

## BLOCK-SYMMETRIC AND BLOCK-LOWER-TRIANGULAR PRECONDITIONERS FOR PDE-CONSTRAINED OPTIMIZATION PROBLEMS\*

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### Abstract

Optimization problems with partial differential equations as constraints arise widely in many areas of science and engineering, in particular in problems of the design. The solution of such class of PDE-constrained optimization problems is usually a major computational task. Because of the complex for directly seeking the solution of PDE-constrained optimization problem, we transform it into a system of linear equations of the saddle-point form by using the Galerkin finite-element discretization. For the discretized linear system, in this paper we construct a block-symmetric and a block-lower-triangular preconditioner, for solving the PDE-constrained optimization problem. Both preconditioners exploit the structure of the coefficient matrix. The explicit expressions for the eigenvalues and eigenvectors of the corresponding preconditioned matrices are derived. Numerical implementations show that these block preconditioners can lead to satisfactory experimental results for the preconditioned GMRES methods when the regularization parameter is suitably small.

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*Key words:* Saddle-point matrix, Preconditioning, PDE-constrained optimization, Eigenvalue and eigenvector, Regularization parameter.

### 1. Introduction

We consider the distributed control problem which consists of a cost functional (1.1) to be minimized subject to a partial differential equation problem posed on a domain  $\Omega \subset \mathbb{R}^2$  or  $\mathbb{R}^3$ :

$$\min_{u, f} \frac{1}{2} \|u - u_*\|_2^2 + \beta \|f\|_2^2, \quad (1.1)$$

$$\text{subject to } -\nabla^2 u = f \quad \text{in } \Omega, \quad (1.2)$$

$$\text{with } u_* = g \quad \text{on } \partial\Omega_1 \quad \text{and} \quad \frac{\partial u_*}{\partial n} = g \quad \text{on } \partial\Omega_2, \quad (1.3)$$

where  $\partial\Omega$  is the boundary of  $\Omega$ ,  $\partial\Omega_1 \cup \partial\Omega_2 = \partial\Omega$  and  $\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$ ,  $\beta \in R^+$  is a regularization parameter, and the function  $u_*$  is a given function that represents the desired state. We want to find  $u$  which satisfies the PDE problem and is as close to  $u_*$  as possible in some norm sense (e.g., the  $L_2$  norm). In order to achieve this aim, the right-hand side  $f$  of the PDE can be varied. The second term in the cost functional (1.1) is added because the problem would be generally ill-posed and then needs this Tikhonov regularization term. Such class of problems

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were introduced by J.L. Lions in [15].

There are two approaches to obtain the solution of the PDE-constrained optimization problem (1.1)-(1.3). The first is optimize-then-discretize and the second is discretize-then-optimize. Following the discretize-then-optimize approach (see [1,16]), we transform (1.1)-(1.3) into a linear system of the saddle-point form. That is to say, firstly, by employing the Galerkin finite-element method to the weak formulation of (1.2) and (1.3), we obtain the finite-dimensional discrete analogue of the minimization problem as follows (see [1,11,13,15,16]):

$$\min_{u,f} \quad \frac{1}{2}u^T M u - u^T b + \alpha + \beta f^T M f, \tag{1.4}$$

$$\text{subject to} \quad K u = M f + d, \tag{1.5}$$

where  $M \in \mathbb{R}^{n \times n}$  is the mass matrix,  $K \in \mathbb{R}^{n \times n}$  is the stiffness matrix (the discrete Laplacian),  $d \in \mathbb{R}^n$  represents the boundary data,  $\alpha = \|u_*\|_2^2$ , and  $b \in \mathbb{R}^n$  is the Galerkin projection of the discrete state  $u_*$ . Then by applying the Lagrangian multiplier method to this minimization problem (1.4)-(1.5) we find that  $f, u$  and  $\lambda$  are defined by the linear system

$$\mathcal{A}x \equiv \begin{pmatrix} 2\beta M & 0 & -M \\ 0 & M & K^T \\ -M & K & 0 \end{pmatrix} \begin{pmatrix} f \\ u \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ d \end{pmatrix} \equiv g, \tag{1.6}$$

where  $\lambda$  is a vector of Lagrange multiplier, see [1,16-17]. Evidently, if we let

$$A = \begin{pmatrix} 2\beta M & 0 \\ 0 & M \end{pmatrix}, \quad B = \begin{pmatrix} -M & K \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} 0 & b^T \end{pmatrix}^T,$$

then the system of linear equations (1.6) can be transformed into the standard saddle-point system:

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}. \tag{1.7}$$

Frequently, iterative methods are more attractive than direct methods for solving the saddle point problem (1.7), because the coefficient matrix of the saddle point problem (1.7) is large and sparse. Many efficient iterative methods have been studied in the literatures. For example, Uzawa-like methods ([8,10,12]), SOR-like methods ([7,14]), RPCG methods ([6, 9]), HSS-like methods ([2-5]) and so on. We refer to [2] for algebraic properties for saddle point problem (1.7). In this paper, we will focus on the systems that arise in the context of PDE-constrained optimization. Systems of the type given in (1.6) are typically very poorly conditioned and large sparse. Therefore preconditioning is usually necessary in practice in order to achieve rapid convergence of Krylov subspace methods.

In this paper, by exploiting the structure of the coefficient matrix, we construct a block-symmetric preconditioner and a block-lower-triangular preconditioner. The explicit expressions for the eigenvalues and eigenvectors of the corresponding preconditioned matrices are derived. Both theoretical analysis and numerical results show that the preconditioned GMRES(20) methods with these block preconditioners are effective and robust linear solvers for the saddle-point problems such as (1.6) from PDE-constrained optimization.

