Journal of Computational Mathematics
Vol.31, No.1, 2013, 88–106.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1210-m3966

A CHARACTERISTIC FINITE ELEMENT METHOD FOR CONSTRAINED CONVECTION-DIFFUSION-REACTION OPTIMAL CONTROL PROBLEMS*

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Abstract

In this paper, we develop a priori error estimates for the solution of constrained convection-diffusion-reaction optimal control problems using a characteristic finite element method. The cost functional of the optimal control problems consists of three parts: The first part is about integration of the state over the whole time interval, the second part refers to final-time state, and the third part is a regularization term about the control. We discretize the state and co-state by piecewise linear continuous functions, while the control is approximated by piecewise constant functions. Pointwise inequality function constraints on the control are considered, and optimal a L^2-norm priori error estimates are obtained. Finally, we give two numerical examples to validate the theoretical analysis.

Mathematics subject classification: 49J20, 65M15, 65M25, 65M60.
Key words: Characteristic finite element method, Constrained optimal control, Convection-diffusion-reaction equations, Pointwise inequality constraints, A priori error estimates.

1. Introduction

Optimal control problems governed by convection-diffusion equations arise in many scientific and engineering applications, such as atmospheric pollution control problems [1, 2]. Efficient numerical methods are essential to successful applications of such optimal control problems. To the best of the authors' knowledge, recently there are some growing published results on optimal control problems governed by steady convection-diffusion equations; see [3,4] of SUPG method, [5] of the standard finite element discretizations with stabilization, [6] of symmetric stabilization method, [7] of edge-stabilization method, [8] of the application of RT mixed DG scheme, [9] of domain decomposition method and so on. However, for the approximation of constrained optimal control problems governed by time-dependent convection-diffusion equations, it is much more complicated and only a few paper has been published, see [10–13] for example. Systematic

* Received December 3, 2011 / Revised version received July 2, 2012 / Accepted October 25, 2012 / Published online January 17, 2013 /

introductions of the finite element method for PDEs and optimal control problems can be found in, for example, [14–17].

In many time-dependent optimal control problems, people are usually interested in an optimization of the final-time state $y(x,T)$. Therefore, in this paper we consider the cost functional consisting of three parts: The first part is about integration of the state over the whole time interval, the second part refers to final-time state, and the third part is a regularization term about the control. Besides, we discuss the pointwise inequality function constraints on the control. In what follows, we shall study in details the following convection-diffusion-reaction state equations:

$$\begin{aligned}&\partial_t y(x,t) - \mu\Delta y(x,t) + \mathbf{a}(x)\cdot\nabla y(x,t) + c(x)y(x,t)\\ &= f(x,t) + u(x,t) \quad \text{in } \Omega\times(0,T],\end{aligned} \tag{1.1}$$

combined with the following boundary and initial conditions

$$y(x,t) = 0 \quad \text{on } \partial\Omega\times(0,T], \quad y(x,0) = y_0(x) \quad \text{in } \Omega, \tag{1.2}$$

and

$$\alpha(x,t) \le u(x,t) \le \beta(x,t) \quad \text{a.e. in } \Omega\times[0,T], \tag{1.3}$$

where $\mathbf{a}$, c, f, α, β are given functions, $\mu > 0$ is a constant diffusion coefficient. Detailed assumptions for model problems will be introduced in Section 2.

It is well known that for the above convection-diffusion-reaction equations standard finite element discretization may not work. The methods of characteristics [19, 20] combine the convection and capacity terms in the governing equations, to carry out the temporal discretization in a Lagrange coordinate. These methods are symmetric and stable, even if large time steps and coarse spatial meshes are used. Thus, in this work we apply a characteristic finite element method to constrained optimal control problems governed by convection-diffusion-reaction equations. Pointwise inequality constraints on the control are also considered, and we obtain optimal a L^2-norm error estimates for both the control and state approximations.

The outline of the paper is as follows: In Section 2, we review the model problems and derive the continuous optimality conditions. In Section 3, we describe the characteristic finite element discretization of (1.1)-(1.3) and formulate a corresponding discretized optimality conditions. In Section 4, we prove a L^2-norm error estimates for the optimal control problems with control constraints. In Section 5, some numerical experiments are presented to observe the convergence behavior of the proposed numerical scheme.

In this paper, we denote C and δ be a generic constant and small positive number, which are independent of the discrete parameters and may have different values in different circumstances, respectively.

2. Model Problems and Optimality Conditions

Let Ω be a bounded open set in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. Just for simplicity of presentation, we assume that Ω is a convex polygon. Throughout this paper, we use the standard notations $L^p(\Omega)$ $(1 \le p \le \infty)$ for Lebesgue space of real-valued functions with norm $\|\cdot\|_{L^p(\Omega)}$, and $W^{m,p}(\Omega)$ $(1 \le p \le \infty)$ for Sobolev spaces endowed with the norm $\|\cdot\|_{W^{m,p}(\Omega)}$ and semi-norm $|\cdot|_{W^{m,p}(\Omega)}$. For $p = 2$, we denote $\|\cdot\|_{L^2(\Omega)} = \|\cdot\|$, $W^{m,2}(\Omega) = H^m(\Omega)$ and we drop the subscript $p = 2$ in the corresponding norms and semi-norms. We also denote by

$L^s(0,T;W^{m,p}(\Omega))$ the Banach space of all L^s integrable functions from $(0,T)$ into $W^{m,p}(\Omega)$ with norm

$$\|v\|_{L^s(0,T;W^{m,p}(\Omega))} = \left(\int_0^T \|v\|_{W^{m,p}(\Omega)} dt\right)^{\frac{1}{s}} \quad \text{for } s \in [1,\infty),$$

and the standard modification for $s = \infty$. Similarly, one can define the spaces $H^l(0,T;W^{m,p}(\Omega))$ and $C^l(0,T;W^{m,p}(\Omega))$. The details can be found in [14,18].

The optimization problems considered in this paper are formulated in the following abstract setting: Let $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$ be Hilbert spaces together with the dual space $V' = H^{-1}(\Omega)$ of V which satisfy $V \hookrightarrow H \hookrightarrow V'$. For a time interval $I = (0,T)$, we shall take the state space

$$W = \Big\{w | w \in L^2(I;V) \text{ and } \partial_t w \in L^2(I;V')\Big\},$$

and the control space

$$X = L^2(I;U) \text{ with } U = L^2(\Omega).$$

It is well known that the space W is continuously embedded in $C(\bar{I};H)$, see, e.g., [21]. Let K be a closed convex set in X.

In the above problems (1.1)-(1.3), we assume

- μ is a positive constant;
- $\mathbf{a}(x) \in C_0^1(\bar{\Omega})^2$ which is divergence-free, i.e., $\nabla \cdot \mathbf{a} = 0$ in Ω;
- $c(x) \in L^\infty(\Omega)$ which satisfies $c(x) \geq 0$;
- $f \in L^2(I;L^2(\Omega))$, $y_0 \in V = H_0^1(\Omega)$;
- α, β are functions in $L^\infty(\Omega)$ for any $t \in I$ such that $\alpha < \beta$.

To formulate the optimal control problem, we introduce the admissible set K collecting the pointwise inequality constraints (1.3) as

$$K = \Big\{v \in X : \alpha(x,t) \leq v(x,t) \leq \beta(x,t) \quad \text{a.e. in } \Omega \times [0,T]\Big\}. \tag{2.1}$$

In this paper, the cost functional $J : W \times X \to \mathbb{R}$ is defined using two three times Gâteaux differentiable functionals $J_1 : V \to \mathbb{R}$ and $J_2 : H \to \mathbb{R}$ by

$$J(y,u) = \int_0^T J_1(y(x,t))\, dt + J_2(y(x,T)) + \frac{\gamma}{2}\int_0^T \|u(x,t) - u_0(x,t)\|^2\, dt, \tag{2.2}$$

where the regularization term is added which involves $\gamma > 0$ and a reference parameter $u_0 = u_0(x,t) \in X$. In many practical applications,

$$J_1(y(x,t)) = \frac{\lambda_1}{2}\|y(x,t) - z_1(x,t)\|^2 \quad \text{and} \quad J_2(y(x,T)) = \frac{\lambda_2}{2}\|y(x,T) - z_2(x)\|^2,$$

where λ_1 and λ_2 are nonnegative parameters satisfying $\lambda_1 + \lambda_2 > 0$.

