)$.*

In the last section, the extrapolation for the finite element approximations is considered. The mesh refinement for the triangulation Π_h of Ω should be a new triangulation which is achieved via dividing each triangle of Π_h into four small equal triangles. Let u^h be the linear finite element approximation over the triangulation Π_h and $u^{h/2}$ the new approximation over a new triangulation. The numerical results show that the accuracy of the extrapolation approximation $\frac{1}{3}(4u^{h/2}-u^h)$ is much better than $u^{h/2}$. However, the theoretical basis for this algorithm still remains an open question, and we will try to give some answer to this question. We prove.

Theorem 3. *Assume that the triangulation Π_h is strongly regular and $u\in C^4(\Omega)\cap H_0^1(\Omega)$. Let u^h, $u^{h/2}$ be the linear finite element approximation over Π_h and the refinement respectively. We have for the nodes of Π_h*

$$u-\frac{1}{3}(4u^{h/2}-u^h)=O\left(h^3\ln\frac{1}{h}\right). \tag{4}$$

§ 2. Maximum Norm Estimate

For simplicity we shall consider the 2-dimensional Poisson equation

$$\begin{aligned}-\Delta u&=f \quad \text{in } \Omega,\\ u&=0 \quad \text{on } \partial\Omega.\end{aligned} \tag{5}$$

Suppose that the triangulation Π_h of $\Omega_h(\subseteq\Omega)$ is strongly regular, i. e. Π_h satisfies the following conditions:

c1: Each triangle $T\in\Pi_h$ contains a circle of radius c_1h and is contained in a circle of radius c_2h, $0<c_1<c_2$ independent of h and T (quasi-uniform).

c2: Any two adjacent triangles of Π_h form an approximate parallelogram, i. e. there exists a constant c independent of h, such that (see Fig. 1)

$$|\overline{p_1p_2}-\overline{p_3p_4}|\leqslant ch^2. \tag{6}$$

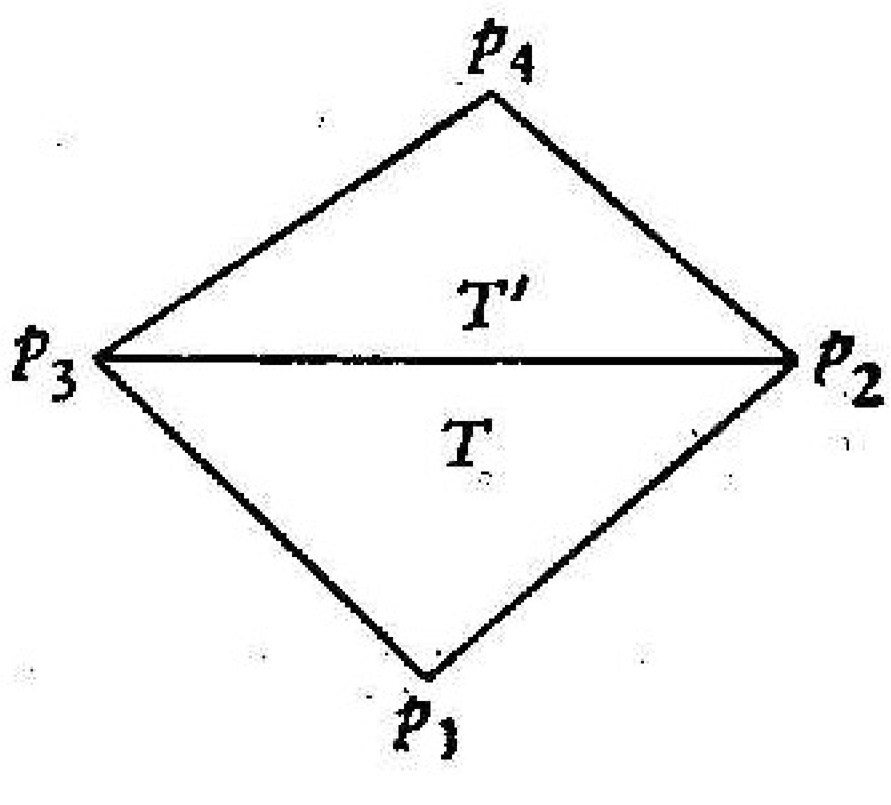

Fig. 1

Let S_h be the piecewise linear finite element space on Ω_h with zero on $\Omega\backslash\Omega_h$, $u^h\in S_h$ the finite element approximation satisfying

$$a(u^h, v_h)=(f, v_h), \quad \forall v_h\in S_h, \tag{7}$$

where $a(u, v)=(\nabla u, \nabla v)$, and u^I the interpolation of u on S_h.

