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STRONG PREDICTOR-CORRECTOR APPROXIMATION FOR STOCHASTIC DELAY DIFFERENTIAL EQUATIONS*

Yuanling Niu
School of Mathematics and Statistics, Central South University, Changsha 410075, China
School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China
Email: yuanlingniu@csu.edu.cn
Chengjian Zhang
School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China
Email: cjzhang@mail.hust.edu.cn
Kevin Burrage
Department of Computer Science, Oxford University, Wolfson Building, Parks Road, Oxford, OX1 3QD, UK
School of Mathematical Sciences, Queensland University of Technology, Brisbane, Australia
Email: kevin.burrage@gmail.com

Abstract

This paper presents a strong predictor-corrector method for the numerical solution of stochastic delay differential equations (SDDEs) of Itô-type. The method is proved to be mean-square convergent of order $\min\{1/2, \hat{p}\}$ under the Lipschitz condition and the linear growth condition, where $\hat{p}$ is the exponent of Hölder condition of the initial function. Stability criteria for this type of method are derived. It is shown that for certain choices of the flexible parameter p the derived method can have a better stability property than more commonly used numerical methods. That is, for some p, the asymptotic MS-stability bound of the method will be much larger than that of the Euler-Maruyama method. Numerical results are reported confirming convergence properties and comparing stability properties of methods with different parameters p. Finally, the vectorised simulation is discussed and it is shown that this implementation is much more efficient.

Mathematics subject classification: 65C30, 60H35.
Key words: Strong predictor-corrector approximation, Stochastic delay differential equations, Convergence, Mean-square stability, Numerical experiments, Vectorised simulation.

1. Introduction

In many scientific fields, such as biology, economics, medicine and finance, stochastic delay differential equations (SDDEs) are often used to model complex dynamics. Such equations generalize both deterministic delay differential equations (DDEs) and stochastic ordinary differential equations (SODEs). For the general theory on SDDEs, one can refer to Mao [22] and Mohammed [24].

Explicit solutions of SDDEs can rarely be obtained. Thus, it has become an important issue to develop numerical methods for SDDEs. In the last several decades, the research in

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the computational implementation and the numerical analysis for SODEs has made a lot of advances. An overview of these results can be found in some monographs and survey papers, see for example $[1, 12, 13, 15, 25, 27]$.

The research into numerical methods for SDDEs is relatively new, compared with that for DDEs and SODEs. In recent years a number of numerical methods have been developed for SDDEs. For an introduction to the numerical analysis of SDDEs see Buckwar [7]. Baker & Buckwar [3] and Buckwar [8] derived several convergence results for one-step methods. Küchler & Platen [18] proposed the adapted low order Taylor methods for SDDEs. Moreover, for linear SDDEs, Baker & Buckwar [4], Cao, Liu & Fan [21] and Wang & Zhang [26] studied the stability properties of Euler-Maruyama method, semi-Euler method and Milstein method, respectively.

As in the deterministic case, using an explicit numerical scheme to solve a stiff system often results in instability and hence generates an inaccurate numerical solution. However, when an implicit method is used, the numerical stability and the computational accuracy can be greatly improved (cf. [14]). Hence, implicit numerical methods are preferred for the effective computation of numerical solutions to stiff systems. In the references $[2, 17, 23]$, for solving stiff SODEs, the authors introduced implicitness into the approximation of the diffusion term and obtained several classes of the balanced implicit method. Here, an SODE is said to be stiff if it has widely varying lyapunov exponents. To implement an implicit method, generally speaking, an algebraic equation has to be solved at each time step, leading to a large computational cost. In order to resolve this difficulty, in papers $[5, 6, 10]$, authors presented a few predictor-corrector schemes. Furthermore, Li et al. developed a family of strong predictor-corrector Euler-Maruyama methods for SODEs with Markovian switching, which were shown to converge with strong order 0.5 in [20]. But they did not take time delays into account. For SDDE with constant delay in Stratonovich form, Cao et al. [11] presented a predictor-corrector scheme using the Wong-Zakai approximation as an intermediate step, and proved the predictor-corrector scheme is of half-order convergence in the mean-square. This method was derived from the trapezoidal rule and does not have any free parameters. However, the performance of the predictor-corrector methods presented in this paper is tunable through the use of a free parameter p that controls the size of its stability region and hence the step size.

So far, to the best of our knowledge, no strong predictor-corrector scheme has been applied to SDDEs in Itô form. Hence in this paper we will focus on such a topic. We attempt to avoid implicit methods by using explicit methods with larger stability regions to deal with moderately stiff problems. The strong Euler predictor-corrector methods will be extended to solve SDDEs of Itô-type. The adapted method will be proved to be convergent of order $\min\{1/2, \hat{p}\}$ under the Lipschitz condition and the linear growth condition, where $\hat{p}$ is the exponent of Hölder condition of the initial function. We also investigate the asymptotic mean-square stability of the extended predictor-corrector method. Numerical stability criterion is derived which shows that this type of method preserves the asymptotic MS-stability of the underlying equation. Numerical examples will be given to illustrate these theoretical results. It is shown that for certain choices of the flexible parameter p the method presented here can have a larger stability bound than the Euler-Maruyama method. We also demonstrate that substantial speed-ups are possible by vectorising across the simulations the implementation of the numerical method.

2. The Strong Predictor-corrector Method

Let $W(t)$ be a one-dimensional standard Wiener process defined on the filtered probability space $(\Omega, \mathscr{A}, P)$, and $C([-\tau, 0]; \mathbb{R})$ denote the Banach space consisting of all continuous paths from $[-\tau, 0]$ to $\mathbb{R}$, equipped with the norm $\|\mu\| = \sup_{s\in[-\tau,0]} |\mu(s)|$. Consider the following Itô-type scalar SDDEs with delay $\tau > 0$:

$$\begin{cases} dX(t) = f(X(t), X(t-\tau))dt + g(X(t), X(t-\tau))dW(t), & t \in [0, T], \\ X(t) = \psi(t), \quad t \in [-\tau, 0], \end{cases} \tag{2.1}$$

where $\psi(t)$ is an $\mathscr{A}_{t_0}$-measurable $C([-\tau, 0]; \mathbb{R})$-valued random variable with $E\|\psi\|^2 < \infty$, and functions $f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous. The unique solvability of equation (2.1) can be judged by the following proposition.

Proposition 2.1. ([22]) *Assume that there exist positive constants L, K such that for all $\xi_1, \xi_2, \eta_1, \eta_2 \in C([-\tau, 0]; \mathbb{R})$ and $t \in [0, T]$,*

$$|f(\xi_1, \eta_1) - f(\xi_2, \eta_2)| \vee |g(\xi_1, \eta_1) - g(\xi_2, \eta_2)| \le L(|\xi_1 - \xi_2| + |\eta_1 - \eta_2|) \tag{2.2}$$

and

$$|f(\xi, \eta)|^2 \vee |g(\xi, \eta)|^2 \le K(1 + |\xi|^2 + |\eta|^2). \tag{2.3}$$

Then equation (2.1) *has a unique strong solution $X(t)$. Here, $\vee$ means the maximum of two values.*

For numerically solving SDDEs (2.1), we take a uniform mesh on $[0, T]$:

$$0 = t_0 < t_1 < t_2 < \ldots < t_N \le T,$$

where $t_n = t_0 + nh$, $h = \tau/m$ $(m \in \mathbb{N})$, X_n denotes a strong approximation to $X(t_n)$, and it is assumed that the increment of the driving Wiener process: $\triangle W_n := W(t_{n+1}) - W(t_n)$ is an $N(0, h)$-distributed Gaussian random variable. With these settings, the strong Euler predictor-corrector method can be applied to SDDEs (2.1). The predictor is given by

$$\overline{X}_{n+1} = X_n + f(X_n, X_{n-m})h + g(X_n, X_{n-m})\triangle W_n, \tag{2.4}$$

and the corrector is given by

$$X_{n+1} = X_n + \{pf(\overline{X}_{n+1}, X_{n+1-m}) + (1-p)f(X_n, X_{n-m})\}h + g(X_n, X_{n-m})\triangle W_n, \tag{2.5}$$

where the parameter $p \in [0, 1]$ is called the degree of implicitness in the drift coefficient. When substituting (2.4) into (2.5), method $\{(2.4), (2.5)\}$ can be written in a compact form:

$$\begin{aligned} X_{n+1} = X_n &+ f(X_n, X_{n-m})h + g(X_n, X_{n-m})\triangle W_n + p\{f(X_n + f(X_n, X_{n-m})h \\ &+ g(X_n, X_{n-m})\triangle W_n, X_{n+1-m}) - f(X_n, X_{n-m})\}h. \end{aligned} \tag{2.6}$$

Throughout this paper, we define that, when a meshpoint t_n falls in the initial interval $[-\tau, 0]$, the corresponding approximation solution equals its explicit solution. The predictor-corrector method $\{(2.4), (2.5)\}$ have some potential advantages. Firstly, the use of the implicit scheme as a corrector can improve numerical stability, while avoiding to solve a nonlinear equation at each time step. Secondly, for certain choices of parameter p the stability bound of the method will be much larger than that of the Euler-Maruyama method.

