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A POSTERIORI ERROR ANALYSIS OF A FULLY-MIXED FINITE ELEMENT METHOD FOR A TWO-DIMENSIONAL FLUID-SOLID INTERACTION PROBLEM *

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Abstract

In this paper we develop an a posteriori error analysis of a fully-mixed finite element method for a fluid-solid interaction problem in 2D. The media are governed by the elastodynamic and acoustic equations in time-harmonic regime, respectively, the transmission conditions are given by the equilibrium of forces and the equality of the corresponding normal displacements, and the fluid is supposed to occupy an annular region surrounding the solid, so that a Robin boundary condition imitating the behavior of the Sommerfeld condition is imposed on its exterior boundary. Dual-mixed approaches are applied in both domains, and the governing equations are employed to eliminate the displacement $\mathbf{u}$ of the solid and the pressure p of the fluid. In addition, since both transmission conditions become essential, they are enforced weakly by means of two suitable Lagrange multipliers. The unknowns of the solid and the fluid are then approximated by a conforming Galerkin scheme defined in terms of PEERS elements in the solid, Raviart-Thomas of lowest order in the fluid, and continuous piecewise linear functions on the boundary. As the main contribution of this work, we derive a reliable and efficient residual-based a posteriori error estimator for the aforedescribed coupled problem. Some numerical results confirming the properties of the estimator are also reported.

Mathematics subject classification: 65N30, 65N15, 74F10, 74B05, 35J05.
Key words: Mixed finite elements, Helmholtz equation, Elastodynamic equation, A posteriori error analysis.

1. Introduction

In the recent paper [14] we introduced and analyzed a fully-mixed finite element method for the two-dimensional fluid-solid interaction problem studied originally in [17] (see also [18]). The respective model consists of an elastic body which is subject to a given incident wave that travels in the fluid surrounding it. Actually, the fluid is supposed to occupy an annular region,

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and hence a Robin boundary condition imitating the behavior of the scattered field at infinity is imposed on its exterior boundary, which is located far from the obstacle. The media are governed by the elastodynamic and acoustic equations in time-harmonic regime, respectively, and the transmission conditions are given by the equilibrium of forces and the equality of the corresponding normal displacements. Differently from the analysis in [17] where dual and primal methods are utilized in the solid and fluid, respectively, dual-mixed approaches are applied in both domains in [14], and the governing equations are employed to eliminate the displacement $\mathbf{u}$ of the solid and the pressure p of the fluid. In addition, since both transmission conditions become essential, they are enforced weakly by means of two suitable Lagrange multipliers. In this way, the Cauchy stress tensor and the rotation of the solid, together with the gradient of p and the traces of $\mathbf{u}$ and p on the boundary of the fluid, constitute the unknowns of the coupled problem. The solvability of the resulting continuous formulation is analyzed in [14] by incorporating first suitable decompositions of the spaces to which the stress and the gradient of p belong, and then by applying the Babuška-Brezzi theory and the Fredholm alternative. The unknowns of the solid and the fluid are approximated by a conforming Galerkin scheme defined in terms of PEERS elements in the solid, Raviart-Thomas of lowest order in the fluid, and continuous piecewise linear functions on the boundary. The analysis of the discrete method relies on a stable decomposition of the corresponding finite element spaces and also on the classical result on projection methods for Fredholm operators of index zero.

On the other hand, it is well known that in order to guarantee a good convergence behaviour of the finite element solutions, specially under the presence of complex geometries leading eventually to singularities, one needs to apply an adaptive strategy based on a posteriori error estimates. These are usually represented by global quantities $\boldsymbol{\theta}$ that are expressed in terms of local estimators θ_T defined on each element T of a given triangulation of the domain. The estimator $\boldsymbol{\theta}$ is said to be reliable (resp. efficient) if there exists $C_{\mathtt{rel}} > 0$ (resp. $C_{\mathtt{eff}} > 0$), independent of the meshsizes, such that

$$C_{\mathtt{eff}}\,\boldsymbol{\theta} + \text{h.o.t.} \;\leq\; \|error\| \;\leq\; C_{\mathtt{rel}}\,\boldsymbol{\theta} + \text{h.o.t.}\,,$$

where h.o.t. is a generic expression denoting one or several terms of higher order. Concerning the Helmholtz and elasticity equations, several approaches have already been developed independently in the literature. In particular, a posteriori error analyses for interior Helmholtz problems, which are based on local computations or explicit residuals, can be found in [7] and [24], respectively. In addition, a reliable residual-based a posteriori error estimator, which follows the nowadays standard approach from [29], is proposed in [25]. In turn, a posteriori error estimators for the mixed finite element formulation of the linear elasticity problem, which are based on residuals and on the solution of local problems, are provided in [2]. The main novelty of the approach there has to do with the utilization of a Helmholtz decomposition of the stress-type unknown to derive the corresponding reliability and efficiency estimates. For related approaches employing the Helmholtz decomposition technique as well we refer to [11] and [26].