The organization of the paper is as follows. In Sections 2 and 3, we use the structure of the linear system (1.6) to give two block preconditioners. The explicit expressions for the eigenvalues of the two preconditioned matrices are derived. Numerical examples are given in Section 4 to show the effectiveness of these new preconditioners. Finally, we draw some conclusions in Section 5.

## 2. Block-Symmetric Preconditioning

For the linear system (1.6), Rees et al.[16, 17] presented the following block-diagonal preconditioner  $P_D$  and the constraint preconditioner  $P_C$  for MINRES and PCG, respectively:

$$P_D = \begin{pmatrix} 2\beta M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & KM^{-1}K^T \end{pmatrix}, \quad (2.1)$$

and

$$P_C = \begin{pmatrix} 0 & 0 & -M \\ 0 & 2\beta K^T M^{-1} K M & K^T \\ -M & K & 0 \end{pmatrix}. \quad (2.2)$$

As pointed out by Bai [1], the main costs in using the block-diagonal preconditioner  $P_D$  and the constraint preconditioner  $P_C$  come from solving linear sub-systems with the coefficient matrices  $M, K$  and  $K^T$ . The mass matrix  $M$  is often well-conditioned and the linear sub-systems associated with it is easily solvable. Thus the difficulty comes from the stiffness matrices  $K$  and  $K^T$ , which are the only parts that contain the partial differential equation, especially when such an equation is more general and complicated. To avoid the difficulty from inverting the stiffness matrices  $K$  and  $K^T$  in  $P_D$  and  $P_C$ , some authors ([1,16-17]) proposed to further approximate  $M$  and  $K$ , with certain strategies and by some matrices, say,  $\tilde{M}$  and  $\tilde{K}$ , respectively, resulting in the approximations:

$$\tilde{P}_D := \begin{pmatrix} 2\beta\tilde{M} & 0 & 0 \\ 0 & \tilde{M} & 0 \\ 0 & 0 & \tilde{K}M^{-1}\tilde{K}^T \end{pmatrix}, \quad (2.3)$$

and

$$\tilde{P}_C := \begin{pmatrix} 0 & 0 & -\tilde{M} \\ 0 & 2\beta\tilde{K}^T M^{-1} \tilde{K} M & K^T \\ -\tilde{M} & K & 0 \end{pmatrix}. \quad (2.4)$$

Recently, Bai [1] proposed the following block-counter-diagonal preconditioner  $P_{BCD}$  and block-counter-triangular preconditioner  $P_{BCT}$ :

$$P_{BCD} := \begin{pmatrix} 0 & 0 & -M \\ 0 & M & 0 \\ -M & 0 & 0 \end{pmatrix}, \quad P_{BCT} := \begin{pmatrix} 0 & 0 & -M \\ 0 & M & K^T \\ -M & K & 0 \end{pmatrix}. \quad (2.5)$$

Bai [1] showed that the block-counter-diagonal and the block-counter-triangular preconditioners are less costly and complicated than the block-diagonal and the constraint preconditioners in actual applications. In [3], Bai et al. developed a PMHSS iteration method and applied it to a class of KKT linear systems arising from the finite element discretization of a class of distributed control problems.

In this paper, by exploiting the structure of the coefficient matrix and using the idea of the preconditioner  $P_{BCD}$ , we construct a block-symmetric preconditioner and a block-lower-triangular preconditioner. We first consider the following block-symmetric preconditioner  $P_{BS}$  for the linear system (1.6):

$$P_{BS} := \begin{pmatrix} 2\beta M & 0 & -M \\ 0 & M & 0 \\ -M & 0 & 0 \end{pmatrix}. \quad (2.6)$$

Note that the preconditioner  $P_{BS}$  is symmetric. In addition, the action of  $P_{BS}$  only requires to solve three linear sub-systems with the coefficient matrix  $M$ , but does not need to solve any linear sub-system with the coefficient matrix  $K$  or  $K^T$ , neither need the exact nor the approximate computation of  $KM^{-1}K^T$ . Hence, we can expect that the block-symmetric preconditioner is less costly and complicated than the block-diagonal and the constraint preconditioners in actual applications. The following theorem gives the clustering of the eigenvalues of the preconditioned matrix with respect to  $P_{BS}$ .

**Theorem 2.1.** *Let  $\mathcal{A} \in \mathbb{R}^{3n \times 3n}$  be the coefficient matrix of the linear system (1.6) and  $P_{BS} \in \mathbb{R}^{3n \times 3n}$  be the block-symmetric preconditioner of  $\mathcal{A}$  defined in (2.6). Denote  $(\sigma_k, \mathbf{x}^{(k)})$  an eigenpair of the matrix  $M^{-1}KM^{-1}K^T \in \mathbb{R}^{n \times n}$ , where  $\mathbf{x}^{(k)} \in \mathbb{C}^n$ ,  $\sigma_k \geq 0, k = 1, \dots, n$ . Then we have the following facts:*

- (i) *The eigenvalues of the preconditioned matrix  $P_{BS}^{-1}\mathcal{A}$  are 1 (with algebraic multiplicity  $n$ ) and  $1 \pm \iota\sqrt{2\beta\sigma_k}$ ,  $k = 1, \dots, n$ , where  $\iota = \sqrt{-1}$  denotes the imaginary unit.*
- (ii) *The corresponding eigenvectors of the preconditioned matrix  $P_{BS}^{-1}\mathcal{A}$  are*