Let $\phi(x) = \sqrt{|\mathbf{a}|^2 + 1}$, and let the characteristic direction associated with the material derivative term $\partial_t y + \mathbf{a} \cdot \nabla y$ be denoted by s, where

$$\phi \partial_s y = \partial_t y + \mathbf{a} \cdot \nabla y. \tag{2.3}$$

Then the corresponding constrained optimization problem is formulated as follows: (QCP)

$$\min_{u\in K} J(y,u) \tag{2.4}$$

subject to the standard weak formulation for the state

$$\begin{cases} (\phi\partial_s y(u),w)+a(y(u),w)=(f+u,w) & \forall w\in V,\ t\in(0,T], \\ y(u)(x,0)=y_0(x). \end{cases}$$

where $a(v,w)=(\mu\nabla v,\nabla w)+(cv,w)$ for any $v,\ w\in V$.

It is well known (see, e.g., [15]) that the control problem (QCP) has a unique solution $(y,u)\in W\times K$. Moreover, a pair (y,u) is the solution of (QCP) if there is a co-state $p\in W$, such that the triplet $(y,p,u)\in W\times W\times K$ satisfies the following optimality conditions: (QCP-OPT)

$$\begin{cases} (\phi\partial_s y,w)+a(y,w)=(f+u,w) & \forall w\in V,\ t\in(0,T], \\ y(0)=y_0, \end{cases} \tag{2.5}$$

$$\begin{cases} -(\phi\partial_s p,q)+a(q,p)=(J_1'(y),q) & \forall q\in V,\ t\in[0,T), \\ p(T)=J_2'(y(T)), \end{cases} \tag{2.6}$$

$$\int_0^T (\gamma(u-u_0)+p,v-u)\,dt\ge 0 \quad \forall v\in K. \tag{2.7}$$

Define the projection [22]

$$\mathbb{P}_{[\alpha,\ \beta]}f(x,t)=\max\Big\{\alpha(x,t),\min\{\beta(x,t),f(x,t)\}\Big\}. \tag{2.8}$$

Then inequality (2.7) is equivalent to

$$u(x,t)=\mathbb{P}_{[\alpha,\ \beta]}\left(u_0-\frac{1}{\gamma}p\right)(x,t). \tag{2.9}$$

3. Discretization with Characteristic Finite Element

In this section, we consider a characteristic finite element approximation for the optimal control problem (QCP).

Let $0=t_0<t_1<t_2<\cdots<t_{N_T}=T$ be a subdivision of $\bar{I}=[0,T]$, with corresponding time intervals $I_n=(t_{n-1},t_n]$ and time steps $\Delta t_n=t_n-t_{n-1}, n=1,2,\cdots,N_T$. Denote $\Delta t=\max_{1\le n\le N_T}\Delta t_n$ and $f^n(x)=f(x,t_n)$. We define, for $1\le q<\infty$, the discrete time-dependent norms

$$|||f|||_{L^q(I;A)}=\left(\sum_{n=1}^{N_T}\Delta t_n\|f^n\|_A^q\right)^{\frac{1}{q}},$$

and the standard modification for $q=\infty$. Define

$$L_D^q(I;A)=\left\{f: |||f|||_{L^q(I;A)}<\infty\right\},\ 1\le q\le\infty.$$

Let $z=G(x^*,t^*;t)$ be an approximate characteristic curve passing through point x^* at time t^*, which is defined by

$$G(x^*,t^*;t)=x^*-\mathbf{a}(x^*)(t^*-t). \tag{3.1}$$

We denote by $\bar{x} = G(x, t_n; t_{n-1})$ the foot at time t_{n-1} of the characteristic curve with head x at time t_n, and $\bar{f}(x) = f(\bar{x})$. Approximate $(\partial_s y^n)(x) = (\partial y/\partial s)(x, t_n)$ by a backward difference quotient in the s-direction,

$$\phi \partial_s y^n \simeq \frac{y^n - \bar{y}^{n-1}}{\Delta t_n}. \tag{3.2}$$

We remark that, for a bounded domain Ω, the mapping $x \to G(x^*, t^*; t)$ is a homeomorphism of Ω onto itself for sufficiently small $|t^* - t|$ (cf. [23]). Thus, $\bar{y}^{n-1}$ is always defined and the tangent to the characteristics (i.e., the s-segment) cannot cross a boundary to an undefined location.

Let $\mathcal{T}^h$ and $\mathcal{T}_U^h$ be two regular triangulations of Ω, such that $\bar{\Omega} = \cup_{\tau \in \mathcal{T}^h} \bar{\tau}$ and $\bar{\Omega} = \cup_{\tau_U \in \mathcal{T}_U^h} \bar{\tau}_U$. Let $h = \max_{\tau \in \mathcal{T}^h} h_\tau$, $h_U = \max_{\tau_U \in \mathcal{T}_U^h} h_{\tau_U}$, where h_τ and h_{τ_U} denote the diameters of the elements τ and τ_U, respectively. Let $\mathcal{P}_k$ denote polynomials of total degree at most k. Then we introduce finite-dimensional subspaces as follows:

$$V^h = \Big\{ w_h \in C(\bar{\Omega}) \cap H_0^1(\Omega) : w_h|_\tau \in \mathcal{P}_1 \quad \text{for } \tau \in \mathcal{T}^h, \text{ and } w_h = 0 \text{ on } \partial\Omega \Big\}, \tag{3.3}$$

$$U^h = \Big\{ v_h \in L^2(\Omega) : v_h|_{\tau_U} \in \mathcal{P}_0 \quad \text{for } \tau_U \in \mathcal{T}_U^h \Big\}, \tag{3.4}$$

$$K_n^h = \Big\{ v_h \in U^h : \overline{\alpha}(t_n)|_{\tau_U} \leq v_h \leq \overline{\beta}(t_n)|_{\tau_U} \quad \text{for } \tau_U \in \mathcal{T}_U^h \Big\}, \tag{3.5}$$

where

$$\overline{\alpha}(t_n)|_{\tau_U} = \frac{1}{|\tau_U|} \int_{\tau_U} \alpha(x, t_n) dx \quad \text{and} \quad \overline{\beta}(t_n)|_{\tau_U} = \frac{1}{|\tau_U|} \int_{\tau_U} \beta(x, t_n) dx.$$

It is obviously that $K_n^h \nsubseteq K$.

Define the following discretized cost functional

$$J_h(y_h, u_h) = \sum_{n=1}^{N_T} \Delta t_n J_1(y_h^n) + J_2(y_h^{N_T}) + \frac{\gamma}{2} \sum_{n=1}^{N_T} \Delta t_n \|u_h^n - u_0^n\|^2.$$

Then the characteristic finite element discretization of (QCP), which will be labeled as (QCP)$^{\mathrm{d}}$, is to find $(y_h^n, u_h^n) \in V^h \times K_n^h, n = 1, 2, \cdots, N_T$, such that

$$\min_{u_h^n \in K_n^h} J_h(y_h, u_h) \tag{3.6}$$

subject to

$$\begin{cases} \left(\dfrac{y_h^n - \bar{y}_h^{n-1}}{\Delta t_n}, w_h \right) + a\left(y_h^n, w_h\right) = \left(f(x, t_n) + u_h^n, w_h\right) \quad \forall w_h \in V^h, \\ y_h^0(x) = y_0^h(x), \quad x \in \Omega, \end{cases}$$

where $y_0^h \in V^h$ is an approximation of y_0 which will be specified later on.

It again follows from [15] that the control problem (QCP)$^{\mathrm{d}}$ has a unique solution $(Y_h^n, U_h^n) \in V^h \times K_n^h$, and that a pair (Y_h^n, U_h^n) is the solution of (QCP)$^{\mathrm{d}}$ iff there is a co-state $P_h^{n-1} \in V^h$, such that the triplet $\left(Y_h^n, P_h^{n-1}, U_h^n\right) \in V^h \times V^h \times K_n^h$ satisfies the following discretized optimality conditions: (QCP-OPT)$^{\mathrm{d}}$:

For $n = 1, 2, \cdots, N_T,$ *solve*

$$\begin{cases} \left(\dfrac{Y_h^n - \bar{Y}_h^{n-1}}{\Delta t_n}, w_h \right) + a\left(Y_h^n, w_h\right) = \left(f^n + U_h^n, w_h\right) \quad \forall w_h \in V^h, \\ \left(Y_h^0, w_h\right) = \left(y_0^h, w_h\right); \end{cases} \tag{3.7}$$

$for\ n = N_T, \cdots, 2, 1, solve$

$$\begin{cases} \left(\dfrac{P_h^{n-1} - \bar{\bar{P}}_h^n \cdot J}{\Delta t_n}, q_h \right) + a\left(q_h, P_h^{n-1}\right) = \left(J_1'(Y_h^n), q_h\right) \quad \forall q_h \in V^h, \\ \left(\bar{\bar{P}}_h^{N_T} \cdot J, q_h \right) = \left(J_2'(Y_h^{N_T}), q_h \right); \end{cases} \tag{3.8}$$

$for\ n = 1, 2, \cdots, N_T, solve$

$$\left(\gamma(U_h^n - u_0^n) + P_h^{n-1}, v_h - U_h^n\right) \geq 0 \quad \forall v_h \in K_n^h, \tag{3.9}$$

where $\bar{\bar{P}}_h^n := P_h^n(\bar{\bar{x}})$, and $\bar{\bar{x}}$ represents the head of the characteristic curve with foot x at time t_{n-1}, namely, $x = G(\bar{\bar{x}}, t_n; t_{n-1})$.