By [2], for any $z_0\in\Omega$, there exist an element $T\in\Pi_h$ and the linear weight function $\omega(\lambda_1, \lambda_2, \lambda_3)$ on T, such that $z_0\in T$ and the relation[1)]

$$\int_T \omega(\lambda_1, \lambda_2, \lambda_3)v(\lambda_1, \lambda_2, \lambda_3)dz=v(z_0) \tag{8}$$

holds uniformly for any linear function v on T, where $(\lambda_1, \lambda_2, \lambda_3)$ is the borycentric coordinates. We will consider the function $\delta=\omega x_T$, where x_T is the characteristic function of T, and the equation

$$\begin{aligned} -\Delta g&=\delta \quad \text{in} \quad \Omega, \\ g&=0 \quad \text{on} \quad \Omega, \end{aligned} \tag{9}$$

where the regularized Green's function g corresponds to G^h in [2].

Suppose that $u\in W^3_\infty(\Omega)\cap H^1_0(\Omega)$. Thus $u\in C^2(\Omega)$ and we have

$$a(g, u^h)=(\delta, u^h)=u^h(z_0), \tag{10}$$

$$\begin{aligned} a(g, u)=(\delta, u)=\int_T \omega u\,dz=\int_T \omega\Big[u(z_0)+\sum_{|\alpha|=1}(z-z_0)^\alpha u^{(\alpha)}(z_0) \\ +\sum_{|\alpha|=2}\frac{1}{2!}u^{(\alpha)}(p)(z-z_0)^\alpha\Big]dz=u(z_0)+O(h^2). \end{aligned} \tag{11}$$

Combining (10) with (11) we obtain

$$a(g, u-u^h)=u(z_0)-u^h(z_0)+O(h^2). \tag{12}$$

Note that the left side of (12) can be written as

$$a(g, u-u^h)=a(g-g^I, u^I-u^h)+a(g-g^I, u-u^I). \tag{13}$$

By using the superconvergence estimate given in (30), (40), etc., we obtain

$$|a(g-g^I, u^I-u^h)|=|a(g^h-g^I, u-u^I)|\leqslant ch^2\|u\|_{3,\infty}\|g^h-g^I\|_{1,1}, \tag{14}$$

where g^h is the Ritz projection of g. According to the results for the regularized Green's function in [2] we have

$$\|g^h-g^I\|_{1,1}\leqslant\|g-g^h\|_{1,1}+\|g-g^I\|_{1,1}\leqslant ch\ln\frac{1}{h}. \tag{15}$$

Hence

$$|a(g-g^I, u^I-u^h)|\leqslant ch^3\ln\frac{1}{h}\|u\|_{3,\infty}. \tag{16}$$

On the other hand, we have

$$\begin{aligned} a(g-g^I, u-u^I)&=-\int_\Omega (u-u^I)\Delta g dz-a(g^I, u-u^I) \\ &=\int_T \omega(u-u^I)dz-a(g^I, u-u^I)\equiv J_1+J_2. \end{aligned} \tag{17}$$

Obviously,

$$|J_1|\leqslant ch^2. \tag{18}$$

By using (30), (40), etc.,

$$|J_2|\leqslant ch^2\|u\|_{3,\infty}\|g^I\|_{1,1}$$

and from [2],

$$\|g\|_{1,1}=O(1).$$

1) For instance, $\omega(\lambda_1, \lambda_2, \lambda_3)=\sum_{i=1}^{3}(12\lambda_i^2-3)\lambda_i/\text{meas}\,T$.

Hence

$$\|g^I\|_{1,1}=O(1). \tag{19}$$

Thus we obtain

$$|J_2|\leqslant ch^2\|u\|_{3,\infty}. \tag{20}$$

Combine (12), (13), (16), (17), (18) with (20) and note that z_0 is any point in Ω. Then Theorem 1 is proved.