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \begin{pmatrix} \frac{\iota\sqrt{2\beta\sigma_k}}{2\beta\sigma_k}M^{-1}K\mathbf{x}^{(k)} \\ \mathbf{x}^{(k)} \\ \frac{\iota\sqrt{2\beta\sigma_k}}{2\beta\sigma_k}M^{-1}K\mathbf{x}^{(k)} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\frac{\iota\sqrt{2\beta\sigma_k}}{2\beta\sigma_k}M^{-1}K\mathbf{x}^{(k)} \\ \mathbf{x}^{(k)} \\ -\frac{\iota\sqrt{2\beta\sigma_k}}{2\beta\sigma_k}M^{-1}K\mathbf{x}^{(k)} \end{pmatrix}, \quad k = 1, \dots, n,$$

where  $v_1 \in \mathbb{C}^n$ ,  $v_2 \in \text{null}(K)$  and  $v_3 \in \text{null}(K^T) \setminus \{0\}$ .

*Proof.* We define the matrix  $R_{BS}$  as follows:

$$R_{BS} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -K^T \\ 0 & -K & 0 \end{pmatrix}.$$

Then we have  $\mathcal{A} = P_{BS} - R_{BS}$ . By concrete operations we have

$$P_{BS}^{-1} = \begin{pmatrix} 0 & 0 & -M^{-1} \\ 0 & M^{-1} & 0 \\ -M^{-1} & 0 & -2\beta M^{-1} \end{pmatrix}.$$

Then it follows that

$$\begin{aligned} P_{BS}^{-1}\mathcal{A} &= P_{BS}^{-1}(P_{BS} - R_{BS}) = I - P_{BS}^{-1}R_{BS} \\ &= I - \begin{pmatrix} 0 & M^{-1}K & 0 \\ 0 & 0 & -M^{-1}K^T \\ 0 & 2\beta M^{-1}K & 0 \end{pmatrix} \\ &= \begin{pmatrix} I & -M^{-1}K & 0 \\ 0 & I & M^{-1}K^T \\ 0 & -2\beta M^{-1}K & I \end{pmatrix}. \end{aligned}$$

Let  $(\mu, \mathbf{v})$  be an eigenpair of the matrix  $P_{BS}^{-1}\mathcal{A}$ , where  $\mathbf{v} = (v_1^*, v_2^*, v_3^*)^* \in \mathbb{C}^{3n}$ . Then

$$P_{BS}^{-1}\mathcal{A}\mathbf{v} = \mu\mathbf{v},$$

or equivalently,

$$\begin{pmatrix} I & -M^{-1}K & 0 \\ 0 & I & M^{-1}K^T \\ 0 & -2\beta M^{-1}K & I \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \mu \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}. \tag{2.7}$$

We can rewrite (2.7) as

$$\begin{cases} v_1 - M^{-1}Kv_2 = \mu v_1, \\ v_2 + M^{-1}K^T v_3 = \mu v_2, \\ v_3 - 2\beta M^{-1}Kv_2 = \mu v_3. \end{cases} \tag{2.8}$$

By inspection (2.8), one solution of this problem is  $\mu = 1$  with the algebraic multiplicity  $n$ .

Next we consider the case that  $\mu \neq 1$ . From (2.8), through simple computations we get

$$-\frac{2\beta}{\mu - 1}M^{-1}KM^{-1}K^T v_3 = (\mu - 1)v_3. \tag{2.9}$$

By multiplying  $\frac{v_3^*}{v_3^* v_3}$  from left on both sides of the third equation in (2.9) and considering the definition of  $\sigma_k$  in Theorem 2.1, we obtain

$$(\mu - 1)^2 = -2\beta\sigma_k, \quad k = 1, \dots, n,$$

which leads to

$$\mu = 1 \pm \iota\sqrt{2\beta\sigma_k}, \quad k = 1, \dots, n.$$

Thus we know that the remaining eigenvalues of the preconditioned matrix  $P_S^{-1}\mathcal{A}$  are  $1 \pm \iota\sqrt{2\beta\sigma_k}$ ,  $k = 1, \dots, n$ .

Now we turn to prove (ii). We consider the following three cases separately: (a)  $\mu = 1$ , (b)  $\mu = 1 + \iota\sqrt{2\beta\sigma_k}$ , and (c)  $\mu = 1 - \iota\sqrt{2\beta\sigma_k}$ .

(a) When  $\mu = 1$ , through direct computations, from (2.8) we have

$$M^{-1}Kv_2 = 0, \quad M^{-1}K^T v_3 = 0.$$

Since the matrix  $M$  is nonsingular, it is easily known that  $v_2 \in \text{null}(K)$  and  $v_3 \in \text{null}(K^T)$ . Therefore, the corresponding eigenvector in this case is

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad \text{where } v_1 \in \mathbb{C}^n, \quad v_2 \in \text{null}(K) \quad \text{and} \quad v_3 \in \text{null}(K^T) \setminus \{0\}.$$

(b) When  $\mu = 1 + \iota\sqrt{2\beta\sigma_k}$ , from the third equation in (2.8) we obtain  $(1 - \mu)v_3 = 2\beta M^{-1}Kv_2$ . As  $\mu \neq 1$ , by substituting this expression with respect to  $v_3$  into the second equation in (2.8) and considering the definition of  $\mathbf{x}^{(k)}$  in Theorem 2.1, we obtain  $v_2 = \mathbf{x}^{(k)}$ . Substituting  $v_2$  into the first and the third equations in (2.8), respectively, after some simple algebra computations we obtain that the corresponding eigenvectors are

$$\begin{pmatrix} \frac{\iota\sqrt{2\beta\sigma_k}}{2\beta\sigma_k}M^{-1}K\mathbf{x}^{(k)} \\ \mathbf{x}^{(k)} \\ \frac{\iota\sqrt{2\beta\sigma_k}}{2\beta\sigma_k}M^{-1}K\mathbf{x}^{(k)} \end{pmatrix}, \quad k = 1, \dots, n.$$