3. Some Basic Lemmas

In order to derive the convergence result of the method, we first introduce some basic lemmas in this section. The following notations and concepts will be used. Write

$$\begin{aligned} & R(h, X_n, X_{n-m}, X_{n-m+1}, \triangle W_n) \\ := & \{f(X_n + f(X_n, X_{n-m})h + g(X_n, X_{n-m})\triangle W_n, X_{n+1-m}) - f(X_n, X_{n-m})\}h, \end{aligned}$$

and

$$\begin{aligned} & \Phi(h, X_n, X_{n-m}, X_{n-m+1}, \triangle W_n) \\ := & f(X_n, X_{n-m})h + g(X_n, X_{n-m})\triangle W_n + pR(h, X_n, X_{n-m}, X_{n-m+1}, \triangle W_n), \\ & \widehat{X}(t_{n+1}) := X(t_n) + \Phi(h, X(t_n), X(t_n - \tau), X(t_{n+1} - \tau), \triangle W_n). \end{aligned}$$

Definition 3.1. *A function φ is called Hölder-continuous with exponent $\hat{p}$ if there exists a positive constant M such that*

$$E|\varphi(t) - \varphi(s)|^2 \leq M|t-s|^{2\hat{p}}, \quad \forall t,\ s \in [-\tau, 0]. \tag{3.1}$$

Definition 3.2. *Method $\{(2.4), (2.5)\}$ is called consistent of order $\hat{q}$ in the mean-square sense if its local error $\delta_n := X(t_n) - \widehat{X}(t_n)$ satisfies*

$$\max_{1\leq n\leq N}(E|\delta_n|^2)^{1/2} = \mathcal{O}(h^{\hat{q}+1/2}) \quad as \quad h \to 0 \tag{3.2}$$

and

$$\max_{1\leq n\leq N}(E|E(\delta_n|\mathscr{A}_{t_{n-1}})|^2)^{1/2} = \mathcal{O}(h^{\hat{q}+1}) \quad as \quad h \to 0. \tag{3.3}$$

Lemma 3.1. *Suppose that functions f, g satisfy the conditions* (2.2)-(2.3). *Then, there exist constants $C_1, C_2, h_1 > 0$ such that, when $0 < h \leq h_1$,*

$$\begin{aligned} & E(|\Phi(h, \xi_1, \eta_1, r_1, \triangle W_n) - \Phi(h, \xi_2, \eta_2, r_2, \triangle W_n)|^2) \\ \leq & C_1 h(|\xi_1 - \xi_2|^2 + |\eta_1 - \eta_2|^2 + |r_1 - r_2|^2), \quad \forall \xi_1, \xi_2, \eta_1, \eta_2, r_1, r_2 \in \mathbb{R} \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} & |E(\Phi(h, \xi_1, \eta_1, r_1, \triangle W_n) - \Phi(h, \xi_2, \eta_2, r_2, \triangle W_n))| \\ \leq & C_2 h(|\xi_1 - \xi_2| + |\eta_1 - \eta_2| + |r_1 - r_2|), \quad \forall \xi_1, \xi_2, \eta_1, \eta_2, r_1, r_2 \in \mathbb{R}. \end{aligned} \tag{3.5}$$

Proof. It follows from condition (2.2) and the Hölder inequality

$$\left(\sum_{i=1}^n x_i\right)^j \leq n^{j-1}\sum_{i=1}^n x_i^j, \quad x_i \in \mathbb{R},\ j \geq 1$$

that

$$
\begin{aligned}
&E(|\Phi(h,\xi_1,\eta_1,r_1,\triangle W_n)-\Phi(h,\xi_2,\eta_2,r_2,\triangle W_n)|^2)\\
\le\ & 6E\left[h^2|f(\xi_1,\eta_1)-f(\xi_2,\eta_2)|^2+h|g(\xi_1,\eta_1)-g(\xi_2,\eta_2)|^2+p^2h^2|f(\xi_1+f(\xi_1,\eta_1)h\right.\\
&\left.+g(\xi_1,\eta_1)\triangle W_n,r_1)-f(\xi_2+f(\xi_2,\eta_2)h+g(\xi_2,\eta_2)\triangle W_n,r_2)|^2+p^2h^2|f(\xi_1,\eta_1)-f(\xi_2,\eta_2)|^2\right]\\
&\le 6E\left[h^2(1+p^2)|f(\xi_1,\eta_1)-f(\xi_2,\eta_2)|^2+h|g(\xi_1,\eta_1)-g(\xi_2,\eta_2)|^2+2p^2h^2L^2(|\xi_1+f(\xi_1,\eta_1)h\right.\\
&\left.+g(\xi_1,\eta_1)\triangle W_n-\xi_2-f(\xi_2,\eta_2)h-g(\xi_2,\eta_2)\triangle W_n|^2+|r_1-r_2|^2)\right]\\
\le\ & 6[2L^2h(1+p^2)+2L^2+6L^2(1+2h^2L^2+2hL^2)p^2h]h|\xi_1-\xi_2|^2+6[2L^2h(1+p^2)+2L^2\\
&+6L^2(2h^2L^2+2hL^2)p^2h]h|\eta_1-\eta_2|^2+12L^2p^2h^2|r_1-r_2|^2]\\
\le\ & C_1h(|\xi_1-\xi_2|^2+|\eta_1-\eta_2|^2+|r_1-r_2|^2),\quad 0<h\le h_1,
\end{aligned}
$$

where

$$
\begin{aligned}
C_1=6L^2\max\{&2h_1(1+p^2)+2+6(1+2h_1^2L^2+2h_1L^2)p^2h_1,\\
&2h_1(1+p^2)+2+6(2h_1^2L^2+2h_1L^2)p^2h_1,\ 2p^2h_1\}.
\end{aligned}
$$

Similarly, by condition (2.2) we have

$$
\begin{aligned}
&|E(\Phi(h,\xi_1,\eta_1,r_1,\triangle W_n)-\Phi(h,\xi_2,\eta_2,r_2,\triangle W_n))|\\
&\le C_2h(|\xi_1-\xi_2|+|\eta_1-\eta_2|+|r_1-r_2|),\quad 0<h\le h_1,
\end{aligned}
$$

where $C_2=L\max\{1+pLh_1^{1/2}+pLh_1,1-p+ph_1^{1/2}+ph_1,p\}$. This completes the proof.