§ 3. Optimal Points of Stresses

Let M be the midpoint of common sides for any two adjacent triangles of Π_h (Fig. 2) and $u\in C^3(\Omega)\cap H_0^1(\Omega)$. We will prove that M is the optimal point of stresses in the sense of [5, 6].

We now employ the arguments of § 2 repeatedly. Let $z_0\in\Omega$ be a node of any element. Then there exist $T\in\Pi_h$, the function ω on T and the corresponding regularized function g, such that

$$\begin{aligned}u(z_0)-u^h(z_0)&+\sum_{|\alpha|=2}\frac{1}{2!}u^{(\alpha)}(z_0)\int_T\omega(z-z_0)^\alpha dz+O(h^3)\\&=a(g^I-g^h,\ u-u^I)+a(g-g^I,\ u-u^I)\\&=\int_T\omega(u-u^I)dz-a(g^I,\ u-u^I)+O\Big(h^3\ln\frac{1}{h}\Big).\end{aligned} \tag{21}$$

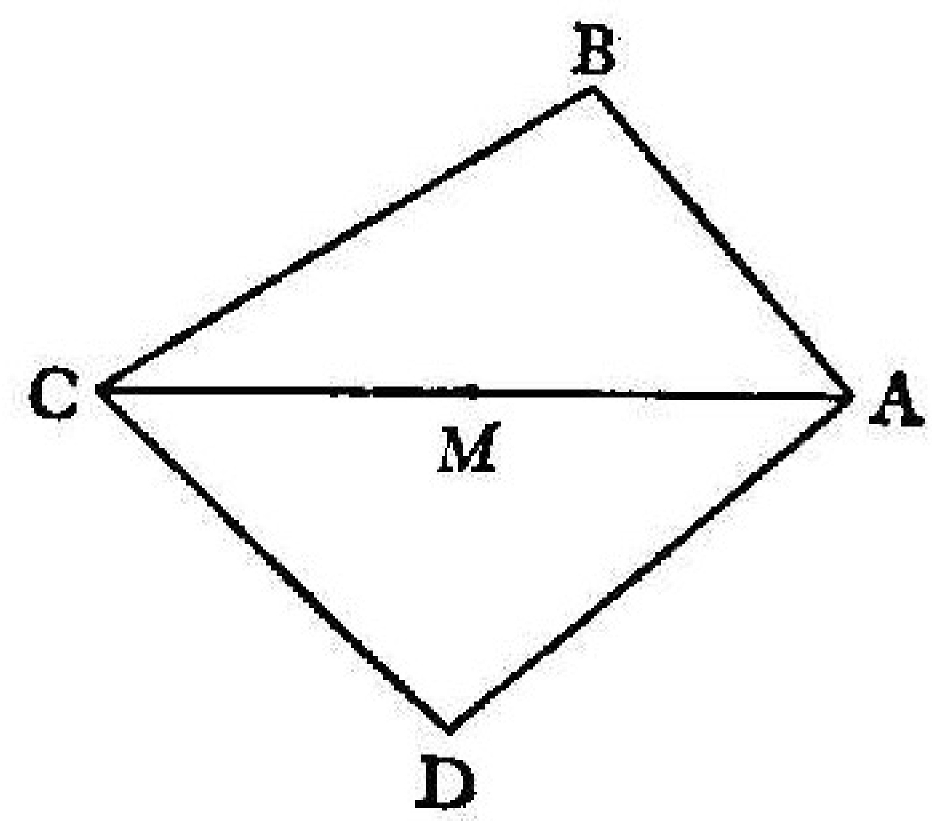

Fig. 2

But

$$\begin{aligned}\int_T\omega(u-u^I)dz&=\int_T\omega\Big[u(z_0)+\sum_{|\alpha|=1}u^{(\alpha)}(z_0)(z-z_0)^\alpha+\sum_{|\alpha|=2}\frac{1}{2!}u^{(\alpha)}(z_0)(z-z_0)^\alpha\Big]dz\\&\quad+O(h^3)-u^I(z_0)=\frac{1}{2}\sum_{|\alpha|=2}u^{(\alpha)}(z_0)\int_T\omega(z-z_0)^\alpha dz+O(h^3).\end{aligned} \tag{22}$$