(c) When  $\mu = 1 - \iota\sqrt{2\beta\sigma_k}$ , based on the definition of  $\sigma_k$  and  $\mathbf{x}^{(k)}$ , from (2.8) we obtain that the corresponding eigenvectors are

$$\begin{pmatrix} -\frac{\iota\sqrt{2\beta\sigma_k}}{2\beta\sigma_k}M^{-1}K\mathbf{x}^{(k)} \\ \mathbf{x}^{(k)} \\ -\frac{\iota\sqrt{2\beta\sigma_k}}{2\beta\sigma_k}M^{-1}K\mathbf{x}^{(k)} \end{pmatrix}, \quad k = 1, \dots, n.$$

It is easily seen that the observations (a)-(c) readily imply the validity of the conclusion (ii).□

### 3. Block-Lower-Triangular Preconditioning

In this section, we consider the block-lower-triangular preconditioner

$$P_{BLT} := \begin{pmatrix} 2\beta M & 0 & 0 \\ 0 & M & 0 \\ -M & K & -\frac{1}{2\beta}M \end{pmatrix}. \tag{3.1}$$

Evidently, when we solve the generalized residual equation  $P_{BLT}\mathbf{Z} = \mathbf{r}$ , where  $\mathbf{r}$  is a prescribed right-hand-side vector, we only need to solve three linear sub-systems with respect to the mass matrix  $M$ , which is often well-conditioned. Thus the linear sub-systems associated with  $P_{BLT}$  can be easily solved when  $P_{BLT}$  is employed to precondition certain Krylov subspace method.

For the block-lower-triangular preconditioner  $P_{BLT}$ , we can demonstrate that the preconditioned matrix  $P_{BLT}^{-1}\mathcal{A}$  has eigenvalue 1 of algebraic multiplicity  $2n$ , and when  $\beta \ll 1$  it holds that  $\lambda \approx 1$ . This property is precisely stated in the following theorem.

**Theorem 3.1.** *Let  $\mathcal{A} \in \mathbb{R}^{3n \times 3n}$  be the coefficient matrix of the linear system (1.6) and  $P_{BLT} \in \mathbb{R}^{3n \times 3n}$  be the block-lower-triangular preconditioner of  $\mathcal{A}$  defined in (3.1). Assume that  $\sigma_k$  is an eigenvalue and  $\mathbf{x}^{(k)} \in \mathbb{C}^n$  is the corresponding eigenvector of the matrix  $M^{-1}KM^{-1}K^T \in \mathbb{R}^{n \times n}$ ,  $k = 1, \dots, n$ , where  $\sigma_k \geq 0$  ( $k = 1, \dots, n$ ). Then we have the following facts:*

(i) *The eigenvalues of the preconditioned matrix  $P_{BLT}^{-1}\mathcal{A}$  are 1 (with algebraic multiplicity  $2n$ ) and  $1 + 2\beta\sigma_k$ ,  $k = 1, \dots, n$ .*

(ii) *The eigenvectors of the preconditioned matrix  $P_{BLT}^{-1}\mathcal{A}$  are*

$$\begin{pmatrix} \mathbf{x}^{(k)} \\ \mathbf{y}^{(k)} \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\frac{1}{4\beta^2\sigma_k}\mathbf{x}^{(k)} \\ \frac{1}{2\beta\sigma_k}M^{-1}K^T\mathbf{x}^{(k)} \\ \mathbf{x}^{(k)} \end{pmatrix}, \quad k = 1, \dots, n,$$

where  $\mathbf{x}^{(k)}, \mathbf{y}^{(k)} \in \mathbb{C}^n \setminus \{0\}$ .

*Proof.* The proof is similar to that of Theorem 2.1. Note that

$$\mathcal{A} = P_{BLT} - R_{BLT}, \quad \text{with} \quad R_{BLT} = \begin{pmatrix} 0 & 0 & M \\ 0 & 0 & -K^T \\ 0 & 0 & -\frac{1}{2\beta}M \end{pmatrix}.$$

By straightforward computations we have

$$P_{BLT}^{-1} = \begin{pmatrix} \frac{1}{2\beta}M^{-1} & 0 & 0 \\ 0 & M^{-1} & 0 \\ -M^{-1} & 2\beta M^{-1}KM^{-1} & -2\beta M^{-1} \end{pmatrix},$$

and

$$\begin{aligned} P_{BLT}^{-1}\mathcal{A} &= P_{BLT}^{-1}(P_{BLT} - R_{BLT}) = I - P_{BLT}^{-1}R_{BLT} \\ &= I - \begin{pmatrix} 0 & 0 & \frac{1}{2\beta}I \\ 0 & 0 & -M^{-1}K^T \\ 0 & 0 & -2\beta M^{-1}KM^{-1}K^T \end{pmatrix} = \begin{pmatrix} I & 0 & -\frac{1}{2\beta}I \\ 0 & I & M^{-1}K^T \\ 0 & 0 & I + 2\beta M^{-1}KM^{-1}K^T \end{pmatrix}. \end{aligned}$$

Let  $\mu$  be an eigenvalue of the preconditioned matrix  $P_{BLT}^{-1}\mathcal{A}$  and  $v = (x^*, y^*, z^*)^*$  be the corresponding eigenvector. Then we have

$$P_{BLT}^{-1}\mathcal{A}v = \mu v,$$

or equivalently

$$\begin{cases} x - \frac{1}{2\beta}z = \mu x, \\ y + M^{-1}K^T z = \mu y, \\ z + 2\beta M^{-1}KM^{-1}K^T z = \mu z. \end{cases} \tag{3.2}$$

Obviously,  $\mu = 1$  is an eigenvalue of  $P_{BLT}^{-1}\mathcal{A}$  with algebraic multiplicity  $2n$ .