Lemma 3.2. ([16]) *Assume that the condition* (2.3) *holds. Then for any given* $T>0$*, there exist constants* $C_3,C_4>0$ *such that the solution of equation* (2.1) *satisfies*

$$
E\left(|X(t)-X(s)|^2\right)\le C_3|t-s|,\quad \textit{for}\ \ t,s:\ 0\le s\le t\le T, \tag{3.6}
$$

and

$$
E\left(\sup_{-\tau\le t\le T}|X(t)|^2\right)\le C_4\left(1+E\|\psi\|^2\right),\quad \forall t\in[-\tau,T]. \tag{3.7}
$$

Lemma 3.3. *Suppose that initial function* ψ *is Hölder-continuous with exponent* $\hat{p}$*. Then the following inequality holds:*

$$
\begin{aligned}
&\int_{t_n}^{t_{n+1}}[E|X(s)-X(t_n)|^2+E|X(s-\tau)-X(t_n-\tau)|^2]ds\\
&\le C_3h^2+Mh^{1+2\hat{p}},\qquad 0\le n\le N-1.
\end{aligned}\tag{3.8}
$$

Proof. The conclusion can be followed by discussion in two cases. When $t_{n+1}\le\tau$, by Lemma 3.2 and Definition 3.1, it holds that

$$
\begin{aligned}
&\int_{t_n}^{t_{n+1}}[E|X(s)-X(t_n)|^2+E|X(s-\tau)-X(t_n-\tau)|^2]ds\\
=&\int_{t_n}^{t_{n+1}}[E|X(s)-X(t_n)|^2+E|\psi(s-\tau)-\psi(t_n-\tau)|^2]ds\\
\le&\int_{t_n}^{t_{n+1}}[C_3(s-t_n)+M(s-t_n)^{2\hat{p}}]ds\\
\le\ & C_3h^2+Mh^{1+2\hat{p}}.
\end{aligned}
$$

When $\tau \leq t_n$, Lemma 3.2 implies that

$$\begin{aligned}&\int_{t_n}^{t_{n+1}} [E|X(s) - X(t_n)|^2 + E|X(s-\tau) - X(t_n-\tau)|^2]ds \\ &\leq 2\int_{t_n}^{t_{n+1}} C_3(s-t_n)ds = C_3h^2.\end{aligned}$$

This shows that (3.8) always holds for $0 \leq n \leq N-1$. Hence the conclusion is proven.

Lemma 3.4. *Suppose that functions f, g satisfy the conditions* (2.2)-(2.3), *and the initial function ψ is Hölder-continuous with exponent $\hat{p}$. Then method* {(2.4), (2.5)} *is consistent of order* $\min\{1/2, \hat{p}\}$ *in the mean-square sense.*

Proof. Let

$$\begin{aligned}I_{1,n} &= \int_{t_n}^{t_{n+1}} f(X(s), X(s-\tau)) - f(X(t_n), X(t_n-\tau))ds, \\ I_{2,n} &= \int_{t_n}^{t_{n+1}} g(X(s), X(s-\tau)) - g(X(t_n), X(t_n-\tau))dW(s).\end{aligned}$$

Then

$$\delta_{n+1} = I_{1,\ n} + I_{2,\ n} - pR(h, X(t_n), X(t_n-\tau), X(t_{n+1}-\tau), \triangle W_n), \tag{3.9}$$

which gives

$$\begin{aligned}&E|\delta_{n+1}|^2 \\ &\leq 4(p^2E|R(h, X(t_n), X(t_n-\tau), X(t_{n+1}-\tau), \triangle W_n)|^2 + E|I_{1,n}|^2 + E|I_{2,n}|^2).\end{aligned} \tag{3.10}$$

In the following, we further estimate the various items of the right-hand side in (3.10). Using the Hölder inequality and the condition (2.2), we obtain

$$\begin{aligned}E|I_{1,n}|^2 &\leq E\left[\int_{t_n}^{t_{n+1}} |f(X(s), X(s-\tau)) - f(X(t_n), X(t_n-\tau))|ds\right]^2 \\ &\leq h\int_{t_n}^{t_{n+1}} E|f(X(s), X(s-\tau)) - f(X(t_n), X(t_n-\tau))|^2ds \\ &\leq 2hL^2\int_{t_n}^{t_{n+1}} [E|X(s) - X(t_n)|^2 + E|X(s-\tau) - X(t_n-\tau)|^2]ds.\end{aligned} \tag{3.11}$$

Inserting (3.8) into (3.11) yields

$$E|I_{1,n}|^2 \leq 2C_3L^2h^3 + 2ML^2h^{2+2\hat{p}}, \quad for \ \ 0 \leq n \leq N-1. \tag{3.12}$$

It follows from the Itô isometry and the inequality (2.2) that

$$\begin{aligned}E|I_{2,n}|^2 &= E\left|\int_{t_n}^{t_{n+1}} g(X(s), X(s-\tau)) - g(X(t_n), X(t_n-\tau))dW(s)\right|^2 \\ &= \int_{t_n}^{t_{n+1}} E|g(X(s), X(s-\tau)) - g(X(t_n), X(t_n-\tau))|^2ds \\ &\leq 2L^2\int_{t_n}^{t_{n+1}} [E|X(s) - X(t_n)|^2 + E|X(s-\tau) - X(t_n-\tau)|^2]ds.\end{aligned} \tag{3.13}$$

This, together with (3.8) gives

$$E|I_{2,n}|^2 \leq 2C_3L^2h^2 + 2ML^2h^{1+2\hat{p}}, \qquad 0 \leq n \leq N-1. \tag{3.14}$$

Moreover, we have

$$\begin{aligned}
&E|R(h, X(t_n), X(t_n-\tau), X(t_{n+1}-\tau), \triangle W_n)|^2\\
&= E\Big|\Big[f(X(t_n)+f(X(t_n), X(t_n-\tau))h + g(X(t_n), X(t_n-\tau))\triangle W_n, X(t_{n+1}-\tau))h\\
&\quad -\int_{t_n}^{t_{n+1}} f(X(s), X(s-\tau))ds\Big] - \Big[f(X(t_n), X(t_n-\tau))h - \int_{t_n}^{t_{n+1}} f(X(s), X(s-\tau))ds\Big]\Big|^2\\
&\leq 2E\Big|\int_{t_n}^{t_{n+1}} [f(X(t_n) + f(X(t_n), X(t_n-\tau))h + g(X(t_n), X(t_n-\tau))\triangle W_n, X(t_{n+1}-\tau))\\
&\quad -f(X(s), X(s-\tau))]ds\Big|^2 + 2EI_{1,n}^2
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
&E\Big|\int_{t_n}^{t_{n+1}} |f(X(t_n) + f(X(t_n), X(t_n-\tau))h + g(X(t_n), X(t_n-\tau))\triangle W_n, X(t_{n+1}-\tau))\\
&\quad -f(X(s), X(s-\tau))|ds\Big|^2\\
&\leq hE\int_{t_n}^{t_{n+1}} |f(X(t_n) + f(X(t_n), X(t_n-\tau))h + g(X(t_n), X(t_n-\tau))\triangle W_n, X(t_{n+1}-\tau))\\
&\quad -f(X(s), X(s-\tau))|^2ds\\
&\leq 2L^2hE\int_{t_n}^{t_{n+1}} [|X(t_n) + f(X(t_n), X(t_n-\tau))h + g(X(t_n), X(t_n-\tau))\triangle W_n - X(s)|^2\\
&\quad +|X(t_{n+1}-\tau) - X(s-\tau)|^2]ds\\
&\leq 2L^2h\int_{t_n}^{t_{n+1}} [2E|X(s) - X(t_n)|^2 + E|X(s-\tau) - X(t_{n+1}-\tau)|^2]ds\\
&\quad +4L^2h\int_{t_n}^{t_{n+1}} E|f(X(t_n), X(t_n-\tau))h + g(X(t_n), X(t_n-\tau))\triangle W_n|^2ds,\\
&\leq 4L^2h\int_{t_n}^{t_{n+1}} E|f(X(t_n), X(t_n-\tau))h + g(X(t_n), X(t_n-\tau))\triangle W_n|^2ds\\
&\leq +4C_3L^2h^3 + 2ML^2h^{2+2\hat{p}}
\end{aligned} \tag{3.16}$$

where the Hölder inequality, condition (2.2) and the similar arguments for Lemma 3.3 have been used. With the Hölder inequality, the Itô isometry and conditions (2.3) and (3.7), the final integral item in (3.16) can be bounded by