Substituting (22) into (21) we obtain, for z_0 being a node,

$$u(z_0)-u^h(z_0)=-a(g^I,\ u-u^I)+O\Big(h^3\ln\frac{1}{h}\Big). \tag{23}$$

In order to emphasize that g is dependent on z_0, we replace g with $g(z_0,\ z)$. We first consider the directional derivative along the edge $\overline{CA}$ of the point M shown in Fig. 2:

$$D_{CA}u(M)=\frac{u(A)-u(C)}{\overline{AC}}+O(h^2) \tag{24}$$

and

$$D_{CA}u^h(M)=\frac{u^h(A)-u^h(C)}{\overline{AC}}. \tag{25}$$

Replacing z_0 with A and C in (23) respectively we obtain

$$\begin{aligned}|D_{CA}u(M)-D_{CA}u^h(M)|&=\Big|\frac{1}{\overline{AC}}\,a(u-u^I,\ g^I(C,\ z)-g^I(A,\ z))\Big|\\&\quad+O\Big(h^2\ln\frac{1}{h}\Big)\leqslant ch\|u\|_{3,\infty}\|g^I(C,\ z)-g^I(A,\ z)\|_{1,1}+O\Big(h^2\ln\frac{1}{h}\Big).\end{aligned}$$

Note that ω_{z_0} defined by (8) is continuously dependent on z_0 for $z_0\in T$. According to [2] and using the continuity of ∇g at $\overline{AC}$ we have

$$\|g^I(C,z)-g^I(A,z)\|_{1,1}\leqslant\|g^I(C,z)-g(C,z)\|_{1,1}+\|g(C,z)-g(A,z)\|_{1,1}$$
$$+\|g(A,z)-g^I(A,z)\|_{1,1}\leqslant\|g(C,z)-g(A,z)\|_{1,1}+O\Big(h\ln\frac{1}{h}\Big)$$
$$\leqslant C\cdot\overline{AC}\|g(p,z)\|_{2,1}+O\Big(h\ln\frac{1}{h}\Big)=O\Big(h\ln\frac{1}{h}\Big),$$

where $p\in\overline{AC}$. Therefore, we obtain

$$|D_{CA}u(M)-D_{CA}u^h(M)|=O\Big(h^2\ln\frac{1}{h}\Big). \tag{26}$$

Next, for the directional derivative along the edge $\overline{MB}$, let us take the mean of gredients of u^h for two adjacent elements as the approximation of gradient of u^h at the point M. Thus we have

$$D_{MB}u^h(M)=\frac{1}{2}\Big[\frac{u^h(B)-u^h(M)}{\overline{MB}}+\frac{u^h(M)-u^h(D)}{\overline{DM}}\Big]=\frac{u^h(B)-u^h(D)}{\overline{DB}}+O(h^2) \tag{27}$$

because the union of two adjacent elements is an approximate parallelogram. Using the same arguments as above we obtain

$$|D_{MB}u(M)-D_{MB}u^h(M)|=O\Big(h^2\ln\frac{1}{h}\Big). \tag{28}$$

Since the accuracies of the directional derivatives at the point M along two different directions are $O\Big(h^2\ln\frac{1}{h}\Big)$, the superconvergence accuracy $O\Big(h^2\ln\frac{1}{h}\Big)$ holds for all directional derivatives at the point M, and then Theorem 2 is proved.

§ 4. Extrapolation for the Finite Element Approximations

Suppose that $u\in C^4(\Omega)\cap H_0^1(\Omega)$, z_0 being a node of any element. Thus we have by (23)

$$u^h(z_0)-u(z_0)=\sum_i\int_{\partial T_i}(u-u^I)\frac{\partial g^I}{\partial n}\,ds+O\Big(h^3\ln\frac{1}{h}\Big). \tag{29}$$

In order to estimate the first term on the right side of (29), we will employ the symbols and notations in [5, 6] (except for the notes below).