When  $\mu \neq 1$ , by multiplying  $\frac{z^*}{z^*z}$  from left on both sides of the third equation in (3.2), we obtain  $\mu - 1 = 2\beta\sigma_k$ , namely,  $\mu = 1 + 2\beta\sigma_k$  ( $k = 1, 2, \dots, n$ ) are the remaining eigenvalues of the matrix  $P_{BLT}^{-1}\mathcal{A}$ . This demonstrates the validity of (i).

Next we consider the eigenvectors of the preconditioned matrix  $P_{BLT}^{-1}\mathcal{A}$ .

(a) When  $\mu = 1$ , since  $\beta \neq 0$ , it is easily known that the corresponding eigenvectors are  $(x^*, y^*, z^*)^*$  with  $z = 0$ ,  $x \in \mathbb{C}^n \setminus \{0\}$  and  $y \in \mathbb{C}^n \setminus \{0\}$ .

(b) When  $\mu = 1 + 2\beta\sigma_k$ , from (3.2) we have  $z = \mathbf{x}^{(k)}$  and

$$x = -\frac{1}{4\beta^2\sigma_k}z, \quad y = \frac{1}{2\beta\sigma_k}M^{-1}K^T z.$$

In other words, when  $\mu = 1 + \iota\sqrt{2\beta\sigma_k}$  the corresponding eigenvectors are

$$\begin{pmatrix} -\frac{1}{4\beta^2\sigma_k}\mathbf{x}^{(k)} \\ \frac{1}{2\beta\sigma_k}M^{-1}K^T\mathbf{x}^{(k)} \\ \mathbf{x}^{(k)} \end{pmatrix}, \quad k = 1, \dots, n.$$

Combining (a)-(b), we have demonstrated the validity of (ii). □

Similarly to the discussion in [1] we can obtain the following remarks.

**Remark 3.1.** If the eigenvalues of the matrix  $M^{-1}KM^{-1}K^T$  are clustered or  $\beta$  is small, then the eigenvalues of the preconditioned matrices  $P_{BS}^{-1}\mathcal{A}$  and  $P_{BLT}^{-1}\mathcal{A}$  are clustered.

**Remark 3.2.** The splitting  $\mathcal{A} = P_{BS} - R_{BS}$  of the saddle-point matrix  $\mathcal{A} \in \mathbb{R}^{3n \times 3n}$  induced by the block-symmetric preconditioning matrix  $P_{BS}$  is convergent if and only if the matrix  $2\beta M^{-1}KM^{-1}K^T$  is convergent. Moreover, it holds that

$$\rho(P_{BS}^{-1}R_{BS}) \leq \sqrt{2\beta\rho(M^{-1}KM^{-1}K^T)} \rightarrow 0, \quad \text{as } \beta \rightarrow 0.$$

**Remark 3.3.** Analogously, the splitting  $\mathcal{A} = P_{BLT} - R_{BLT}$  of the saddle-point matrix  $\mathcal{A} \in \mathbb{R}^{3n \times 3n}$  induced by the block-lower-triangular preconditioning matrix  $P_{BLT}$  is convergent if and only if the matrix  $2\beta M^{-1}KM^{-1}K^T$  is convergent. Moreover, it holds that

$$\rho(P_{BLT}^{-1}R_{BLT}) \leq 2\beta\rho(M^{-1}KM^{-1}K^T) \rightarrow 0, \quad \text{as } \beta \rightarrow 0.$$

**Remark 3.4.** To reduce the computational complexity from inverting the matrix  $M$ , we propose to further approximate  $M$  with  $\tilde{M}$  in  $P_{BS}$  and  $P_{BLT}$ , resulting in the preconditioners,

$$\tilde{P}_{BS} = \begin{pmatrix} 2\beta\tilde{M} & 0 & -\tilde{M} \\ 0 & \tilde{M} & 0 \\ -\tilde{M} & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{P}_{BLT} = \begin{pmatrix} 2\beta\tilde{M} & 0 & 0 \\ 0 & \tilde{M} & 0 \\ -\tilde{M} & \tilde{K} & -\frac{1}{2\beta}\tilde{M} \end{pmatrix}.$$

The approximation  $\tilde{M}$  can be produced by a few steps of the Chebyshev semi-iteration, see[1, 16-17]. In this way, if  $\tilde{M}$  is a good approximation to  $M$ , then  $\tilde{P}_{BLT}$  should be a good approximation to  $P_{BLT}$ . It then follows that using  $\tilde{P}_{BS}$  and  $\tilde{P}_{BLT}$  as preconditioners will take only slightly more numbers of the Krylov subspace iteration steps than using  $P_{BS}$  and  $P_{BLT}$ .