$$\begin{aligned}
&\int_{t_n}^{t_{n+1}} E|f(X(t_n),X(t_n-\tau))h+g(X(t_n),X(t_n-\tau))\triangle W_n|^2 ds\\
&= \int_{t_n}^{t_{n+1}} E\left|\int_{t_n}^{t_{n+1}}[f(X(t_n),X(t_n-\tau))d\mu+g(X(t_n),X(t_n-\tau))dW(\mu)]\right|^2 ds\\
&\le 2\int_{t_n}^{t_{n+1}} E\left|\int_{t_n}^{t_{n+1}} f(X(t_n),X(t_n-\tau))d\mu\right|^2 ds\\
&\qquad +2\int_{t_n}^{t_{n+1}} E\left|\int_{t_n}^{t_{n+1}} g(X(t_n),X(t_n-\tau))dW(\mu)\right|^2 ds\\
&\le 2hE\int_{t_n}^{t_{n+1}}\int_{t_n}^{t_{n+1}} |f(X(t_n),X(t_n-\tau))|^2 d\mu ds\\
&\qquad +2E\int_{t_n}^{t_{n+1}}\int_{t_n}^{t_{n+1}} |g(X(t_n),X(t_n-\tau))|^2 d\mu ds\\
&\le 2KhE\int_{t_n}^{t_{n+1}}\int_{t_n}^{t_{n+1}} |1+X(t_n)^2+X(t_{n+1})^2| d\mu ds\\
&\qquad +2KE\int_{t_n}^{t_{n+1}}\int_{t_n}^{t_{n+1}} |1+X(t_n)^2+X(t_{n+1})^2| d\mu ds\\
&\le 2K[1+2C_4\left(1+E\|\psi\|^2\right)]h^2(1+h). \qquad (3.17)
\end{aligned}$$

A combination of (3.12), (3.15), (3.16) and (3.17) yields

$$\begin{aligned}
&E|R(h,X(t_n),X(t_n-\tau),X(t_{n+1}-\tau),\triangle W_n)|^2\\
&\le 4L^2[3C_3+4K[1+2C_4\left(1+E\|\psi\|^2\right)](1+h)]h^3+8ML^2h^{2+2\hat{p}}, \qquad (3.18)
\end{aligned}$$

which shows that there is a constant $h_2>0$ such that

$$E|R(h,X(t_n),X(t_n-\tau),X(t_{n+1}-\tau),\triangle W_n)|^2 \le C_5h^3+\tilde{C}_5h^{2+2\hat{p}},\ 0<h\le h_2, \qquad (3.19)$$

where

$$C_5 = 4L^2\{3C_3+4K[1+2C_4\left(1+E\|\psi\|^2\right)](1+h_2)\},\ \tilde{C}_5 = 8ML^2.$$

Therefore, substituting (3.12), (3.14) and (3.19) into (3.10) concludes

$$\max_{1\le n\le N}(E|\delta_{n+1}|^2)^{1/2} = \mathcal{O}(h^{\min\{1/2+\hat{p},1\}})\ \ as\ h\to 0.$$

Next, we begin to prove that

$$\max_{1\le n\le N}(E|E(\delta_{n+1}|\mathscr{A}_{t_n})|^2)^{1/2} = \mathcal{O}(h^{\min\{\hat{p},\ 1/2\}+1})\quad as\quad h\to 0. \qquad (3.20)$$

It follows from

$$E(I_{2,n}|\mathscr{A}_{t_n}) = 0,$$

and (3.9) that

$$\begin{aligned}
&E(E(\delta_{n+1}|\mathscr{A}_{t_n}))\\
&= E(E(I_{1,n}|\mathscr{A}_{t_n})) - PE(E(R(h,X(t_n),X(t_n-\tau),X(t_{n+1}-\tau),\triangle W_n)|\mathscr{A}_{t_n})). \qquad (3.21)
\end{aligned}$$

Applying the inequality $(a+b)^2 \le 2(a^2+b^2)$ $(\forall a, b \in \mathbb{R})$ to (3.21) yields

$$\begin{aligned} &E|E(\delta_{n+1}|\mathscr{A}_{t_n})|^2 \\ &\le 2E|E(I_{1,n}|\mathscr{A}_{t_n})|^2 + 2p^2 E|E(R(h, X(t_n), X(t_n-\tau), X(t_{n+1}-\tau), \triangle W_n)|\mathscr{A}_{t_n})|^2. \quad (3.22) \end{aligned}$$

By the Jensen inequality, the Hölder inequality, the properties of conditional expectation and the condition (2.2), the following estimation for $E|E(I_{1,n}|\mathscr{A}_{t_n})|^2$ holds:

$$\begin{aligned} &E|E(I_{1,n}|\mathscr{A}_{t_n})|^2 \\ &\le E\left[E\bigg(\int_{t_n}^{t_{n+1}} |f(X(s), X(s-\tau)) - f(X(t_n), X(t_n-\tau))|ds)^2|\mathscr{A}_{t_n}\bigg)\right] \\ &\le hE\left[E\bigg(\int_{t_n}^{t_{n+1}} |f(X(s), X(s-\tau)) - f(X(t_n), X(t_n-\tau))|^2 ds|\mathscr{A}_{t_n}\bigg)\right] \\ &= hE\left[\int_{t_n}^{t_{n+1}} |f(X(s), X(s-\tau)) - f(X(t_n), X(t_n-\tau))|^2 ds\right] \\ &\le 2hL^2 E\left[\int_{t_n}^{t_{n+1}} [|X(s) - X(t_n)|^2 + |X(s-\tau) - X(t_n-\tau)|^2] ds\right]. \end{aligned} \tag{3.23}$$

With (3.8), we further have for $0 \le n \le N-1$:

$$E|E(I_{1,n}|\mathscr{A}_{t_n})|^2 \le 2C_3 L^2 h^3 + 2ML^2 h^{2+2\hat{p}}. \tag{3.24}$$

Moreover, it derives from the similar arguments for (3.23) that

$$\begin{aligned} &E|E(R(h, X(t_n), X(t_n-\tau), X(t_{n+1}-\tau), \triangle W_n)|\mathcal{A}_{t_n})|^2 \\ &\le 4hL^2 E\bigg(\int_{t_n}^{t_{n+1}} [|X(t_n) + f(X(t_n), X(t_n-\tau))h + g(X(t_n), X(t_n-\tau))\triangle W_n \\ &\qquad -X(s)|^2 + |X(t_{n+1}-\tau) - X(s-\tau)|^2] ds\bigg) \\ &\qquad +4hL^2 E\bigg(\int_{t_n}^{t_{n+1}} [|X(t_n) - X(s)|^2 + |X(t_n-\tau) - X(s-\tau)|^2] ds\bigg). \end{aligned} \tag{3.25}$$

Also, from (3.16) we can find the following inequality

$$\begin{aligned} &E\int_{t_n}^{t_{n+1}} [|X(t_n) + f(X(t_n), X(t_n-\tau))h + g(X(t_n), X(t_n-\tau))\triangle W_n - X(s)|^2 \\ &\qquad +|X(t_{n+1}-\tau) - X(s-\tau)|^2] ds \\ &\le 2\int_{t_n}^{t_{n+1}} E|f(X(t_n), X(t_n-\tau))h + g(X(t_n), X(t_n-\tau))\triangle W_n|^2 ds + 2C_3 h^2 \\ &\qquad +Mh^{1+2\hat{p}}. \end{aligned} \tag{3.26}$$

Inserting (3.17) into (3.26) yields

$$\begin{aligned} &E(\int_{t_n}^{t_{n+1}} [|X(t_n) + f(X(t_n), X(t_n-\tau))h + g(X(t_n), X(t_n-\tau))\triangle W_n - X(s)|^2 \\ &\qquad +|X(t_{n+1}-\tau) - X(s-\tau)|^2] ds) \\ &\le 2C_3 h^2 + Mh^{1+2\hat{p}} + 4K[1 + 2C_4(1 + E\|\psi\|^2)]h^2(1+h). \end{aligned} \tag{3.27}$$

This, as well as (3.8), leads to a further estimation of (3.25)

$$\begin{aligned}&E|E(R(h, X(t_n), X(t_n-\tau), X(t_{n+1}-\tau), \triangle W_n)|\mathscr{A}_{t_n})|^2\\ &\leq 4h^3L^2\{3C_3 + 2Mh^{2\hat{p}-1} + 4K[1+2C_4(1+E\|\psi\|^2)](1+h)\}.\end{aligned} \tag{3.28}$$

A combination of (3.22), (3.24) and (3.28) shows that there exist constants $h_3, C_6 > 0$ such that

$$E|E(\delta_{n+1}|\mathscr{A}_{t_n})|^2 \leq C_6 h^{\min\{2+2\hat{p},\ 3\}}$$

for $0 < h \leq h_3$ and $0 \leq n \leq N-1$, which implies (3.20). This completes the proof.