Let T be a standard element of Π_h (Fig. 1). We have

$$\sum_i\int_{\partial T_i}(u-u^I)\frac{\partial g^I}{\partial n}\,ds=\sum\int_{\overline{p_3p_2}}(u-u^I)\Big[\frac{\partial\bar g^I}{\partial n}-\frac{\partial g^I}{\partial n}\Big]ds$$
$$\doteq\sum\int_{\overline{p_3p_2}}(u-u^I)\sin\theta_{23}(-\bar g_x^I+g_x^I)ds+\sum\int_{\overline{p_3p_2}}(u-u^I)\cos\theta_{23}(\bar g_y^I-g_y^I)ds$$
$$\equiv J_3+J_4, \tag{30}$$

where $\sum$ denotes the sum in which $\overline{p_3p_2}$ runs though the common sides of all two adjacent triangles, θ_{23} is the directional angle of $\overline{p_2p_3}$, and $\bar g_x^I=(g^I)'_x$.

Note that g^I is the piecewise linear function and that

$$\bar g_x^I-g_x^I=-\eta_1\frac{\delta^2g^I}{\Box_{23}}+\frac{\delta^2y}{\Box_{23}}\cdot w_1+\frac{s'-s}{\Box_{23}}(\bar g_x^I+g_x^I), \tag{31}$$

where $\Box_{23}=s+s'$ denotes the area of the approximate parallelogram $(T+T')$ and $w_1=g_2^I-g_3^I$. Thus, by using the remainder expression of the trapezoid quadrature formula for $u-u^I$, we obtain

$$J_3=\sum\frac{l_{23}^3}{12}\left[\frac{\partial^2u(M_{23})}{\partial s_{23}^2}+O(l_{23}^2)\right]\cdot\sin\theta_{23}\left[\frac{\delta^2y}{\Box_{23}}\cdot w_1+\frac{s'-s}{\Box_{23}}(\bar g_x^I+g_x^I)\right]$$
$$+\sum\int_{\overline{p_3p_1}}(u-u^I)\cdot\eta_1\frac{\delta^2g^I}{\Box_{23}}\cdot\sin\theta_{23}\,ds$$
$$=\sum\frac{l_{23}^3}{12}\frac{\partial^2u(M_{23})}{\partial s_{23}^2}\sin\theta_{23}(s'-s)(\bar g_x^I+g_x^I)/\Box_{23}$$
$$+\sum\frac{l_{23}^3}{12}\frac{\partial^2u(M_{23})}{\partial s_{23}^2}\sin\theta_{23}\cdot\delta^2y\cdot w_1/\Box_{23}$$
$$+\sum\int_{\overline{p_3p_1}}(u-u^I)\cdot\eta_1\frac{\delta^2g^I}{\Box_{23}}\cdot\sin\theta_{23}\,ds+O(h^3)\equiv J_5+J_6+J_7+O(h^3),\quad(32)$$

where $l_{23}=\overline{p_2p_3}$, and M_{23} is the midpoint of side $\overline{p_2p_3}$.

In order to calculate $u^{h/2}$, we need mesh refinement for Π_h, where an approximate parallelogram $\Box_{k1}$ is divided into four small approximate parallelograms $\Box_{k1}^{(\lambda)}(\lambda=1, 2, 3, 4)$. As shown in Fig. 3, $\Box_{23}^{(2)}$, $\Box_{23}^{(4)}$ are the parallelograms and the small approximate parallelograms $\Box_{23}^{(1)}$, $\Box_{23}^{(3)}$ are similar contraction with respect to $\Box_{23}$. To prove (4), it is sufficient to show that