We denote by $J = |\det \mathcal{D}G(x, t_n; t_{n-1})^{-1}|$ the determinant of the Jacobian transformation from G to x. It is clear that for the incompressible flow, the determinant can be expressed as

$$\det \mathcal{D}G(x, t_n; t_{n-1}) = 1 - (\nabla \cdot \mathbf{a})\Delta t_n + \mathcal{O}(\Delta t_n^2) = 1 + \mathcal{O}(\Delta t_n^2),$$

which shows that $J = 1 + \mathcal{O}(\Delta t_n^2)$ for sufficiently small Δt_n.

In the rest of the paper, we shall use two auxiliary intermediate variables. For the control function $u \in K$, we define the discrete state solution $(Y_h^n(u), P_h^n(u)) \in V^h \times V^h$, $n = 1, 2, \cdots, N_T$, associated with u that

$$\begin{cases} \left(\dfrac{Y_h^n(u) - \bar{Y}_h^{n-1}(u)}{\Delta t_n}, w_h \right) + a\left(Y_h^n(u), w_h\right) = \left(f^n + u^n, w_h\right) \quad \forall w_h \in V^h, \\ \left(Y_h^0(u), w_h\right) = \left(y_0^h, w_h\right), \end{cases} \tag{3.10}$$

$$\begin{cases} \left(\dfrac{P_h^{n-1}(u) - \bar{\bar{P}}_h^n(u) \cdot J}{\Delta t_n}, q_h \right) + a\left(q_h, P_h^{n-1}(u)\right) = \left(J_1'(Y_h^n(u)), q_h\right) \quad \forall q_h \in V^h, \\ \left(\bar{\bar{P}}_h^{N_T}(u) \cdot J, q_h \right) = \left(J_2'(Y_h^{N_T}(u)), q_h \right). \end{cases} \tag{3.11}$$

Let Π_h be the L^2-projection from $U = L^2(\Omega)$ to U^h such that for any $v \in U$

$$(v - \Pi_h v, \phi) = 0 \quad \forall \phi \in U^h. \tag{3.12}$$

It is easy to check that $\Pi_h u^n := \Pi_h u(t_n) \in K_n^h$ for the optimal control $u \in K$, and inequality (3.9) is equivalent to

$$U_h^n = \mathbb{P}_{[\overline{\alpha}(t_n),\ \overline{\beta}(t_n)]} \Pi_h \left(u_0^n - \frac{1}{\gamma} P_h^{n-1} \right). \tag{3.13}$$

4. A Priori Error Estimates

In this section, we develop a priori error estimates for the optimal control problem (QCP-OPT) and its characteristic finite element approximation (QCP-OPT)$^{\mathrm{d}}$. Set

$$\theta^n = Y_h^n - Y_h^n(u), \quad \eta^n = y^n - Y_h^n(u), \quad n = 0, 1, \cdots, N_T,$$

$$\zeta^n = P_h^n - P_h^n(u), \quad \xi^n = p^n - P_h^n(u), \quad n = N_T, \cdots, 1, 0.$$

To derive the main results for the state and control, some useful lemmas are needed.

Lemma 4.1. ([14]) *For the L^2-projection operator Π_h defined by* (3.12)*, there is a constant $C > 0$ independent of h_U such that*

$$\|v - \Pi_h v\| \le C h_U \|v\|_1. \tag{4.1}$$

for any $v \in H^1(\Omega)$.

Lemma 4.2. ([19]) *Suppose that $f \in L^2(\Omega)$ and $\bar{f}(x) = f(x - g(x)\Delta t)$, where g and ∇g are bounded on $\bar{\Omega}$. Then for sufficiently small Δt, we have*

$$\|f(x) - \bar{f}(x)\| \le C\Delta t \|f\|_1, \tag{4.2}$$

$$\|f(x) - \bar{f}(x)\|_{-1} \le C\Delta t \|f\|, \tag{4.3}$$

where the constant C depends only on $\|g\|_{L^\infty(\Omega)}$ and $\|\nabla g\|_{L^\infty(\Omega)}$, and the negative-norm $\|\cdot\|_{-1}$ is defined as follows:

$$\|v\|_{-1} = \sup_{0 \ne \phi \in H_0^1(\Omega)} \frac{(v, \phi)}{\|\phi\|_1}.$$

Lemma 4.3. *Let (Y_h, P_h) and $(Y_h(u), P_h(u))$ be the solutions of* (3.7)-(3.8) *and* (3.10)-(3.11)*, respectively. Assume that $J_1'(\cdot)$ and $J_2'(\cdot)$ are uniformly Lipschitz continuous. Then the following estimate holds*

$$\begin{aligned} & |||Y_h - Y_h(u)|||_{L^\infty(I;L^2(\Omega))} + |||P_h - P_h(u)|||_{L^\infty(I;L^2(\Omega))} \\ \le\ & C|||u - U_h|||_{L^2(I;L^2(\Omega))}. \end{aligned} \tag{4.4}$$

Proof. The proof of Lemma 4.3 is divided into two parts.
Part I. It follows from (3.7) and (3.10) that for $\forall w_h \in V^h$

$$\begin{cases} \left(\dfrac{\theta^n - \bar{\theta}^{n-1}}{\Delta t_n}, w_h\right) + a\left(\theta^n, w_h\right) = \left(U_h^n - u^n, w_h\right), \\ (\theta^0, w_h) = 0. \end{cases} \tag{4.5}$$

Select $w_h = \theta^n$ as a test function. The inequality $a(a-b) \ge \frac{1}{2}(a^2 - b^2)$ and a direct calculation show that

$$\left(\frac{\theta^n - \bar{\theta}^{n-1}}{\Delta t_n}, \theta^n\right) \ge \frac{1}{2\Delta t_n}\left(\|\theta^n\|^2 - \|\bar{\theta}^{n-1}\|^2\right), \tag{4.6}$$

$$\|\bar{\theta}^{n-1}\|^2 \le (1 + C\Delta t_n)\|\theta^{n-1}\|^2, \tag{4.7}$$

$$\|\theta^0\| = 0. \tag{4.8}$$

Then inequalities (4.6)-(4.7) can be combined with (4.5) to give the recursion relation

$$\frac{1}{2\Delta t_n}\left(\|\theta^n\|^2 - \|\theta^{n-1}\|^2\right) + \|\theta^n\|_a^2 \le C(\|\theta^n\|^2 + \|\theta^{n-1}\|^2) + C\|u^n - U_h^n\|^2, \tag{4.9}$$

where the $\|\cdot\|_a$-norm is defined by $\|\cdot\|_a = \sqrt{a(\cdot,\cdot)}$ which is equivalent to H_0^1-norm on Ω.

If we multiply both sides of (4.9) by $2\Delta t_n$ and sum over n from 1 to N ($1 \le N \le N_T$), then it follows from (4.8) that

$$\|\theta^N\|^2 + 2\sum_{n=1}^{N} \Delta t_n \|\theta^n\|_a^2 \le C\sum_{n=1}^{N} \Delta t_n \|\theta^n\|^2 + C\sum_{n=1}^{N} \Delta t_n \|u^n - U_h^n\|^2. \tag{4.10}$$

We apply the discrete Gronwall's lemma to conclude

$$|||Y_h - Y_h(u)|||_{L^\infty(I;L^2(\Omega))} + |||Y_h - Y_h(u)|||_{L^2(I;H_0^1(\Omega))} \leq C|||u - U_h|||_{L^2(I;L^2(\Omega))}. \tag{4.11}$$

Part II. We derive from (3.8) and (3.11) that for $\forall q_h \in V^h$

$$\begin{cases} \left(\dfrac{\zeta^{n-1} - \bar{\bar{\zeta}}^n \cdot J}{\Delta t_n}, q_h\right) + a\left(q_h, \zeta^{n-1}\right) = \left(J_1'(Y_h^n) - J_1'(Y_h^n(u)), q_h\right), \\ \left(\bar{\bar{\zeta}}^{N_T} \cdot J, q_h\right) = \left(J_2'(Y_h^{N_T}) - J_2'(Y_h^{N_T}(u)), q_h\right). \end{cases} \tag{4.12}$$