4. Mean-Square Convergence of the Method

In this section, we will deal with mean-square convergence of the method $\{(2.4), (2.5)\}$.

Definition 4.1. *Method* $\{(2.4), (2.5)\}$ *is called convergent of order* q *in the mean-square sense if its global error* $\varepsilon_n := X(t_n) - X_n$ *satisfies*

$$\max_{1\leq n\leq N}(E|\varepsilon_n|^2)^{1/2} = \mathcal{O}(h^q) \quad as \quad h \to 0. \tag{4.1}$$

Theorem 4.1. *Suppose that functions* f, g *satisfy the conditions* (2.2)-(2.3), *and the initial function* ψ *is Hölder-continuous with exponent* $\hat{p}$. *Then method* $\{(2.4), (2.5)\}$ *is convergent of order* $\min\{1/2, \hat{p}\}$ *in the mean-square sense.*

Proof. Let

$$u_n = \Phi(h, X(t_n), X(t_n-\tau), X(t_{n+1}-\tau), \triangle W_n) - \Phi(h, X_n, X_{n-m}, X_{n+1-m}, \triangle W_n). \tag{4.2}$$

Then, by the definitions of δ_n and ε_n,

$$\varepsilon_{n+1} = \delta_{n+1} + \varepsilon_n + u_n.$$

Hence, we have

$$\begin{aligned}&E(|\varepsilon_{n+1}|^2|\mathscr{A}_{t_0})\\ &\leq E(|\varepsilon_n|^2|\mathscr{A}_{t_0}) + E(|\delta_{n+1}|^2|\mathscr{A}_{t_0}) + E(|u_n|^2|\mathscr{A}_{t_0})||\\ &\quad +2|E(\varepsilon_n\cdot\delta_{n+1}|\mathscr{A}_{t_0}) + 2|E(\delta_{n+1}\cdot u_n|\mathscr{A}_{t_0}) + 2|E(\varepsilon_n\cdot u_n|\mathscr{A}_{t_0})|.\end{aligned} \tag{4.3}$$

Since by Lemma 3.4 method $\{(2.4), (2.5)\}$ is consistent of order $\min\{1/2, \hat{p}\}$ in the mean-square sense, there exist constants $\hat{h}_1, \hat{C}_1, \hat{C}_2 > 0$ such that

$$E(|\delta_{n+1}|^2|\mathscr{A}_{t_0}) \leq \hat{C}_1\ h^{\min\{2,1+2\hat{p}\}}, \quad 0 < h \leq \hat{h}_1 \tag{4.4}$$

and

$$\begin{aligned}&2|E(\varepsilon_n\cdot\delta_{n+1}|\mathscr{A}_{t_0})|\\ &\leq 2E(|\varepsilon_n|\cdot|E(\delta_{n+1}|\mathscr{A}_{t_n})||\mathscr{A}_{t_0})\\ &\leq 2[E(|\varepsilon_n|^2|\mathscr{A}_{t_0})]^{1/2}[E|E(\delta_{n+1}|\mathscr{A}_{t_n})|^2]^{1/2}\\ &\leq 2[E(|\varepsilon_n|^2|\mathscr{A}_{t_0})]^{1/2}\cdot[\hat{C}_2 h^{\min\{3,2+2\hat{p}\}}]^{1/2}\\ &\leq hE[|\varepsilon_n|^2|\mathscr{A}_{t_0}] + \hat{C}_2 h^{\min\{2,1+2\hat{p}\}}, \quad 0 < h \leq \hat{h}_1,\end{aligned} \tag{4.5}$$

where some common properties of conditional expectation have been used. Also, it follows from Lemma 3.1 that there exist constants $\hat{h}_2, \hat{C}_3, \hat{C}_4 > 0$ such that, for $0 < h \le \hat{h}_2$, the following inequalities hold:

$$E(|u_n|^2|\mathscr{A}_{t_0}) \le \hat{C}_3 h[E(|\varepsilon_n|^2|\mathscr{A}_{t_0}) + E(|\varepsilon_{n-m}|^2|\mathscr{A}_{t_0}) + E(|\varepsilon_{n-m+1}|^2|\mathscr{A}_{t_0})] \tag{4.6}$$

and

$$\begin{aligned} &2|E(\varepsilon_n \cdot u_n|\mathscr{A}_{t_0})| \\ \le\ & 2E(|\varepsilon_n| \cdot |E(u_n|\mathscr{A}_{t_n})||\mathscr{A}_{t_0}) \\ \le\ & 2\hat{C}_4 h[E(|\varepsilon_n|^2|\mathscr{A}_{t_0}) + E(|\varepsilon_n||\varepsilon_{n-m}||\mathscr{A}_{t_0}) + E(|\varepsilon_n||\varepsilon_{n-m+1}||\mathscr{A}_{t_0})] \\ \le\ & 2\hat{C}_4 h[E(|\varepsilon_n|^2|\mathscr{A}_{t_0}) + (E(|\varepsilon_n|^2|\mathscr{A}_{t_0}))^{1/2} \cdot (E(|\varepsilon_{n-m}|^2|\mathscr{A}_{t_0}))^{1/2} \\ & \quad +(E(|\varepsilon_n|^2|\mathscr{A}_{t_0}))^{1/2} \cdot (E(|\varepsilon_{n-m+1}|^2|\mathscr{A}_{t_0}))^{1/2}] \\ \le\ & \hat{C}_4 h[4E(|\varepsilon_n|^2|\mathscr{A}_{t_0}) + E(|\varepsilon_{n-m}|^2|\mathscr{A}_{t_0}) + E(|\varepsilon_{n-m+1}|^2|\mathscr{A}_{t_0})]. \end{aligned} \tag{4.7}$$

Let $h_0 = \min\{\hat{h}_1, \hat{h}_2\}$. Then, when $0 < h \le h_0$, (4.4) and (4.6) implies

$$\begin{aligned} &2|E(\delta_{n+1} \cdot u_n|\mathscr{A}_{t_0})| \\ \le\ & 2(E(|\delta_{n+1}|^2|\mathscr{A}_{t_0}))^{1/2} \cdot (E(|u_n|^2|\mathscr{A}_{t_0}))^{1/2} \\ \le\ & E(|\delta_{n+1}|^2|\mathscr{A}_{t_0}) + E(|u_n|^2|\mathscr{A}_{t_0}) \\ \le\ & \hat{C}_3 h[E(|\varepsilon_n|^2|\mathscr{A}_{t_0}) + E(|\varepsilon_{n-m}|^2|\mathscr{A}_{t_0}) + E(|\varepsilon_{n-m+1}|^2|\mathscr{A}_{t_0})] + \hat{C}_1 h^{\min\{2,1+2\hat{p}\}}. \end{aligned} \tag{4.8}$$

Substituting (4.4)-(4.8) into (4.3), yields

$$\begin{aligned} &E(|\varepsilon_{n+1}|^2|\mathscr{A}_{t_0}) \\ \le\ & E(|\varepsilon_n|^2\mathscr{A}_{t_0}|)+d_1hE(|\varepsilon_n|^2\mathscr{A}_{t_0}|)+d_2hE(|\varepsilon_{n-m}|^2|\mathscr{A}_{t_0})+d_2hE(|\varepsilon_{n-m+1}|^2|\mathscr{A}_{t_0}) \\ & \quad +d_3h^{\min\{2,1+2\hat{p}\}}, \quad 0 < h \le h_0, \end{aligned} \tag{4.9}$$

where $d_1 = 1 + 2\hat{C}_3 + 4\hat{C}_4$, $d_2 = 2\hat{C}_3 + \hat{C}_4$, $d_3 = 2\hat{C}_1 + \hat{C}_2$. An induction to the inequality (4.9) generates

$$\begin{aligned} &E(|\varepsilon_{n+1}|^2|\mathcal{A}_{t_0}) \\ \le\ & E(|\varepsilon_0|^2|\mathcal{A}_{t_0}) + d_1 h \sum_{i=0}^{n} E(|\varepsilon_i|^2|\mathcal{A}_{t_0}) + d_2 h \sum_{i=0}^{n} E(|\varepsilon_{i-m}|^2|\mathcal{A}_{t_0}) \\ & \quad +d_2 h \sum_{i=0}^{n} E(|\varepsilon_{i-m+1}|^2|\mathcal{A}_{t_0})+(n+1)d_3h^{\min\{2,1+2\hat{p}\}}, \ 0 < h \le h_0. \end{aligned} \tag{4.10}$$