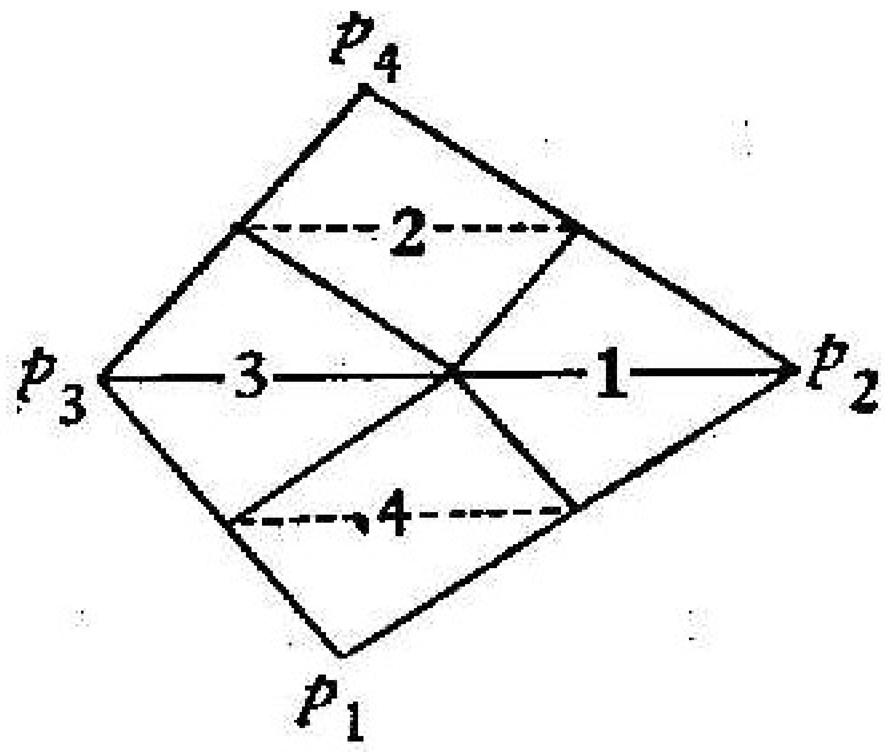

Fig. 3

$$4\sum_{\lambda=1}(J_3^{(\lambda)}+J_4^{(\lambda)})-J_3-J_4=O\left(h^3\ln\frac{1}{h}\right),\quad(33)$$

where the super script λ denotes the terms corresponding to (30), (32) obtained by the mesh refinement.

We now consider J_5. Since $\Box_{k1}^{(2)}$ and $\Box_{k1}^{(4)}$ are the parallelograms, $\delta^2y=s'-s=0$; thus

$$J_5^{(2)}=J_5^{(4)}=0.\quad(34)$$

Next, let the constant c_{23} satisfy

$$\frac{1}{12}l_{23}^3[\sin\theta_{23}(s'-s)/\Box_{23}]=c_{23}l_{23}^2\Box_{23}/8.\quad(35)$$

Obviously, the values in $[\cdots]$ are invariant under the similar contraction, and $\Box_{k1}^{(\lambda)}$ $(\lambda=1, 3)$ are similar contraction with respect to $\Box_{23}$. Thus

$$J_5^{(\lambda)}=\sum\frac{1}{12}\left(\frac{l_{23}}{2}\right)^3\frac{\partial^2u(M_{23})}{\partial s_{23}^2}\cdot\sin\theta_{23}(s'-s)(\bar g_x^{I/2}+g_x^{I/2})/\Box_{23}$$
$$=\frac{1}{8}\sum c_{23}l_{23}^2\int_{\Box_{k1}^{(\lambda)}}\frac{\partial^2u(z)}{\partial s_{23}^2}g_x^{I/2}\,dz+O(h^3)$$
$$=\frac{1}{8}\sum c_{23}l_{23}^2\int_{\Box_{k1}^{(\lambda)}}\frac{\partial^2u(z)}{\partial s_{23}^2}G_x(z_0, z)dz+O\left(h^3\ln\frac{1}{h}\right),\quad\lambda=1, 3,\quad(36)$$

where $\bar g_x^{I/2}$, $g_x^{I/2}$ denote the derivatives of the interpolating function of g on the refined mesh, $G(z_0, z)$ is Green's function with respect to (5), and the last step in (36) is obtained by $g^h=G^h$, (15), and

$$\|G-G^h\|_{1,1}=O\left(h\ln\frac{1}{h}\right).$$

On the other hand, by $M_{23}=\frac{1}{2}(M_{23}^{(1)}+M_{23}^{(3)})$, we have

$$
\begin{aligned}
J_5 &= \sum c_{23} l_{23}^2 \square_{23} \frac{\partial^2 u(M_{23})}{\partial s_{23}^2} (\bar{g}_x^I + g_x^I)/8 \\
&= \sum c_{23} l_{23}^2 \square_{23} \left(\frac{\partial^2 u(M_{23}^{(1)})}{\partial s_{23}^2} + \frac{\partial^2 u(M_{23}^{(3)})}{\partial s_{23}^2} \right) (\bar{g}_x^I + g_x^I)/16 + O\left(h^3 \ln \frac{1}{h}\right) \\
&= \frac{1}{2} \sum c_{23} l_{23}^2 \left[\sum_{\lambda=1,3} \int_{\square_{23}^{(\lambda)}} \frac{\partial^2 u(z)}{\partial s^2} G_x(z_0, z)\, dz + O\left(h^3 \ln \frac{1}{h}\right) \right. \\
&= 4(J_5^{(1)} + J_5^{(3)}) + O\left(h^3 \ln \frac{1}{h}\right).
\end{aligned}
$$