Then with $q_h = \zeta^{n-1}$ and $q_h = \bar{\bar{\zeta}}^{N_T}$, respectively, in (4.12) we have

$$\begin{cases} \left(\dfrac{\zeta^{n-1} - \bar{\bar{\zeta}}^n}{\Delta t_n}, \zeta^{n-1}\right) + \|\zeta^{n-1}\|_a^2 = \left(J_1'(Y_h^n) - J_1'(Y_h^n(u)), \zeta^{n-1}\right) - \left(\dfrac{\bar{\bar{\zeta}}^n - \bar{\bar{\zeta}}^n \cdot J}{\Delta t_n}, \zeta^{n-1}\right), \\ \left(\bar{\bar{\zeta}}^{N_T} \cdot J, \bar{\bar{\zeta}}^{N_T}\right) = \left(J_2'(Y_h^{N_T}) - J_2'(Y_h^{N_T}(u)), \bar{\bar{\zeta}}^{N_T}\right). \end{cases} \tag{4.13}$$

Note that

$$\left(\bar{\bar{\zeta}}^{N_T} \cdot J, \bar{\bar{\zeta}}^{N_T}\right) = \|\zeta^{N_T}\|^2, \tag{4.14}$$

and $J_1'(\cdot)$ and $J_2'(\cdot)$ are uniformly Lipschitz continuous. Then

$$\left|\left(J_1'(Y_h^n) - J_1'(Y_h^n(u)), \zeta^{n-1}\right)\right| \leq C\|\theta^n\|\|\zeta^{n-1}\| \leq C\|\theta^n\|^2 + C\|\zeta^{n-1}\|^2, \tag{4.15}$$

$$\left|\left(J_2'(Y_h^{N_T}) - J_2'(Y_h^{N_T}(u)), \bar{\bar{\zeta}}^{N_T}\right)\right| \leq C\|\theta^{N_T}\|\|\zeta^{N_T}\|, \tag{4.16}$$

where (4.14) and (4.16) imply that

$$\|\zeta^{N_T}\| \leq C\|\theta^{N_T}\|. \tag{4.17}$$

Moreover, it follows from $J = 1 + \mathcal{O}(\Delta t_n^2)$ that

$$\left|\left(\dfrac{\bar{\bar{\zeta}}^n - \bar{\bar{\zeta}}^n \cdot J}{\Delta t_n}, \zeta^{n-1}\right)\right| \leq C\Delta t_n\|\zeta^n\|\|\zeta^{n-1}\| \leq C\Delta t_n\|\zeta^n\|^2 + C\Delta t_n\|\zeta^{n-1}\|^2. \tag{4.18}$$

Finally, we substitute the above estimates (4.15)-(4.18) into (4.13), by using of a similar results of (4.6)-(4.7), and multiply both sides of (4.13) by $2\Delta t_n$, then sum over n from N_T to $M+1$ $(0 \leq M \leq N_T - 1)$, we get

$$|||\zeta|||_{L^\infty(I;L^2(\Omega))} + |||\zeta|||_{L^2(I;H_0^1(\Omega))} \leq C|||Y_h - Y_h(u)|||_{L^\infty(I;L^2(\Omega))}. \tag{4.19}$$

Therefore Lemma 4.3 is proved from (4.11) and (4.19). □

In the following, it is necessary to estimate $|||u - U_h|||_{L^2(I;L^2(\Omega))}$. Before that, we recall the following result.

Lemma 4.4. ([24,25]) *For $n = 1, 2, \cdots, N_T$ and any $v_h \in K_n^h$, there exists a function $v_* \in K$ such that*

$$\overline{v_*}(t_n)\big|_{\tau_U} = v_h\big|_{\tau_U} \quad \text{for } \tau_U \in \mathcal{T}_U^h, \tag{4.20}$$

where $\overline{v_*}(t_n)\big|_{\tau_U} = \frac{1}{|\tau_U|}\int_{\tau_U} v_*(x, t_n)dx$. Moreover, for all $\tau_U \in \mathcal{T}_U^h$ the following estimate holds

$$\|v_h - v_*(t_n)\|_{-1} \leq Ch_U^2\left(\|\alpha(t_n)\|_1^2 + \|\beta(t_n)\|_1^2\right)^{1/2}. \tag{4.21}$$

Theorem 4.1. *Let (y, p, u) and (Y_h, P_h, U_h) be the solutions of (QCP-OPT) and (QCP-OPT)$^{\mathrm{d}}$, respectively. Let U^h be the piecewise constant element space. Suppose that $u, u_0, \alpha, \beta \in L^2_D$ $(I; H^1(\Omega))$, and $p \in L^2_D(I; H^1_0(\Omega)) \cap H^1(I; L^2(\Omega))$. Furthermore, assume that $J_1'(\cdot)$ and $J_2'(\cdot)$ satisfy the following convexity conditions:*

$$\Big(J_1'(v) - J_1'(w), v - w\Big) \geq 0 \quad \textit{and} \quad \Big(J_2'(v) - J_2'(w), v - w\Big) \geq 0 \quad \forall v, w \in V. \tag{4.22}$$

Then we have

$$\begin{aligned} &|||u - U_h|||_{L^2(I;L^2(\Omega))} \\ &\leq C_1(u, u_0, p, \gamma, \alpha, \beta)\Big(h_U + \Delta t + |||p - P_h(u)|||_{L^2(I;L^2(\Omega))}\Big), \end{aligned} \tag{4.23}$$

where the constant C_1 depends on some spatial and temporal derivatives of $u, u_0, p, \gamma, \alpha$ and β, and $P_h(u)$ is defined in (3.11).

Proof. Let $\Pi_h u^n \in K_n^h$ be an approximation of $u(t_n)$. Then we have

$$\begin{aligned} &\gamma|||u - U_h|||^2_{L^2(I;L^2(\Omega))} \\ =&\sum_{n=1}^{N_T} \Delta t_n\Big(\gamma(u^n - u_0^n), u^n - U_h^n\Big) - \sum_{n=1}^{N_T} \Delta t_n\Big(\gamma(U_h^n - u_0^n), u^n - U_h^n\Big) \\ =&\sum_{n=1}^{N_T} \Delta t_n\Big(\gamma(u^n - u_0^n) + p^n, u^n - U_h^n\Big) + \sum_{n=1}^{N_T} \Delta t_n\Big(\gamma(U_h^n - u_0^n) + P_h^{n-1}, U_h^n - \Pi_h u^n\Big) \\ &+ \sum_{n=1}^{N_T} \Delta t_n\Big(\gamma(U_h^n - u^n), \Pi_h u^n - u^n\Big) + \sum_{n=1}^{N_T} \Delta t_n\Big(\gamma(u^n - u_0^n) + p^n, \Pi_h u^n - u^n\Big) \\ &+ \sum_{n=1}^{N_T} \Delta t_n\Big(p^{n-1} - p^n, \Pi_h u^n - u^n\Big) + \sum_{n=1}^{N_T} \Delta t_n\Big(p^{n-1} - P_h^{n-1}(u), u^n - \Pi_h u^n\Big) \\ &+ \sum_{n=1}^{N_T} \Delta t_n\Big(P_h^{n-1}(u) - P_h^{n-1}, u^n - \Pi_h u^n\Big) + \sum_{n=1}^{N_T} \Delta t_n\Big(P_h^{n-1} - P_h^{n-1}(u), u^n - U_h^n\Big) \\ &+ \sum_{n=1}^{N_T} \Delta t_n\Big(P_h^{n-1}(u) - p^{n-1}, u^n - U_h^n\Big) + \sum_{n=1}^{N_T} \Delta t_n\Big(p^{n-1} - p^n, u^n - U_h^n\Big) \\ =:& \sum_{j=1}^{10} I_j. \end{aligned} \tag{4.24}$$

First, from Lemma 4.4 we know that for every $U_h^n \in K_n^h$ there exists a function $u_* \in K$ such that for all $\tau_U \in \mathcal{T}_U^h$

$$\overline{u_*}^n\big|_{\tau_U} = U_h^n\big|_{\tau_U}, \tag{4.25}$$

$$\|U_h^n - u_*^n\|_{-1} \leq C h_U^2 \left(\|\alpha^n\|_1^2 + \|\beta^n\|_1^2\right)^{1/2}. \tag{4.26}$$

Then we deduce from inequalities (2.7) and (4.26) that

$$
\begin{aligned}
I_1 =& \sum_{n=1}^{N_T} \Delta t_n \left(\gamma(u^n - u_0^n) + p^n, u^n - u_*^n\right) + \sum_{n=1}^{N_T} \Delta t_n \left(\gamma(u^n - u_0^n) + p^n, u_*^n - U_h^n\right) \\
\le& 0 + \sum_{n=1}^{N_T} \Delta t_n \left\|\gamma(u^n - u_0^n) + p^n\right\|_1 \left\|u_*^n - U_h^n\right\|_{-1} \\
\le& C(\gamma) h_U^2 \sum_{v=u,u_0,p,\alpha,\beta} |||v|||^2_{L^2(I;H^1(\Omega))}. \qquad (4.27)
\end{aligned}
$$