### 4. Numerical Results

We examine our block preconditioners by using the following distributed control problem, which is Example 5.1 in [16-17] or Example 4.1 in [1]. Let  $\Omega = [0, 1]^2$  be a unit square and consider

$$\begin{aligned} \min_{u, f} \quad & \frac{1}{2} \|u - u_*\|_2^2 + \beta \|f\|_2^2, \\ \text{s.t.} \quad & -\nabla^2 u = f \quad \text{in } \Omega, \\ & u = u_* \quad \text{on } \partial\Omega, \end{aligned}$$

where

$$u_* = \begin{cases} (2x - 1)^2(2y - 1)^2, & \text{if } (x, y) \in [0, \frac{1}{2}]^2, \\ 0, & \text{otherwise.} \end{cases}$$

We solve the correspondingly discretized linear system by the preconditioned MINRES with the block-diagonal preconditioning matrix  $P_D$  and its approximation  $\tilde{P}_D$ , as well as by GMRES(20)

Table 4.1: Values of  $h$  and the correspondingly dimensions of  $M$ ,  $K$  and  $\mathcal{A}$ .

$h$	sizes of $M$	sizes of $K$	sizes of $\mathcal{A}$
$2^{-3}$	$49 \times 49$	$49 \times 49$	$147 \times 147$
$2^{-4}$	$225 \times 225$	$225 \times 225$	$675 \times 675$
$2^{-5}$	$961 \times 961$	$961 \times 961$	$2883 \times 2883$
$2^{-6}$	$3969 \times 3969$	$3969 \times 3969$	$11007 \times 11007$

preconditioned with the block-symmetric preconditioner  $P_{BS}$ , the block-counter-diagonal preconditioner  $P_{BCD}$ , the block-counter-triangular preconditioner  $P_{BCT}$ , the block-lower-triangular preconditioner  $P_{BLT}$  and their approximations  $\tilde{P}_{BS}$ ,  $\tilde{P}_{BCD}$ ,  $\tilde{P}_{BCT}$ ,  $\tilde{P}_{BLT}$ , respectively. Here, the approximation  $\tilde{K}$  to the matrix block  $K$  is set to be a two-geometric-AMG V-cycles generated by the amg operator in the software COMSOL Multiphysics, while the approximation  $\tilde{M}$  denotes 20 steps of the Chebyshev semi-iteration method. For this example we compare the CPU time and the number of iteration steps.

We discretize the optimality system using  $Q1$  finite elements. The tolerance for all methods

Table 4.2: Iteration numbers and CPU times for different mesh sizes and  $\beta$  for block-diagonal, block-count-diagonal and block-symmetric preconditioners.

$\beta$	$h$	$P_D$	$P_{BCD}$	$P_{BS}$	$\tilde{P}_D$	$\tilde{P}_{BCD}$	$\tilde{P}_{BS}$
$10^{-2}$	$2^{-3}$	8 (0.0204)	203 (4.0968)	116 (2.3330)	9 (0.0278)	282 (4.7263)	97 (2.1039)
	$2^{-4}$	8 (0.2541)	- -	- -	9 (0.0445)	9 (0.0815)	- -
	$2^{-5}$	8 (4.0586)	- -	- -	11 (0.7743)	- -	- -
	$2^{-6}$	8 (11.120)	- -	- -	17 (4.5437)	- -	- -
$10^{-4}$	$2^{-3}$	16 (0.0433)	39 (0.4404)	23 (0.2904)	19 (0.0389)	35 (0.6250)	20 (0.2644)
	$2^{-4}$	16 (0.3283)	98 (12.926)	91 (11.5415)	23 (0.1307)	98 (14.058)	93 (13.7941)
	$2^{-5}$	16 (6.9378)	- -	- -	34 (1.2512)	- -	- -
	$2^{-6}$	16 (203.04)	- -	- -	40 (24.8632)	- -	- -
$10^{-6}$	$2^{-3}$	55 (0.0856)	5 (0.0364)	4 (0.0267)	90 (0.2003)	5 (0.0335)	4 (0.0245)
	$2^{-4}$	65 (1.1850)	16 (0.9185)	24 (1.0115)	278 (4.3357)	16 (0.9264)	22 (0.7015)
	$2^{-5}$	71 (27.696)	- -	757 (737.54)	- -	- -	- -
	$2^{-6}$	78 (312.41)	- -	- -	- -	- -	- -
$10^{-8}$	$2^{-3}$	82 (0.1477)	6 (0.0142)	7 (0.0080)	138 (0.6311)	6 (0.0183)	5 (0.0077)
	$2^{-4}$	275 (4.7718)	9 (0.0728)	13 (0.0972)	599 (26.488)	9 (0.0429)	13 (0.0858)
	$2^{-5}$	464 (173.30)	11 (1.4729)	15 (2.1116)	- -	17 (1.5363)	18 (1.5722)
	$2^{-6}$	- -	15 (62.043)	20 (73.307)	- -	22 (70.5114)	21 (68.357)
$10^{-10}$	$2^{-3}$	82 (0.1277)	3 (0.0061)	5 (0.0066)	140 (0.6425)	6 (0.0159)	5 (0.0067)
	$2^{-4}$	349 (6.1208)	3 (0.0342)	5 (0.0344)	- -	6 (0.0729)	5 (0.0224)
	$2^{-5}$	- -	6 (0.3946)	7 (0.3180)	- -	9 (0.5784)	5 (0.2954)
	$2^{-6}$	- -	6 (2.1743)	8 (2.1425)	- -	10 (3.3918)	8 (2.3775)
$10^{-12}$	$2^{-3}$	80 (0.1291)	3 (0.0055)	3 (0.0044)	140 (0.6315)	4 (0.0058)	4 (0.0048)
	$2^{-4}$	352 (6.7259)	3 (0.0351)	3 (0.0331)	- -	6 (0.0271)	5 (0.0402)
	$2^{-5}$	- -	3 (0.2491)	3 (0.2110)	- -	6 (0.3017)	5 (0.2993)
	$2^{-6}$	- -	5 (1.0199)	5 (0.9948)	- -	6 (1.0013)	5 (0.9877)
$10^{-14}$	$2^{-3}$	82 (0.1308)	3 (0.0045)	3 (0.0040)	140 (0.6332)	4 (0.0057)	4 (0.0041)
	$2^{-4}$	358 (6.9011)	3 (0.0327)	3 (0.0212)	- -	4 (0.0389)	4 (0.0273)
	$2^{-5}$	- -	3 (0.1525)	3 (0.1105)	- -	4 (0.2014)	4 (0.1459)
	$2^{-6}$	- -	5 (1.1482)	3 (1.0072)	- -	4 (1.0035)	4 (0.9925)