Therefore,

$$
4(J_5^{(1)} + J_5^{(3)} + J_5^{(2)} + J_5^{(4)}) - J_5 = O\left(h^3 \ln \frac{1}{h}\right). \tag{37}
$$

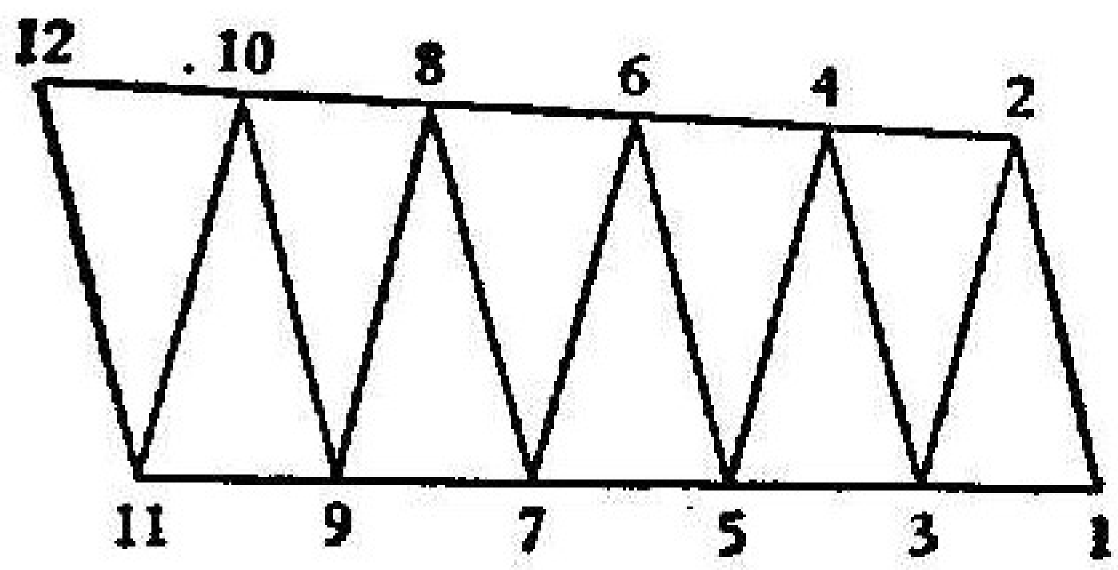

Fig. 4

Note that $J_6^{(2)} = J_6^{(4)} = 0$ and $[\sin \theta_{23} \cdot \delta^2 y w_1] / \square_{23}$ is invariant under the similar contraction. Repeating the above arguments, we can obtain

$$
4 \sum_{\lambda=1}^{4} J_6^{(\lambda)} - J_6 = O\left(h^3 \ln \frac{1}{h}\right). \tag{38}
$$

Now consider J_7. We will employ the techniques of united elements in [5, 6]. As an example, consider the case shown by Fig. 4. We obtain

$$
\begin{aligned}
J_7 &= \sum \frac{\delta^2 g^I}{\square_{23}} \int_{\overline{p_2 p_3}} (u - u^I) \eta_1 \sin \theta_{23}\, ds \\
&= \frac{1}{\square_{23}} (g_2^I - g_1^I) \int_{\overline{p_2 p_3}} + \frac{1}{\square_{43}} (g_4^I - g_3^I) \left[\int_{\overline{p_5 p_4}} - \int_{\overline{p_2 p_3}} \right] + \cdots \\
&= \frac{1}{\square_{43}} (g_4^I - g_3^I) \left[\int_{\overline{p_5 p_4}} - \int_{\overline{p_2 p_3}} \right] + \cdots \\
&= -(g_4^I - g_3^I) \left[\frac{l_{23}^3}{12 \square_{43}} + O(l_{23}^3) \right] \left[\frac{\partial^2 u(M_{45})}{\partial s_{23}^2} \sin \theta_{45} \eta_1^{(2)} - \frac{\partial^2 u(M_{23})}{\partial s_{23}^2} \sin \theta_{23} \eta_1^{(1)} \right] + \cdots
\end{aligned} \tag{39}
$$

where the values of η_1 on $\square_{23}$ and $\square_{45}$ are denoted by $\eta_1^{(1)}$, $\eta_1^{(2)}$ respectively. Note that

$$
|\sin \theta_{45} \eta_1^{(2)} - \sin \theta_{23} \eta_1^{(1)}| = O(l_{43}^2).
$$