Second, we see that inequality (3.9) implies that

$$
I_2 \le 0. \qquad (4.28)
$$

Third, it follows from the L^2-projection (3.12) and Lemma 4.1 that

$$
\begin{aligned}
I_4 =& \sum_{n=1}^{N_T} \Delta t_n \left(\left(\gamma(u^n - u_0^n) + p^n\right) - \Pi_h\left(\gamma(u^n - u_0^n) + p^n\right), \Pi_h u^n - u^n\right) \\
\le& \sum_{n=1}^{N_T} \Delta t_n \left\|\left(\gamma(u^n - u_0^n) + p^n\right) - \Pi_h\left(\gamma(u^n - u_0^n) + p^n\right)\right\| \left\|\Pi_h u^n - u^n\right\| \\
\le& C h_U^2 \left(|||u|||^2_{L^2(I;H^1(\Omega))} + |||u_0|||^2_{L^2(I;H^1(\Omega))} + |||p|||^2_{L^2(I;H^1(\Omega))}\right). \qquad (4.29)
\end{aligned}
$$

Fourth, by letting $w_h = \zeta^{n-1}$ in (4.5) and $q_h = \theta^n$ in (4.12), we derive form (4.22) that

$$
\begin{aligned}
I_8 =& -\sum_{n=1}^{N_T} \Delta t_n \left(\frac{\theta^n - \bar{\theta}^{n-1}}{\Delta t_n}, \zeta^{n-1}\right) - \sum_{n=1}^{N_T} \Delta t_n a\left(\theta^n, \zeta^{n-1}\right) \\
=& -\sum_{n=1}^{N_T} \Delta t_n \left(\frac{\zeta^{n-1} - \bar{\bar{\zeta}}^n \cdot J}{\Delta t_n}, \theta^n\right) - \sum_{n=1}^{N_T} \Delta t_n a\left(\theta^n, \zeta^{n-1}\right) - \left(\bar{\bar{\zeta}}^{N_T} \cdot J, \zeta^{N_T}\right) \\
=& -\sum_{n=1}^{N_T} \Delta t_n \left(J_1'(Y_h^n) - J_1'(Y_h^n(u)), \theta^n\right) - \left(J_2'(Y_h^{N_T}) - J_2'(Y_h^{N_T}(u)), \theta^{N_T}\right) \le 0. \qquad (4.30)
\end{aligned}
$$

Finally, note that

$$
|p^n - p^{n-1}|^2 = \left|\int_{t_{n-1}}^{t_n} \partial_t p \, dt\right|^2 \le \Delta t_n \int_{t_{n-1}}^{t_n} |\partial_t p|^2 \, dt.
$$

It then follows from Lemmas 4.1 and 4.3, the Cauchy-Schwarz inequality, and the inequalities (4.27)-(4.30) that

$$
\begin{aligned}
& \gamma |||u - U_h|||^2_{L^2(I;L^2(\Omega))} \\
\le& C(u, u_0, p, \gamma, \alpha, \beta) h_U^2 + C\Delta t^2 \left\|\partial_t p\right\|^2_{L^2(I;L^2(\Omega))} + C|||p - P_h(u)|||^2_{L^2(I;L^2(\Omega))} \\
& + C\delta_1 |||P_h - P_h(u)|||^2_{L^2(I;L^2(\Omega))} + C\delta_2 |||u - U_h|||^2_{L^2(I;L^2(\Omega))} \qquad (4.31) \\
\le& C(u, u_0, p, \gamma, \alpha, \beta)\left(h_U^2 + \Delta t^2\right) + C|||p - P_h(u)|||^2_{L^2(I;L^2(\Omega))} + \frac{\gamma}{2} |||u - U_h|||^2_{L^2(I;L^2(\Omega))},
\end{aligned}
$$

where δ_1 and δ_2 are sufficiently small positive number. Thus Theorem 4.1 follows immediately from (4.31). $\square$

In the following, let $R_h : H_0^1(\Omega) \to V^h$ be the Ritz projection of (y, p) which satisfies

$$a(R_h y, w_h) = a(y, w_h) \quad \forall w_h \in V^h, \tag{4.32}$$

$$a(q_h, R_h p) = a(q_h, p) \quad \forall q_h \in V^h. \tag{4.33}$$

Then, we recall some well-known results in [26], which are useful for our work.

Lemma 4.5. *For every $t \in I$ and h sufficiently small, there is a positive constant C such that*

$$\|y - R_h y\| + h\|y - R_h y\|_1 \le Ch^2\|y\|_2, \tag{4.34a}$$

$$\|p - R_h p\| + h\|p - R_h p\|_1 \le Ch^2\|p\|_2. \tag{4.34b}$$

Lemma 4.6. *Let $(R_h y, R_h p)$ and $(Y_h(u), P_h(u))$ be the solutions of* (4.32)-(4.33) *and* (3.10)-(3.11)*, respectively. Assume that $y, p \in L^\infty_D(I; H_0^1(\Omega) \cap H^2(\Omega)) \cap H^1(I; H^2(\Omega)) \cap H^2(I; L^2(\Omega))$. Furthermore, suppose that $J_1'(\cdot)$ and $J_2'(\cdot)$ are uniformly Lipschitz continuous. Then the following estimates hold*

$$|||R_h y - Y_h(u)|||_{L^\infty(I;L^2(\Omega))} \le C_2(y)\left(h^2 + \Delta t\right), \tag{4.35}$$

$$|||R_h p - P_h(u)|||_{L^\infty(I;L^2(\Omega))} \le C_3(y,p)\left(h^2 + \Delta t\right), \tag{4.36}$$

where the constants C_2 and C_3 depend on some spatial and temporal derivatives of y and y, p respectively.

Proof. Let

$$\eta = (y - R_h y) + (R_h y - Y_h(u)) = \mu + \kappa,$$

$$\xi = (p - R_h p) + (R_h p - P_h(u)) = \rho + \pi.$$

In the following, we also divide the proof into two parts.

Part I. We derive an error equation from (2.5) and (3.10) on $\eta = y - Y_h(u)$ that

$$\left(\frac{\eta^n - \bar{\eta}^{n-1}}{\Delta t_n}, w_h\right) + a\left(\eta^n, w_h\right) = -(\sigma^n, w_h) \quad \forall w_h \in V^h,\ n = 1, 2, \cdots, N_T,$$

where

$$\sigma^n = \phi\partial_s y^n - (y^n - \bar{y}^{n-1})/\Delta t_n.$$

Note that the estimate for μ is known, we need only to derive an estimate for κ. By choosing $w_h = \kappa^n$ and making use of Ritz projection (4.32), we can rewrite the above equation in terms of μ and κ:

$$\left(\frac{\kappa^n - \bar{\kappa}^{n-1}}{\Delta t_n}, \kappa^n\right) + a\left(\kappa^n, \kappa^n\right) = -(\sigma^n, \kappa^n) - \left(\frac{\mu^n - \bar{\mu}^{n-1}}{\Delta t_n}, \kappa^n\right). \tag{4.37}$$

Firstly, for the first term on the right-hand sides of (4.37), it follows from [19] that

$$\|\sigma^n\|^2 \le C\Delta t_n \left\|\partial_{ss} y\right\|^2_{L^2(I_n;L^2(\Omega))}. \tag{4.38}$$

Then for the second term on the right-hand sides of (4.37), we obtain from Lemmas 4.2 and 4.5 that

$$\begin{aligned} &\left|(\mu^n - \bar{\mu}^{n-1}, \kappa^n)\right| \le \left|(\mu^n - \mu^{n-1}, \kappa^n)\right| + \left|(\mu^{n-1} - \bar{\mu}^{n-1}, \kappa^n)\right| \\ \le\ & \|\kappa^n\| \int_{t_{n-1}}^{t_n} \|\partial_t \mu\| dt + \|\mu^{n-1} - \bar{\mu}^{n-1}\|_{-1}\|\kappa^n\|_1 \\ \le\ & C\|\partial_t \mu\|^2_{L^2(I_n;L^2(\Omega))} + C(\delta)\Delta t_n\|\mu^{n-1}\|^2 + C\Delta t_n\|\kappa^n\|^2 + C\delta\Delta t_n\|\kappa^n\|_1^2 \\ \le\ & Ch^4\|\partial_t y\|^2_{L^2(I_n;H^2(\Omega))} + C\Delta t_n h^4|||y|||^2_{L^\infty(I;H^2(\Omega))} + C\Delta t_n\|\kappa^n\|^2 + C\delta\Delta t_n\|\kappa^n\|_a^2. \end{aligned} \tag{4.39}$$

Multiply both sides of (4.37) by Δt_n and sum over $1 \le n \le N$ $(1 \le N \le N_T)$. We then conclude by (4.38)-(4.39) and the same estimates as (4.6)-(4.7) that

$$\begin{aligned}
&\frac{1}{2}\|\kappa^N\|^2 + \sum_{n=1}^{N} \Delta t_n \|\kappa^n\|_a^2 \\
&\le C\Delta t^2 \|\partial_{ss} y\|^2_{L^2(I;L^2(\Omega))} + Ch^4 \left[\|\partial_t y\|^2_{L^2(I;H^2(\Omega))} + |||y|||^2_{L^\infty(I;H^2(\Omega))}\right] \\
&\quad + C\sum_{n=1}^{N} \Delta t_n \|\kappa^n\|^2 + C\delta \sum_{n=1}^{N} \Delta t_n \|\kappa^n\|_a^2,
\end{aligned} \tag{4.40}$$

where y_0^h is chosen to be the Ritz projection of y_0 which satisfies (4.32), i.e., $\kappa^0 = 0$.