The right-hand side of (4.10) can be further bounded by

$$\begin{aligned} &E(|\varepsilon_0|^2|\mathscr{A}_{t_0}) + 2d_2 h \sum_{i=-m}^{-1} E(|\varepsilon_i|^2|\mathscr{A}_{t_0}) + d_3 T h^{\min\{1,2\hat{p}\}} + (d_1 + 2d_2)h \sum_{i=0}^{n} E(|\varepsilon_i|^2|\mathscr{A}_{t_0}) \\ =\ & d_3 T h^{\min\{1,2\hat{p}\}} + (d_1 + 2d_2)h \sum_{i=0}^{n} E(|\varepsilon_i|^2|\mathscr{A}_{t_0}), \quad 0 < h \le h_0. \end{aligned}$$

Hence

$$E(|\varepsilon_{n+1}|^2|\mathscr{A}_{t_0}) \leq d_3 T h^{\min\{1,2\hat{p}\}} + (d_1 + 2d_2)h \sum_{i=0}^{n} E(|\varepsilon_i|^2|\mathscr{A}_{t_0}), \quad 0 < h \leq h_0. \tag{4.11}$$

Applying the discrete Bellman inequality (cf. [19]) to (4.11) yields

$$\begin{aligned} E(|\varepsilon_{n+1}|^2|\mathscr{A}_{t_0}) &\leq d_3 T h^{\min\{1,2\hat{p}\}} \exp[(d_1 + 2d_2)(n+1)h] \\ &\leq \{d_3 T \exp[(d_1 + 2d_2)T]\} h^{\min\{1,2\hat{p}\}}, \quad 0 < h \leq h_0, \end{aligned} \tag{4.12}$$

which implies

$$\max_{1 \leq n \leq N} (E|\varepsilon_n|^2)^{1/2} = \mathcal{O}(h^{\min\{1/2,\hat{p}\}}), \quad as \quad h \to 0.$$

Therefore, the theorem is proven.

5. Asymptotic MS-stability of the Method

The subsequent investigation will focus on the asymptotic MS-stability of the method $\{(2.4), (2.5)\}$. It is common to consider stability results for the analytical and numerical solutions of simple test equations in numerical analysis. Thus we choose as a test equation the following linear scalar SDDE

$$\begin{cases} dX(t) = [aX(t) + bX(t-\tau)]dt + [cX(t) + dX(t-\tau)]dW(t), t \in [0, +\infty), \\ X(t) = \psi(t), \quad t \in [-\tau, 0], \end{cases} \tag{5.1}$$

where $a, b, c, d \in \mathbb{R}$.

Definition 5.1. *A numerical method is called asymptotic MS-stable for* (5.1) *if there exists an* $h_0(a, b, c, d, p) > 0$ *such that the generated numerical solution* X_n *satisfies*

$$\lim_{n \to \infty} EX_n^2 = 0, \quad \forall h \in (0, h_0(a, b, c, d, p)). \tag{5.2}$$

Proposition 5.1. ([21]) *Suppose that the condition*

$$a < -|b| - \frac{1}{2}(|c| + |d|)^2, \tag{5.3}$$

holds. Then the solution of equation (5.1) *is mean square stable, i.e.*

$$\lim_{t \to \infty} E|X(t)|^2 = 0.$$

Theorem 5.1. *Assume that the condition* (5.3) *holds. Then the strong predictor-corrector method* $\{(2.4), (2.5)\}$ *is asymptotic MS-stable for* (5.1) *with stepsize* $h < h_0(a, b, c, d, p)$, *where* $h_0(a, b, c, d, p)$ *is a positive constant that can be computed through the proof of this theorem.*

Proof. Applying the strong predictor-corrector method $\{(2.4), (2.5)\}$ to (5.1) follows that

$$\begin{aligned} X_{n+1} = (1 + ah + c\triangle W_n + a^2ph^2 + acp\triangle W_n h)X_n + [bh + d\triangle W_n \\ +p\ (abh + ad\triangle W_n - b)h]X_{n-m} + pbhX_{n+1-m}. \end{aligned} \tag{5.4}$$

Using inequality $2xy \le x^2 + y^2$ ($\forall x, y \in \mathbb{R}$), the quantity X_{n+1}^2 can be bounded by

$$\begin{aligned}
&[(1 + ah + c\triangle W_n + a^2ph^2 + acp\triangle W_n h)^2 + (1 + ah + c\triangle W_n + a^2ph^2 + acp\triangle W_n h)\\
&\times(bh + d\triangle W_n + p(abh + ad\triangle W_n - b)h) + (1 + ah + c\triangle W_n + a^2ph^2 + acp\triangle W_n h)pbh]X_n^2\\
&+[(bh + d\triangle W_n + p(abh + ad\triangle W_n - b)h)^2 + (1 + ah + c\triangle W_n + a^2ph^2 + acp\triangle W_n h)\\
&\times(bh + d\triangle W_n + p(abh + ad\triangle W_n - b)h) + (bh + d\triangle W_n\\
&+p(abh + ad\triangle W_n - b)h)pbh]X_{n-m}^2 + [p^2b^2h^2 + (1 + ah + c\triangle W_n + a^2ph^2 + acp\triangle W_n h)pbh\\
&+(bh + d\triangle W_n + p(abh + ad\triangle W_n - b)h)pbh]X_{n+1-m}^2. \qquad (5.5)
\end{aligned}$$

Further, the following equalities hold for all $i \in \{n, n-m, n+1-m\}$:

$$\begin{aligned}
&E(\triangle W_n X_i^2) = E[X_i^2 E(\triangle W_n|\mathcal{A}_{t_n})] = 0,\\
&E[(\triangle W_n)^2 X_i^2] = E[X_i^2 E\big((\triangle W_n)^2|\mathcal{A}_{t_n}\big)] = hE(X_i^2),
\end{aligned}$$

since X_n, X_{n-m}, X_{n+1-m} are all $\mathcal{A}_{t_n}$-measurable and

$$E(\triangle W_n) = 0, \qquad E(\triangle W_n^2) = h.$$

Taking expectation on both sides of equality (5.5) and setting $Y_n = EX_n^2$, We obtain

$$Y_{n+1} \le P(a,b,c,d,p,h)Y_n + Q(a,b,c,d,p,h)Y_{n-m} + R(a,b,c,d,p,h)Y_{n+1-m},$$

with

$$\begin{aligned}
P(a,b,c,d,p,h) =& 1 + p^2a^3(a+b)h^4 + pa^2(pcd + pc^2 + 2a + 2b)h^3\\
&+a(2pcd + 2pc^2 + 2ap + a + b + pb)h^2 + (cd + c^2 + 2a + b)h,\\
Q(a,b,c,d,p,h) =& p^2a^2b(a+b)h^4 + a(p^2acd + p^2ad^2 + p^2b^2 + 2p(1-p)b^2 + 2pab - p^2ab)h^3\\
&+(2pacd + 2pad^2 - pb^2 + b^2 + ab)h^2 + (cd + d^2 + b - pb)h,\\
R(a,b,c,d,p,h) =& p^2ab(a+b)h^3 + pb(a+b)h^2 + pbh.
\end{aligned}$$

If

$$P(a,b,c,d,p,h) \ge 0,\ Q(a,b,c,d,p,h) \ge 0,\ R(a,b,c,d,p,h) \ge 0, \qquad (5.6)$$

then

$$Y_{n+1} \le S(h)\max\{Y_n, Y_{n-m}, Y_{n+1-m}\}, \quad n \ge 0, \qquad (5.7)$$

where

$$\begin{aligned}
S(h) &= P(a,b,c,d,p,h) + Q(a,b,c,d,p,h) + R(a,b,c,d,p,h)\\
&= p^2a^2(a+b)^2h^4 + pa(pa(c+d)^2 + 2(a+b)^2)h^3\\
&\quad +(2pa(c+d)^2 + (a+b)^2 + 2pa(a+b))h^2 + 2(a + b + \frac{1}{2}(c+d)^2)h + 1.
\end{aligned}$$

Let: $pa = \Theta_1,\ \ a + b = \Theta_2,\ \ c + d = \Theta_3$ and write:

$$\begin{aligned}
&C_1 := (\Theta_1\Theta_2)^2,\ \ C_2 := \Theta_1(\Theta_1\Theta_3^2 + 2\Theta_2^2),\\
&C_3 := 2\Theta_1\Theta_3^2 + \Theta_2^2 + 2\Theta_1\Theta_2,\ \ C_4 := 2(\Theta_2 + \frac{1}{2}\Theta_3^2).
\end{aligned}$$

Table 5.1: Stability bounds for system (5.1).