is set to be  $10^{-6}$ . That is to say, all iteration processes are terminated when the current residuals satisfy

$$\frac{\|\mathbf{b} - \mathcal{A}x^{(k)}\|_2}{\|\mathbf{b}\|_2} \leq 10^{-6}$$

or the numbers of iteration steps are over  $k_{max} = 1000$ .  $x^{(0)} = 0$  is the initial guess and  $x^{(k)}$  is the  $k$ th iterates of the corresponding iteration processes, respectively. In addition, the mesh step-size  $h$  is determined by  $h = 1/(\sqrt{n} + 1)$ , where  $n$  is the dimension of the matrix  $M$ . We list the dimensions of the matrices  $M, K$  and  $\mathcal{A}$  in Table 4.1.

In Table 4.2, we report the numbers of iteration steps and the computing times (in braces) with respect to the block-diagonal preconditioner, the block-counter-triangular preconditioner, the block-symmetric preconditioner and their approximated variants, which are employed to precondition MINRES and GMRES(20), respectively.

From Table 4.2 we see that the computing efficiency of these preconditioned iteration methods are dependent on the regularization parameter  $\beta$ . We observe that when  $\beta$  is large, e.g.,  $\beta = 10^{-2}$  and  $10^{-4}$ , the block-diagonal preconditioner  $P_D$  is efficient. However, when  $\beta$  is

Table 4.3: Iteration counts and CPU timings for block-counter-triangular and block-lower-triangular preconditioners.

$\beta$	$h$	$P_{BCT}$		$P_{BLT}$		$\tilde{P}_{BCT}$		$\tilde{P}_{BLT}$	
$10^{-2}$	$2^{-3}$	130	(3.2747)	112	(2.9458)	131	(3.3901)	97	(2.5516)
	$2^{-4}$	-	-	-	-	-	-	-	-
	$2^{-5}$	-	-	-	-	-	-	-	-
	$2^{-6}$	-	-	-	-	-	-	-	-
$10^{-4}$	$2^{-3}$	20	(0.1610)	31	(0.4317)	22	(0.1904)	28	(0.2475)
	$2^{-4}$	70	(11.317)	-	-	42	(5.5694)	-	-
	$2^{-5}$	-	-	-	-	-	-	-	-
	$2^{-6}$	-	-	-	-	-	-	-	-
$10^{-6}$	$2^{-3}$	13	(0.0439)	17	(0.0376)	13	(0.0582)	16	(0.0225)
	$2^{-4}$	16	(0.7136)	115	(2.2915)	16	(0.8031)	110	(1.9357)
	$2^{-5}$	33	(21.073)	-	-	32	(19.883)	-	-
	$2^{-6}$	-	-	-	-	-	-	-	-
$10^{-8}$	$2^{-3}$	7	(0.0179)	4	(0.0163)	6	(0.0154)	4	(0.0090)
	$2^{-4}$	12	(0.1280)	8	(0.1221)	12	(0.1617)	6	(0.0945)
	$2^{-5}$	14	(2.9503)	7	(2.8063)	14	(1.9902)	9	(1.0304)
	$2^{-6}$	17	(67.183)	16	(61.332)	17	(70.241)	11	(32.850)
$10^{-10}$	$2^{-3}$	3	(0.0139)	3	(0.0098)	3	(0.0108)	4	(0.0064)
	$2^{-4}$	5	(0.0756)	5	(0.0584)	5	(0.0938)	5	(0.0830)
	$2^{-5}$	5	(0.4818)	5	(0.4133)	5	(0.8793)	5	(0.6941)
	$2^{-6}$	11	(6.6868)	7	(5.0334)	11	(7.7745)	7	(6.1131)
$10^{-12}$	$2^{-3}$	2	(0.0088)	2	(0.0081)	2	(0.0102)	2	(0.0061)
	$2^{-4}$	3	(0.0512)	2	(0.0386)	3	(0.0887)	2	(0.0400)
	$2^{-5}$	3	(0.2792)	3	(0.2620)	3	(0.3317)	3	(0.2600)
	$2^{-6}$	5	(2.2386)	4	(1.5520)	5	(2.4315)	4	(1.3792)
$10^{-14}$	$2^{-3}$	2	(0.0076)	2	(0.0077)	2	(0.0080)	2	(0.0059)
	$2^{-4}$	2	(0.0511)	2	(0.0384)	2	(0.0514)	2	(0.0328)
	$2^{-5}$	2	(0.2320)	2	(0.2049)	2	(0.6717)	2	(0.2141)
	$2^{-6}$	3	(1.6589)	3	(1.0959)	3	(1.7753)	3	(0.9989)

Table 4.4: Numerical results for optimal regularization parameter  $\beta$ .