Let $C\delta = 1/2$ and apply the discrete Gronwall's lemma to (4.40) yields that

$$\begin{aligned}
&|||\kappa|||_{L^\infty(I;L^2(\Omega))} \\
&\le C\Delta t \|\partial_{ss} y\|_{L^2(I;L^2(\Omega))} + Ch^2 \Big[\|\partial_t y\|_{L^2(I;H^2(\Omega))} + |||y|||_{L^\infty(I;H^2(\Omega))}\Big].
\end{aligned} \tag{4.41}$$

Part II. In this part, we consider the estimate for the difference π between the projection solution $R_h p$ and the intermediate solution $P_h(u)$. We conclude from (2.6) and (3.11) that for $q_h \in V^h$ and $n = N_T, \cdots, 2, 1$:

$$\begin{aligned}
&\left(\frac{\xi^{n-1} - \bar{\bar{\xi}}^n \cdot J}{\Delta t_n}, q_h\right) + a\left(q_h, \xi^{n-1}\right) \\
&= -(\chi^{n-1}, q_h) + \left(\frac{\bar{\bar{p}}^n - \bar{\bar{p}}^n \cdot J}{\Delta t_n}, q_h\right) + \left(J_1'(y^{n-1}) - J_1'(Y_h^n(u)), q_h\right),
\end{aligned}$$

where

$$\chi^{n-1} = -\phi \partial_s p^{n-1} - (p^{n-1} - \bar{\bar{p}}^n)/\Delta t_n.$$

Select $q_h = \pi^{n-1}$, it then follows that

$$\begin{aligned}
&\left(\frac{\pi^{n-1} - \bar{\bar{\pi}}^n}{\Delta t_n}, \pi^{n-1}\right) + a\left(\pi^{n-1}, \pi^{n-1}\right) \qquad (4.42) \\
&= -(\chi^{n-1}, \pi^{n-1}) - \left(\frac{\rho^{n-1} - \bar{\bar{\rho}}^n}{\Delta t_n}, \pi^{n-1}\right) - \left(\frac{\bar{\bar{\rho}}^n - \bar{\bar{\rho}}^n \cdot J}{\Delta t_n}, \pi^{n-1}\right) - \left(\frac{\bar{\bar{\pi}}^n - \bar{\bar{\pi}}^n \cdot J}{\Delta t_n}, \pi^{n-1}\right) \\
&\quad + \left(\frac{\bar{\bar{p}}^n - \bar{\bar{p}}^n \cdot J}{\Delta t_n}, \pi^{n-1}\right) + \Big(J_1'(y^n) - J_1'(Y_h^n(u)), \pi^{n-1}\Big) + \Big(J_1'(y^{n-1}) - J_1'(y^n), \pi^{n-1}\Big).
\end{aligned}$$

Firstly, we can estimate the first and second terms on the right-hand sides of (4.42) just as (4.38)-(4.39), that is

$$|(\chi^{n-1}, \pi^{n-1})| \le C\Delta t_n \|\partial_{ss} p\|^2_{L^2(I_n;L^2(\Omega))} + C\|\pi^{n-1}\|^2, \tag{4.43}$$

$$\begin{aligned}
\left|\left(\frac{\rho^{n-1} - \bar{\bar{\rho}}^n}{\Delta t_n}, \pi^{n-1}\right)\right| &\le C\Delta t_n^{-1} \|\partial_t \rho\|^2_{L^2(I_n;L^2(\Omega))} + C\|\rho^n\|^2 \\
&\quad + C\|\pi^{n-1}\|^2 + \frac{1}{2}\|\pi^{n-1}\|_a^2.
\end{aligned} \tag{4.44}$$

Then for the next three terms, note that $J = 1 + \mathcal{O}(\Delta t_n^2)$, we have

$$\left|\left(\frac{\bar{\rho}^n - \bar{\rho}^n \cdot J}{\Delta t_n}, \pi^{n-1}\right)\right| \le C\Delta t_n \|\bar{\rho}^n\| \|\pi^{n-1}\| \le C\Delta t_n \|\rho^n\|^2 + C\Delta t_n \|\pi^{n-1}\|^2, \quad (4.45)$$

$$\left|\left(\frac{\bar{\bar{\pi}}^n - \bar{\bar{\pi}}^n \cdot J}{\Delta t_n}, \pi^{n-1}\right)\right| \le C\Delta t_n \|\pi^n\|^2 + C\Delta t_n \|\pi^{n-1}\|^2, \quad (4.46)$$

$$\left|\left(\frac{\bar{p}^n - \bar{p}^n \cdot J}{\Delta t_n}, \pi^{n-1}\right)\right| \le C\Delta t_n^2 \|p^n\|^2 + C\|\pi^{n-1}\|^2. \quad (4.47)$$

Finally, since $J_1'(\cdot)$ is uniformly Lipschitz continuous, we obtain

$$\begin{aligned}&\left|\left(J_1'(y^n) - J_1'(Y_h^n(u)), \pi^{n-1}\right)\right| \\ &\quad \le C\|y^n - Y_h^n(u)\| \|\pi^{n-1}\| \le C\|\mu^n\|^2 + C\|\kappa^n\|^2 + C\|\pi^{n-1}\|^2, \quad (4.48)\\ &\left|\left(J_1'(y^{n-1}) - J_1'(y^n), \pi^{n-1}\right)\right| \\ &\quad \le C\|y^{n-1} - y^n\| \|\pi^{n-1}\| \le C\Delta t_n \|\partial_t y\|^2_{L^2(I_n;L^2(\Omega))} + C\|\pi^{n-1}\|^2. \quad (4.49)\end{aligned}$$

Therefore, similar to the estimate of κ, we first insert the estimates (4.43)-(4.49) into (4.42) and multiply (4.42) by Δt_n, then sum over n from N_T to $M+1$ ($0 \le M \le N_T - 1$) to conclude

$$\begin{aligned}&\left\|\pi^M\right\|^2 + \sum_{n=M+1}^{N_T} \Delta t_n \|\pi^{n-1}\|_a^2 \\ \le& \left\|\pi^{N_T}\right\|^2 + C\Delta t^2 \Big[\|\partial_{ss} p\|^2_{L^2(I;L^2(\Omega))} + \|\partial_t y\|^2_{L^2(I;L^2(\Omega))} + \||p\||^2_{L^\infty(I;L^2(\Omega))}\Big] \\ &+ C \sum_{v=\rho,\mu,\kappa} \sum_{n=M+1}^{N_T} \Delta t_n \|v^n\|^2 + C\|\partial_t \rho\|^2_{L^2(I;L^2(\Omega))} + C\sum_{n=M}^{N_T} \Delta t_n \|\pi^n\|^2. \quad (4.50)\end{aligned}$$

Before applying the discrete Gronwall's lemma to (4.50), we should pay special attention on the term $\|\pi^{N_T}\|$. From (2.6), it is easy to see that

$$(\bar{\bar{p}}(T) \cdot J, q) = (J_2'(\bar{\bar{y}}(T)) \cdot J, q). \quad (4.51)$$

Then we subtract the second equation in (3.11) from (4.51) that

$$\begin{aligned}\left(\bar{\bar{\xi}}^{N_T} \cdot J, q_h\right) &= \left(J_2'(\bar{\bar{y}}(T)) \cdot J - J_2'(Y_h^{N_T}(u)), q_h\right) \\ &= \left(J_2'(\bar{\bar{y}}(T)) \cdot J - J_2'(\bar{\bar{y}}(T)), q_h\right) + \left(J_2'(\bar{\bar{y}}(T)) - J_2'(y(T)), q_h\right) \\ &\quad + \left(J_2'(y(T)) - J_2'(Y_h^{N_T}(u)), q_h\right). \quad (4.52)\end{aligned}$$