Stability bounds \ p	0	0.1	1/7	0.2	1/4	1/2	3/4	1
$h_0(-4,1/5,1/10,1,p)$	0.2632	0.2989	0.3226	0.3767	0.8806	0.4017	0.2468	0.1713
$h_0(-4,1,1/2,1,p)$	0.3333	0.3961	0.4480	1.0711	0.8278	0.3494	0.1958	0.1216
$h_0(-4,1/10,1/5,1,p)$	0.2564	0.2900	0.3120	0.3602	0.9065	0.4257	0.2692	0.1924
$h_0(-5,3/2,1/2,1,p)$	0.2857	0.3453	0.4000	0.8446	0.6495	0.2656	0.1423	0.0828
$h_0(-6,1/2,1/5,1/2,p)$	0.1818	0.2077	0.2253	0.2680	0.5393	0.2162	0.1113	0.0603
$h_0(-7,1/3,1/6,1,p)$	0.1500	0.1703	0.1838	0.2143	0.4943	0.2207	0.1324	0.0896
$h_0(-8,1,1/5,1/2,p)$	0.1429	0.1645	0.1798	0.2210	0.3940	0.1494	0.1667	0.1250
$h_0(-9,2,1,1/2,p)$	0.1429	0.1684	0.1886	0.5556	0.3493	0.1327	0.0621	0.0276
$h_0(-10,5,1/5,1,p)$	0.1053	0.1187	0.1275	0.1467	0.3079	0.1123	0.0486	0.0176

By recursive calculation we conclude that $\lim_{n\to\infty} Y_n = 0$ whenever

$$S(h) = C_1h^4 + C_2h^3 + C_3h^2 + C_4h + 1 < 1. \tag{5.8}$$

That is,

$$C_1h^3 + C_2h^2 + C_3h + C_4 < 0. \tag{5.9}$$

Condition (5.3) implies that

$$C_1h^3 + C_2h^2 + C_3h + C_4 = 0 \tag{5.10}$$

has at least one positive real root. Furthermore, note that $P(a, b, c, d, p, h)$, $Q(a, b, c, d, p, h)$ and $R(a, b, c, d, p, h)$ are required to be nonnegative, we have $b \geq 0$, $cd + d^2 + b - pb \geq 0$ and $h \leq h_4$, where $h_4 := \min\{h_P, h_Q, h_R\}$, and

$$h_P := \begin{cases} \min\{h > 0 : P(a,b,c,d,p,h) = 0\}, & \text{when } P(a,b,c,d,p,h) \text{ has positive roots;} \\ \infty, & \text{when } P(a,b,c,d,p,h) \text{ has no positive roots.} \end{cases}$$

Similarly, we can define h_Q and h_R. Moreover, we write

$$h_5 := \text{ The minimum positive real root of } (5.10).$$

Then, $\min\{h_4, h_5\}$ is the stability bound $h_0(a, b, c, d, p)$ of the underlying method. Thus, we obtain easily from condition (5.3) that, there exists an $h_0(a, b, c, d, p) > 0$ such that (5.8) holds for $h \in (0, h_0(a, b, c, d, p))$.

Remark 5.1. Although we have an approach for calculating $h_0(a, b, c, d, p)$ as a function of the parameters, there is no explicit representation and so we attempt to maximize the stability region by computing $h_0(a, b, c, d, p)$ for a number of parameter values.

From the theorem one can see that the asymptotic MS-stability condition of the numerical method $\{(2.4), (2.5)\}$ and that of equation (5.1) keep uniform. With stability condition (5.9), we can present the stability bound for a given concrete method, which is illustrated in Table 5.1. It is shown that the derived method can have better stability property because of the flexible parameter p. When $p \in [1/7, 1/4]$, the stability bounds are about 3 times larger than that of Euler-Maruyama method for a wide range of parameter values for problem 5.1.

6. Numerical illustration

6.1. Numerical examples

In order to illustrate the convergence result obtained in section 4, we consider the following stochastic system with delay $\tau = 1$:

$$\begin{cases} dX(t) = [-4X(t) + \frac{1}{5}X(t-1)]dt + [\frac{1}{10}X(t) + X(t-1)]dW(t), t \in [0, 10] \\ \psi(t) = t+1, \quad t \in [-1, 0). \end{cases} \tag{6.1}$$

Obviously, $\psi(t) = t+1$ in (6.1) is Hölder-continuous with exponent $\hat{p} = 1$. Moreover, it is easy to check that this system satisfies the Lipschitz condition (2.2) and the linear growth condition (2.3), which shows that a unique solution exists. By Proposition (5.1), we know that the system (6.1) is mean-square stable. In fact, the explicit solution of the above system on $[0, 1]$ can be given by

$$\begin{aligned} X(t) = &\left[1 + \frac{1}{10}\int_0^t s\exp\left(\frac{801}{200}s - \frac{1}{10}\int_0^s dW(r)\right)ds + \int_0^t s\exp\left(\frac{801}{200}s - \frac{1}{10}\int_0^s dW(r)\right)dW(s)\right] \\ &\times \exp\left[-\frac{801}{200}t + \frac{1}{10}\int_0^t dW(s)\right], \end{aligned} \tag{6.2}$$

and the explicit solutions on the subsequent intervals can be obtained with the so-called *step method*.

For a given stepsize h, when applying a strong predictor-corrector method with parameters $p = 0$, $p = 1/4$, $p = 1/2$, $p = 3/4$, $p = 1$ and $p = 1/7$ to the system (6.1) we obtain the corresponding numerical solutions. Obviously, when $p = 0$, method $\{(2.4), (2.5)\}$ is the Euler-Maruyama method. It follows from Theorem 4.1 that the mean-square convergent order of these six methods equals 1/2.

Table 6.1: Mean-square errors for solving system (6.1).

$p \backslash h$	1	$1/2$	$1/2^2$	$1/2^3$	$1/2^4$	$1/2^5$	$1/2^6$	$1/2^7$
0	$1.76e+002$	$8.28e-003$	$7.60e-003$	$1.86e-003$	$1.40e-003$	$1.00e-003$	$4.80e-004$	$2.55e-004$
1/7	$5.94e-003$	$2.95e-003$	$5.65e-003$	$1.36e-003$	$1.48e-003$	$9.23e-004$	$4.42e-004$	$2.40e-004$
1/4	$1.02e-002$	$3.34e-003$	$4.44e-003$	$1.50e-003$	$1.54e-003$	$9.20e-004$	$4.13e-004$	$2.38e-004$
1/2	$2.18e+004$	$1.15e-002$	$4.28e-003$	$2.00e-003$	$1.67e-003$	$9.56e-004$	$3.78e-004$	$2.33e-004$
3/4	$4.25e+006$	$5.57e+003$	$7.67e-003$	$2.73e-003$	$1.81e-003$	$9.91e-004$	$3.89e-004$	$2.29e-004$
1	$1.16e+008$	$1.16e+007$	$1.38e-002$	$3.92e-003$	$1.95e-003$	$1.03e-003$	$4.01e-004$	$2.24e-004$