$h$	$2^{-3}$			$2^{-4}$			$2^{-5}$			$2^{-6}$		
	$\beta$	CPU	IT									
$P_D$	$10^{-2}$	0.0204	8	$10^{-2}$	0.2541	8	$10^{-2}$	4.0586	8	$10^{-2}$	11.120	8
$\tilde{P}_D$	$10^{-2}$	0.0278	8	$10^{-2}$	0.0445	9	$10^{-2}$	0.7743	11	$10^{-2}$	4.5437	17
$P_{BCD}$	$10^{-14}$	0.0045	3	$10^{-14}$	0.0327	3	$10^{-14}$	0.1525	3	$10^{-12}$	1.0199	5
$\tilde{P}_{BCD}$	$10^{-14}$	0.0057	4	$10^{-12}$	0.0271	6	$10^{-14}$	0.2014	4	$10^{-12}$	1.0013	6
$P_{BS}$	$10^{-14}$	0.0040	3	$10^{-14}$	0.0212	3	$10^{-14}$	0.1105	3	$10^{-12}$	0.9948	5
$\tilde{P}_{BS}$	$10^{-14}$	0.0041	4	$10^{-14}$	0.0273	4	$10^{-14}$	0.1459	4	$10^{-14}$	0.9877	5
$P_{BCT}$	$10^{-14}$	0.0076	2	$10^{-14}$	0.0511	2	$10^{-14}$	0.2320	2	$10^{-14}$	1.6589	3
$\tilde{P}_{BCT}$	$10^{-14}$	0.0080	2	$10^{-14}$	0.0514	2	$10^{-14}$	0.6717	2	$10^{-14}$	1.7753	3
$P_{BLT}$	$10^{-14}$	0.0077	2	$10^{-14}$	0.0384	2	$10^{-14}$	0.2049	2	$10^{-14}$	1.0959	3
$\tilde{P}_{BLT}$	$10^{-14}$	0.0059	2	$10^{-14}$	0.0328	2	$10^{-14}$	0.2141	2	$10^{-14}$	0.9989	3

small, e.g.,  $\beta = 10^{-8}, 10^{-10}, 10^{-12}$  and  $10^{-14}$ , the block-count-diagonal preconditioner  $P_{BCD}$ , the block-symmetric preconditioner  $P_{BS}$  and their approximation  $\tilde{P}_{BCD}$  and  $\tilde{P}_{BS}$  yield very good computing results. The corresponding preconditioned iteration methods require very small iteration steps and computing times to achieve the stopping criterion, and the numbers of iteration steps are almost  $h$ -independent. That is to say, as preconditioners for MINRES and GMRES(20),  $P_D$  and  $\tilde{P}_D$  show nice preconditioning effect when  $\beta$  is large, while  $P_{BCD}$ ,  $P_{BS}$  and  $\tilde{P}_{BCD}$ ,  $\tilde{P}_{BS}$  show nice preconditioning effect when  $\beta$  is small.

In Table 4.3, the computing times are given in braces after the numbers of iterations for the block-counter-diagonal preconditioner  $P_{BCD}$ , the block-lower-triangular preconditioner  $P_{BLT}$  and their approximated variants, which are all employed to precondition GMRES(20) methods.

Similarly, from Table 4.3 we see that the computing efficiency of the preconditioned iteration methods are also dependent on the regularization parameter  $\beta$ . When  $\beta$  is large, e.g.,  $\beta = 10^{-2}$  and  $10^{-4}$ , both preconditioners  $P_{BCT}$  and  $P_{BLT}$  as well as their approximations  $\tilde{P}_{BCT}$  and  $\tilde{P}_{BLT}$  are almost invalid, and the numbers of failures for these preconditioners are almost the same. However, when  $\beta \ll 1$ , for example,  $\beta = 10^{-8}, 10^{-10}, 10^{-12}$  or  $10^{-14}$ , the block-counter-diagonal preconditioner  $P_{BCT}$  and the block-lower-triangular preconditioner  $P_{BLT}$  yield very nice computing results, and the numbers of iteration steps are almost  $h$ -independent. But comparing with the numbers of iteration steps and the computing times, the block-lower-triangular preconditioner  $P_{BLT}$  has an advantage over  $P_{BCT}$  in totally, though, there are some oppose situation in locally and the same results appear in the approximation preconditioner. In addition, when  $\beta = 10^{-12}$  and  $10^{-14}$ , the numbers of iteration steps of  $P_{BLT}$  are almost equal to 2 and are  $h$ -independent. Therefore, as preconditioners of GMRES(20), when  $\beta$  is large,  $P_D$  and  $\tilde{P}_D$  yield nice preconditioning effect, while  $P_{BLT}$  and  $P_{BCT}$  show very good preconditioning effect when  $\beta$  is small,  $P_{BLT}$  is more effective than  $P_{BCT}$  at most situations.

In Table 4.4, we list numerical results for optimal regularization parameter  $\beta$ . This numerical phenomenon coincides with our theoretical analysis about the eigenvalues and eigenvectors of the preconditioned matrices.

## 5. Concluding Remarks

PDE-constrained optimization problems arise widely in many fields, and efficient solution of the problem heavily depends on preconditioning techniques. In this paper we have presented

two efficient preconditioning strategies for solving linear systems from PDE-constrained optimization problems. The preconditioners are employed to appropriate Krylov subspace methods. Both preconditioners avoid the solution of an ill-conditioned matrix  $K$ . We have demonstrated that our preconditioners work effectively with regularization parameter  $\beta \leq 10^{-6}$ . In addition, when  $\beta \geq 10^{-8}$ , the block-symmetric preconditioner is good, and when  $\beta \leq 10^{-10}$  the block-lower-triangular preconditioner is effective.

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