Let $q_h = \bar{\bar{\pi}}^{N_T}$ and by the uniformly Lipschitz continuous of $J_2'(\cdot)$, we have

$$\begin{aligned}\left\|\pi^{N_T}\right\|^2 &= \left(\bar{\bar{\pi}}^{N_T} \cdot J, \bar{\bar{\pi}}^{N_T}\right) \\ &\le \Big[\left\|\bar{\bar{\rho}}^{N_T} \cdot J\right\| + \|J_2'(\bar{\bar{y}}(T)) \cdot J - J_2'(\bar{\bar{y}}(T))\| + \|J_2'(\bar{\bar{y}}(T)) - J_2'(y(T))\| \\ &\qquad + \left\|J_2'(y(T)) - J_2'(Y_h^{N_T}(u))\right\|\Big] \|\bar{\bar{\pi}}^{N_T}\| \\ &\le \Big[C\left\|\rho^{N_T}\right\| + C\Delta t^2 \left\|J_2'(y(T))\right\| + C\Delta t \left\|y(T)\right\|_1 + \left\|\eta^{N_T}\right\|\Big] \left\|\pi^{N_T}\right\|.\end{aligned}$$

That is

$$\left\|\pi^{N_T}\right\| \leq C\left\|\rho^{N_T}\right\| + C\Delta t^2\left\|J_2'(y(T))\right\| + C\Delta t\left\|y(T)\right\|_1 + \left\|\eta^{N_T}\right\|. \tag{4.53}$$

Thus it follows from Lemma 4.5, (4.41), (4.50) and (4.53) that

$$\begin{aligned}
&|||\pi|||_{L^\infty(I;L^2(\Omega))}\\
\leq{}& C\Delta t\Bigg[\sum_{v=y,p}\|\partial_{ss}v\|_{L^2(I;L^2(\Omega))} + \sum_{v=y,p}\|\partial_t v\|_{L^2(I;L^2(\Omega))} + |||p|||_{L^\infty(I;L^2(\Omega))} \\
&+|||y|||_{L^\infty(I;H^1(\Omega))} + \|J_2'(y(T))\|\Bigg] + Ch^2\sum_{v=y,p}\Big[\|\partial_t v\|_{L^2(I;H^2(\Omega))} + |||v|||_{L^\infty(I;H^2(\Omega))}\Big],
\end{aligned} \tag{4.54}$$

which completed the proof of the lemma. □

From Lemmas 4.5-4.6, we can easily obtain the following main error estimate for the intermediate error.

Theorem 4.2. *Let (y,p) and $(Y_h(u),P_h(u))$ be the solutions of (2.5)-(2.6) and (3.10)-(3.11), respectively. Assume all conditions in Lemmas 4.5-4.6 are valid. Then we have*

$$|||y - Y_h(u)|||_{L^\infty(I;L^2(\Omega))} + |||p - P_h(u)|||_{L^\infty(I;L^2(\Omega))} \leq C_4(y,p)\left(h^2 + \Delta t\right), \tag{4.55}$$

where the constant C_4 depends on some spatial and temporal derivatives of y and p.

Gather the bounds given by Theorems 4.1 and 4.2, we can easily establish the main result for the control, state and co-state.

Theorem 4.3. *Suppose that $\{y,p,u\}$ and $\{Y_h,P_h,U_h\}$ are the solutions of (2.5)-(2.7) and (3.7)-(3.9), respectively. Moreover, we assume that all conditions in Theorems 4.1 and 4.2 hold. Then we have*

$$\begin{aligned}
&|||y - Y_h|||_{L^\infty(I;L^2(\Omega))} + |||p - P_h|||_{L^\infty(I;L^2(\Omega))} + |||u - U_h|||_{L^2(I;L^2(\Omega))}\\
\leq{}& C_5(u,u_0,y,p,\gamma,\alpha,\beta)\left(h_U + h^2 + \Delta t\right),
\end{aligned} \tag{4.56}$$

where the constant C_5 depends on some spatial and temporal derivatives of $u, u_0, y, p, \gamma, \alpha$ and β.

5. Numerical Experiments

In this section, we carry out two numerical examples to demonstrate the theoretical results showed in Theorem 4.3. To solve the optimal control problems numerically, we use the C++ software package:AFEPack, it is freely available at http://dsec.pku.edu.cn/~rli.

In our numerical test, we consider the following optimal control problem:

$$\min\ \frac{1}{2}\int_0^T\|y(t) - z_1(t)\|^2 dt + \frac{1}{2}\|y(T) - z_2\|^2 + \frac{1}{2}\int_0^T\|u - u_0\|^2\,dt, \tag{5.1}$$

subject to

$$\left\{\begin{array}{l}\partial_t y - \mu\Delta y + \mathbf{a}\cdot\nabla y + cy = f + u \quad \text{in } \Omega\times I,\\ y(0) = y_0 \quad \text{in } \Omega.\end{array}\right. \tag{5.2}$$

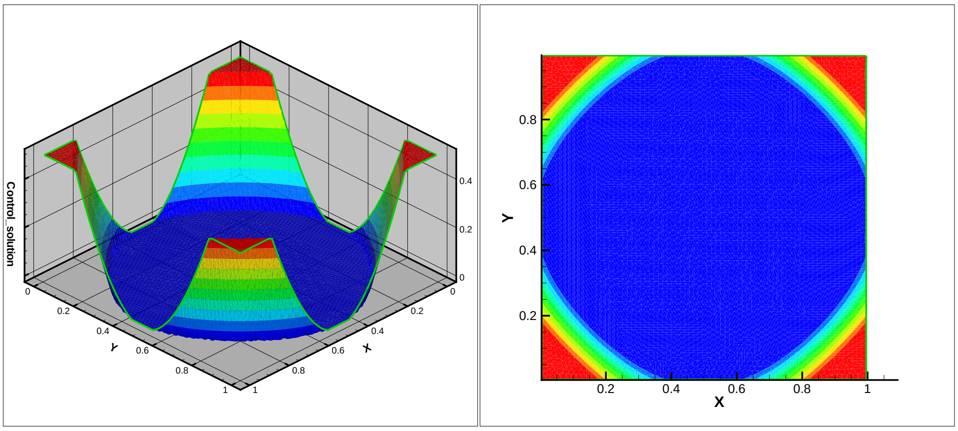

Fig. 5.1. Example 5.1: the numerical solution of the control (left) and its contour-line (right).

The corresponding co-state equation is

$$\begin{cases} -\partial_t p - \mu\Delta p - \mathbf{a}\cdot\nabla p + cp = y - z_1 & \text{in } \Omega\times I, \\ p(T) = y(T) - z_2 & \text{in } \Omega. \end{cases} \tag{5.3}$$

Both equations (5.2) and (5.3) are combined with homogeneous Dirichlet boundary conditions.

For simplicity, in this work we use the same mesh for $\mathcal{T}^h$ and $\mathcal{T}_U^h$. For constrained optimal control problems governed by convection-diffusion equations, people usually pay their attention on the state and the control. Therefore in the following numerical examples, we mostly center on the state variable y, which is approximated by piecewise linear elements; and the control variable u, which is discretized using piecewise constant elements.

Example 5.1. For the first example, the spatial domain is $\Omega = (0,1)^2$, the time interval is $\bar{I} = [0,1]$, the velocity field is imposed as $\mathbf{a} = (x_2 - 0.5, 0.5 - x_1)$, the diffusion coefficient $\mu = 1.0\text{e-}5$, the reaction coefficient $c = 1$. Functions f, z_1 and z_2 are chosen such that the analytical solutions for problems (5.2)-(5.3) are as follows:

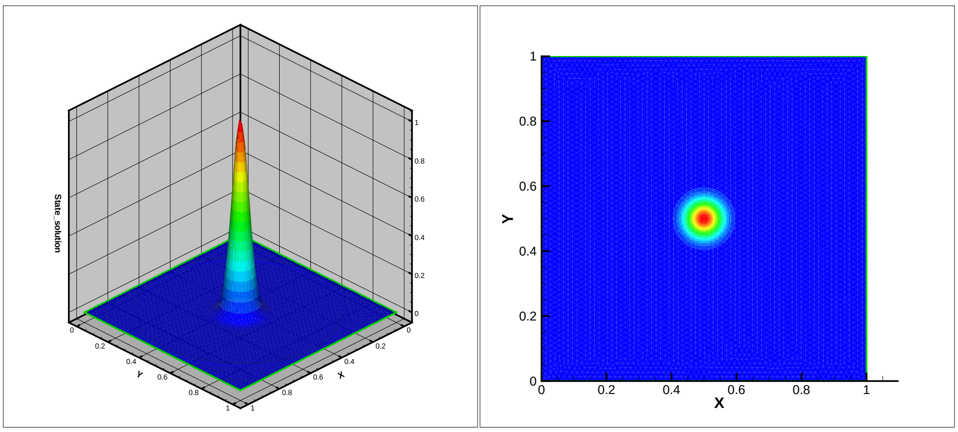

Fig. 5.2. Example 5.1: the numerical solution of the state (left) and its contour-line (right).