For numerically verifying the convergence, we use the approximation formula

$$err \approx \max_{1 \le n \le N}\left(\frac{1}{10000}\sum_{j=1}^{10000}|X(t_n, \omega_j) - X_n(\omega_j)|^2\right)^{1/2}$$

to characterize the mean-square errors

$$err := \max_{1 \le n \le N}(E|\varepsilon_n|^2)^{1/2}$$

of the methods on $[0, 10]$. When taking stepsizes $h = 1, 1/2, 1/2^2, 1/2^3, 1/2^4, 1/2^5, 1/2^6, 1/2^7$, respectively, the mean-square errors of the six methods for (6.1) are displayed in Table 6.1,

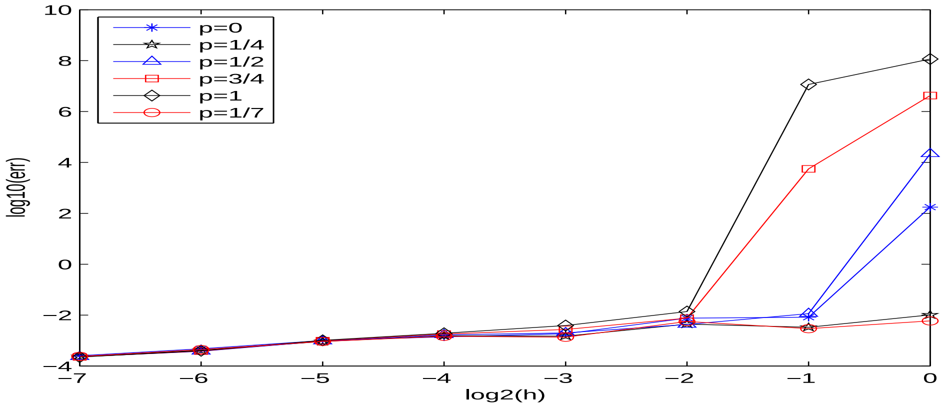

Fig. 6.1. Mean-square errors for solving test equation (6.1) . (h=$1/2^7, 1/2^6, ...1/2, 1$)

Table 6.2: Mean-square errors for solving system (6.3).

p \ h	$1/2$	$1/2^2$	$1/2^3$	$1/2^4$	$1/2^5$	$1/2^6$	$1/2^7$
0	$1.62e+009$	$1.35e+005$	$1.57e-002$	$2.73e-003$	$1.73e-003$	$1.60e-003$	$7.30e-004$
$1/4$	$3.97e+004$	$4.33e-003$	$4.07e-003$	$2.38e-003$	$1.58e-003$	$1.41e-003$	$6.84e-004$
$1/2$	$3.21e+015$	$1.73e+006$	$3.83e-003$	$3.00e-003$	$1.75e-003$	$1.24e-003$	$6.39e-004$
$3/4$	$1.10e+020$	$5.87e+017$	$1.03e-002$	$3.68e-003$	$1.93e-003$	$1.07e-003$	$5.94e-004$
1	$9.04e+022$	$3.28e+024$	$2.33e+007$	$4.45e-003$	$2.11e-003$	$9.77e-004$	$6.45e-004$
$1/7$	$3.57e-003$	$1.16e-002$	$5.91e-003$	$2.51e-003$	$1.54e-003$	$1.49e-003$	$7.04e-004$

which indicate that the strong predictor-corrector method $\{(2.4),(2.5)\}$ is effective. We can see the accuracy of method with $p \in [1/7, 1/4]$ is much better than that of the Euler-Maruyama method when using larger stepsizes.

In figure 6.1, mean-square errors are plotted against stepsizes on a log-log scale. It illustrates that the method is convergent of order $1/2$ in the mean-square sense when stepsizes are small enough. This further confirms Theorem 4.1.

Next, we show what happens when applying the underlying method to nonlinear systems. Consider the following nonlinear SDDE:

$$\begin{cases} dX(t) = [-10X(t) + X(t-1)\cos(X(t-1))]dt \\ \qquad\qquad +[\sin(X(t)) + 2\sin(X(t-1))]dW(t), t \in [0, 10] \\ \psi(t) = t + 1, \qquad t \in [-1, 0). \end{cases} \tag{6.3}$$

The nonlinear test equation is solved by the above 6 methods with stepsizes h=$1/2^7, 1/2^6, \ldots$ $1/2$, respectively. We can see that the strong predictor-corrector method is also effective for the nonlinear case. Table 6.2 and Figure 6.2 show that the stability property of the strong predictor-corrector method is much better than that of the Euler-Maruyama method for solving (6.3). All of the data used here are based on 10000 simulated trajectories just as the linear test equation.

6.2. Implementation via Vectorisation

In subsection 6.1, numerical results were obtained step by step, for each trajectory. The procedure can be terrifically time consuming and this has provided the motivation to develop a

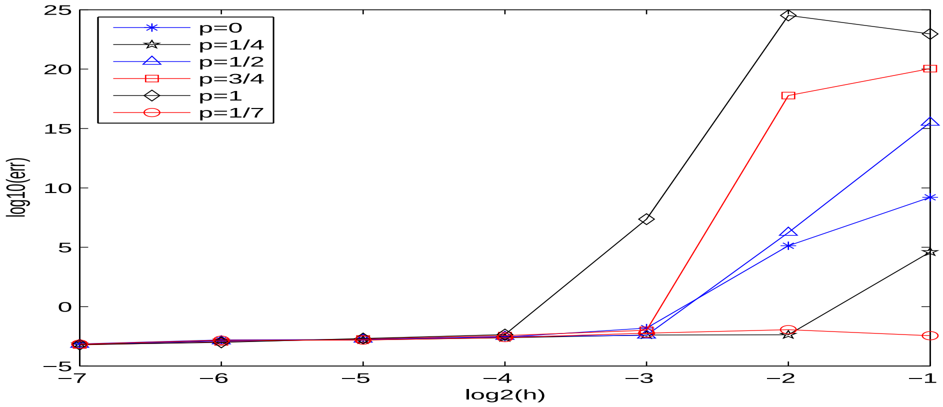

Fig. 6.2. Mean-square errors for solving test equation (6.3) .(h=$1/2^7, 1/2^6, ...1/2$)

vectorised implementation for stochastic numerical methods. Just like the sequential approach, we still implement the method step by step but we do this for all trajectories simultaneously in the vectorised approach. This is a vectorisation across the simulations and works effectively for explicit as well as implicit methods. Similar technique arose in [9]. Applying both the vectorised implementation and sequential implementation to system (6.1) and (6.3) in the interval of $[0, 1]$, we see in Table 6.3 and Table 6.4 speed-ups are substantial. Actual solutions of system (6.1) and (6.3) are unknown, they are approximated by using the Euler-Maruyama method with stepsize $h = 2^{-8}$. Stepsizes used in this section were $h = 2^{-6}, 2^{-5}, 2^{-4}$ and 2^{-3} with 200000 simulations each.

Table 6.3: Times with Euler-Maruyama method ($p = 0$).

	vectorised	sequential	speed-up factor
System (6.1)	7.4117	314.2920	42.4048
System (6.3)	15.3660	327.5115	21.3140

Table 6.4: Times with Predictor-Corrector method($p = \frac{1}{7}$).

	vectorised	sequential	speed-up factor
System (6.1)	7.0318	290.7650	41.3500
System (6.3)	16.3485	338.9233	20.7312

7. Conclusions

In this paper, we have constructed a strong predictor-corrector method with a parameter p that can be varied as appropriate. It is shown that for all values of $p \in [0, 1]$, this method has strong order 1/2 for SDDEs. Furthermore, we have explored the asymptotic MS-stability of this method and shown that a value of $p = 1/7$ gives much superior stability properties than the Euler-Maruyama method ($p = 0$). This theoretical analysis and two numerical simulations (a linear and a nonlinear scalar problem) demonstrate that the predictor-corrector method is very promising for problems that are moderately stiff, thus avoiding the use of implicit methods.

Finally, we discussed a way to vectorise the calculations across the simulations and hence produced an efficient implementation that is a useful tool for generating many simulations of the numerical solution of any SDDE efficiently.

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