Thus

$$
\begin{aligned}
J_7 &= -\left(l_{43} \frac{\partial}{\partial s_{43}} g^I\right) \sin \theta_{45} \cdot \eta_1^{(2)}\, \overline{M_{23} M_{45}}\, \frac{\partial^3 u(M_{43})}{\partial s_{23}^2 \partial t_{23}} \cdot \frac{l_{23}^3}{12 \square_{43}} + \cdots + O\left(h^3 \ln \frac{1}{h}\right) \\
&= c_{43} l_{23}^2 \int_{\square_{43}} \frac{\partial}{\partial s_{43}} g^I \frac{\partial^3 u(z)}{\partial s_{23}^2 \partial t_{23}}\, dz + \cdots + O\left(h^3 \ln \frac{1}{h}\right) \\
&= c_{43} l_{23}^2 \int_{\square_{43}} \frac{\partial}{\partial s_{43}} G(z_0, z) \frac{\partial^3 u(z)}{\partial s_{23}^2 \partial t_{23}}\, dz + \cdots + O\left(h^3 \ln \frac{1}{h}\right),
\end{aligned} \tag{40}
$$

where $\frac{\partial}{\partial t_{23}}$ denotes the directional derivative along $\overline{M_{23} M_{45}}$ and the constants $c_{43}, \cdots$ are invariant under the similar contraction. Hence, via refinement, we have $c_{ij} = c_{ij}^{(\lambda)} (\lambda = 1, 3)$ since $\square_{ij}^{(\lambda)} (\lambda = 1, 3)$ are the similar contraction with respect to $\square_{ij}$, and $c_{ij} = c_{ij}^{(\lambda)} + O(l_{ij}) (\lambda = 2, 4)$ since $\square_{ij}^{(\lambda)} (\lambda = 2, 4)$ are the approximate parallelograms. Therefore, we obtain

$$
4 \sum_{\lambda=1}^{4} J_7^{(\lambda)} - J_7 = O\left(h^3 \ln \frac{1}{h}\right). \tag{41}
$$

Combining (37), (38) with (41) we conclude that

$$4\sum_{\lambda=1}^{4} J_3^{(\lambda)} - J_3 = O\left(h^3 \ln \frac{1}{h}\right). \tag{42}$$

Similarly, we have

$$4\sum_{\lambda=1}^{4} J_4^{(\lambda)} - J_4 = O\left(h^3 \ln \frac{1}{h}\right). \tag{43}$$

Combining (42) with (43), we obtain (33).

Note from (29) that

$$u^h(z_0) - u(z_0) = J_3 + J_4 + O\left(h^3 \ln \frac{1}{h}\right)$$

and

$$u^{h/2}(z_0) - u(z_0) = \sum_{\lambda=1}^{4} J(_3^{(\lambda)} + J_4^{(\lambda)}) + O\left(h^3 \ln \frac{1}{h}\right).$$

Therefore (4) holds. However the constant in (4) is not only dependent on the original division and this will be left to a separate paper.

Finally, we remark that the extrapolation procedure is also effective for the elliptic eigenvalue problem:

$$\lambda - \frac{4}{3}\lambda_{h/2} + \frac{1}{3}\lambda_h = O(h^4).$$

The proof is based on the estimates

$$\lambda_h(p_h u, u_h) = \lambda(u, u_h) = \Lambda,$$

$$\lambda - \lambda_h = a(u - p_h u, u - p_h u) + O(h^4) = a(u - u^I, u - u^I) + O(h^4),$$

the Green formula for $a(u-u^I, u-u^I)$ and (30), (32), etc. We will discuss it in detail in a separate paper.

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Remark. There is a simple proof for Theorem 3 in the case of parallelogram meshes. See Research Report IMS-10, Chengdu Branch of Academia Sinica, 1983.

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