Table 5.1: Numerical results for the state, co-state and control for Example 5.1.

h	$\|\|\|y - Y_h\|\|\|$	Order	$\|\|\|p - P_h\|\|\|$	Order	$\|\|\|u - U_h\|\|\|$	Order
$\frac{1}{8}$	3.980545E-2	——	4.245534E-2	——	4.190737E-2	——
$\frac{1}{16}$	1.144331E-2	1.8037	1.395041E-2	1.6020	2.288011E-2	0.8716
$\frac{1}{32}$	2.533822E-3	2.1890	3.706113E-3	1.9198	1.133799E-2	1.0190
$\frac{1}{64}$	9.476185E-4	1.4739	9.644054E-4	1.8875	5.651237E-3	0.9873

$$
\begin{aligned}
&p(x,t) = \sin(\pi t/2)\sin(\pi x_1)\sin(\pi x_2)\mathrm{exp}\left(-\frac{(x_1-0.5)^2+(x_2-0.5)^2}{\sqrt{\mu}}\right),\\
&u_0(x,t) = 1-\sin(\pi x_1)-\sin(\pi x_2),\\
&u(x,t) = \max(0,\min(u_0-p,0.5)),\\
&y(x,t) = p\left(1+4\sqrt{\mu}+2\mu\pi^2-4\left((x_1-0.5)^2+(x_2-0.5)^2\right)\right).
\end{aligned}
$$

It is clear that the state, co-state and control solutions are strictly dependent on the diffusion parameter μ, and for small μ the state equation is strongly convection-dominated.

In this example, the solutions describe an internal layer problem for the state and co-state. Although the layer does not move with time, it is very sharp near the center-point $(0.5, 0.5)$ and the function value is varying at this point with the time marching. Beside, it is easy to see that for those points far away from the center-point, the solutions y and p are almost zero. However, the characteristic finite element discretization used in this paper shows a good approximation to the control problem. It can be seen that in Table 5.1 numerical convergence order is presented with μ=1.0e-5 and the time step size $\Delta t = h$. Fig. 5.1 shows the numerical solution of the control and its contour-line at $T = 1$. The elevation plot of the numerical state solution and its corresponding contour-line at $T = 1$ are presented in Fig. 5.2.

Example 5.2. The second example considered is the transport of a two-dimensional rotating Gaussian pulse in $\Omega = (-0.5, 0.5)^2$ and $\bar{I} = [0, 1]$. Problems (5.2)-(5.3) are given with a rotating velocity field $\mathbf{a} = (-x_2, x_1)$, a diffusion coefficient μ = 1.0e-4 and a reaction coefficient $c = 0$. Let $f = -u$, $z_1 = y$ and $z_2 = y(T) - p(T)$ such that the analytical solutions are as follows:

$$
\begin{aligned}
&y(x,t) = \frac{2\sigma^2}{2\sigma^2+4\mu t}\exp\left(-\frac{(\bar{x}_1-x_{1c})^2+(\bar{x}_2-x_{2c})^2}{2\sigma^2+4\mu t}\right),\\
&p(x,t) = 0,\\
&u_0(x,t) = \sin(\pi t/2)\sin(\pi x_1)\sin(\pi x_2),\\
&u(x,t) = \max(-0.5,\min(u_0-p,0.5)),
\end{aligned}
$$

where x_{1c}, x_{2c}, and σ are the centered and standard deviations, respectively, and $\bar{x}_1 = x_1\cos t + x_2\sin t$, $\bar{x}_2 = x_2\cos t - x_1\sin t$.

In this numerical test, the data are chosen as follows: $x_{1c} = -0.25$, $x_{2c} = 0$, $\sigma = 0.0447$ which gives $2\sigma^2 = 0.0040$. This problem provides an example for a homogeneous two-dimensional convection-diffusion equation with a variable velocity field and a known analytical solution. It has been used widely to test for numerical artifacts of different schemes, such as numerical stability and numerical dispersion, spurious oscillations, and phase errors. In Table 5.2 numerical results are presented with spatial step sizes $h = 1/8, 1/16, 1/32, 1/64$, which show that the characteristic finite element scheme maintains a first-order accuracy in space for the control. In Figs. 5.3-5.4, we also show the numerical solutions and their corresponding contour-lines for the control and state at $T = 1$, respectively.

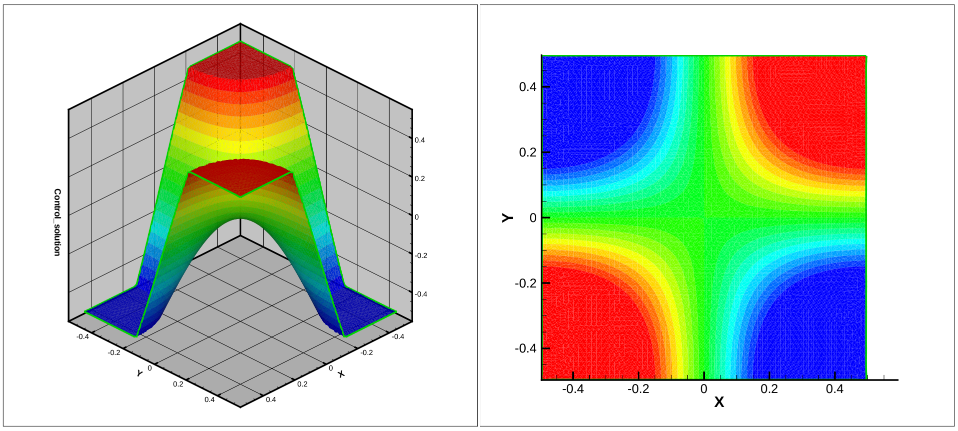

Fig. 5.3. Example 5.2: the numerical solution of the control (left) and its contour-line (right).

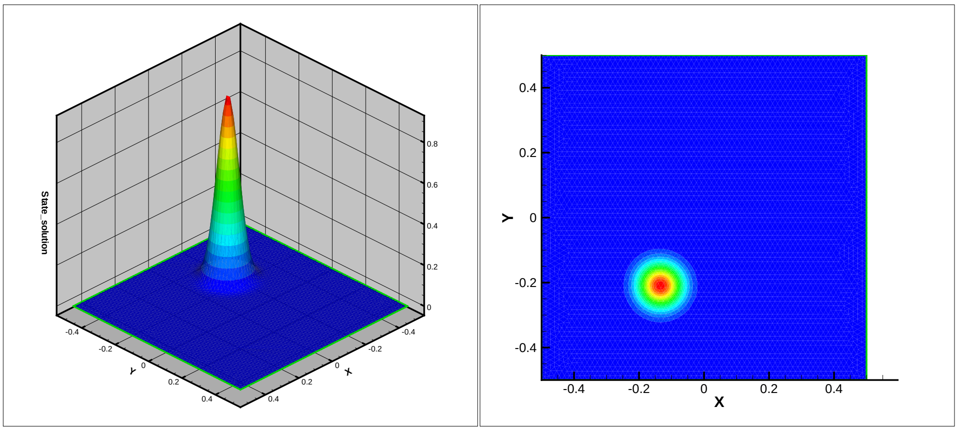

Fig. 5.4. Example 5.2: the numerical solution of the state (left) and its contour-line (right).

Table 5.2: Numerical results for the state, co-state and control for Example 5.2.

h	$\vert\vert\vert y - Y_h \vert\vert\vert$	Order	$\vert\vert\vert p - P_h \vert\vert\vert$	Order	$\vert\vert\vert u - U_h \vert\vert\vert$	Order
$\frac{1}{8}$	4.958123E-2	——	1.744643E-2	——	3.996715E-2	——
$\frac{1}{16}$	2.078207E-2	1.2545	1.074794E-2	0.6989	1.972796E-2	1.0186
$\frac{1}{32}$	5.939231E-3	1.8070	3.614890E-3	1.5720	9.365068E-3	1.0749
$\frac{1}{64}$	2.276362E-3	1.3835	1.246851E-3	1.5357	4.587253E-3	1.0297

From these results we find that using the scheme given in this paper we can approximate the analytical solutions with high accuracy.

Acknowledgments. The authors would like to thank the anonymous reviewers for their valuable comments and suggestions on an earlier version of this paper. The first author was supported by the National Natural Science Foundation of China (No.11126086,11201485) and the Fundamental Research Funds for the Central Universities (No.12CX04083A); The second author was supported by the National Natural Science Foundation of China (No.11171190); The third author was supported by the National Natural Science Foundation of China (No.11101431).

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