Furtermore, to the best of our knowledge, [16] is the only work available in the literature dealing with the a posteriori error analysis of fluid-solid interaction problems involving the acoustic and elastodynamic equations in time-harmonic regime. In fact, a reliable and efficient residual-based a posteriori error estimator for the dual-mixed/primal formulation of the model problem analyzed in [17] was derived in [16]. More precisely, suitable auxiliary problems, the continuous inf-sup conditions satisfied by the bilinear forms involved, a discrete Helmholtz decomposition, and the local approximation properties of the Clément interpolant and Raviart-Thomas operator are the main tools for proving the reliability of the estimator in [16]. Then, Helmholtz decomposition, inverse inequalities, and the localization technique based on triangle-bubble and edge-bubble functions are employed to show the efficiency. According to the preceding remarks, and in order to additionaly contribute in this direction, the main purpose of the present paper is to derive a reliable and efficient residual-based a posteriori error estimator for the fully-mixed formulation introduced and analyzed in [14]. The rest of this work is organized as follows. In Section 2 we recall from [14] the fluid-solid interaction problem and its continuous and discrete fully-mixed variational formulations. The kernel of the present work is given by Section 3, where we develop the a posteriori error analysis. Our tools for showing reliability and efficiency are basically the same ones utilized in [16]. More precisely, in Section 3.2 we employ the global inf-sup condition for the continuous variational formulation, discrete Helmholtz decompositions in both domains, and the above mentioned properties of the Clément interpolant and Raviart-Thomas operator, to derive a reliable residual-based a posteriori error estimator. Even, at some point of this analysis we are able to identify independent terms related to the fluid and solid, respectively, which allows us to apply, separately, some of the arguments employed for the a posteriori error analyses of each equation. Next, in Section 3.3 we apply discrete trace and inverse inequalities, and the localization technique based on triangle-bubble and edge-bubble functions to show the efficiency of the estimator. In this part we take advantage of the fact that either the efficiency estimates for some terms or the way to derive them, are already available in the literature (see, e.g. [11], [16], and [29]). However, and for sake of completeness, we sketch at least most of the corresponding proofs. For the remaining terms defining the a posteriori error estimator we certainly provide full proofs. Finally, some numerical examples confirming the reliability and efficiency of the a posteriori error estimator, and showing the good performance of the associated adaptive algorithm are provided in Section 4.

We end this section with further notations to be used below. Since in the sequel we deal with complex valued functions, we let $\mathbb{C}$ be the set of complex numbers, use the symbol $\imath$ for $\sqrt{-1}$, denote by $\overline{z}$ and $|z|$ the conjugate and modulus, respectively, of each $z \in \mathbb{C}$, and let $\mathbf{I}$ be the identity matrix of $\mathbb{C}^{2\times 2}$. On the other hand, in what follows tr denotes the matrix trace and $^{\mathtt{t}}$ stands for the transpose of a matrix. Also, given $\boldsymbol{\tau}_s := (\tau_{ij})$, $\boldsymbol{\zeta}_s := (\zeta_{ij}) \in \mathbb{C}^{2\times 2}$, we define the deviator tensor $\boldsymbol{\tau}_s^{\mathtt{d}} := \boldsymbol{\tau}_s - \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\tau}_s)\,\mathbf{I}$, the tensor product $\boldsymbol{\tau}_s : \boldsymbol{\zeta}_s := \sum_{i,j=1}^{2} \tau_{ij}\,\zeta_{ij}$, and the conjugate tensor $\overline{\boldsymbol{\tau}_s} := (\overline{\tau}_{ij})$. In turn, in what follows we utilize standard simplified terminology for Sobolev spaces and norms. In particular, if $\mathcal{O}$ is a domain, $\mathcal{S}$ is a closed Lipschitz curve, and $r \in \mathbb{R}$, we define

$$\mathbf{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^2, \quad \mathbb{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^{2\times 2}, \quad \text{and} \quad \mathbf{H}^r(\mathcal{S}) := [H^r(\mathcal{S})]^2 .$$

However, when $r = 0$ we usually write $\mathbf{L}^2(\mathcal{O})$, $\mathbb{L}^2(\mathcal{O})$, and $\mathbf{L}^2(\mathcal{S})$ instead of $\mathbf{H}^0(\mathcal{O})$, $\mathbb{H}^0(\mathcal{O})$, and $\mathbf{H}^0(\mathcal{S})$, respectively. The corresponding norms are denoted by $\|\cdot\|_{r,\mathcal{O}}$ (for $H^r(\mathcal{O})$, $\mathbf{H}^r(\mathcal{O})$, and $\mathbb{H}^r(\mathcal{O})$) and $\|\cdot\|_{r,\mathcal{S}}$ (for $H^r(\mathcal{S})$ and $\mathbf{H}^r(\mathcal{S})$). In general, given any Hilbert space H, we use $\mathbf{H}$ and $\mathbb{H}$ to denote H^2 and $H^{2\times 2}$, respectively. In addition, we use $\langle\cdot,\cdot\rangle_{\mathcal{S}}$ to denote the usual duality pairings between $H^{-1/2}(\mathcal{S})$ and $H^{1/2}(\mathcal{S})$, and between $\mathbf{H}^{-1/2}(\mathcal{S})$ and $\mathbf{H}^{1/2}(\mathcal{S})$. Furthermore, the Hilbert space

$$\mathbf{H}(\mathrm{div};\mathcal{O}) := \left\{ \mathbf{w} \in \mathbf{L}^2(\mathcal{O}) : \quad \mathrm{div}\,\mathbf{w} \in L^2(\mathcal{O}) \right\}, \tag{1.1}$$

is standard in the realm of mixed problems (see [8], [23]). The space of matrix valued functions whose rows belong to $\mathbf{H}(\mathrm{div};\mathcal{O})$ will be denoted $\mathbb{H}(\mathbf{div};\mathcal{O})$. Note that if $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\mathcal{O})$, then

$\mathbf{div}\,\boldsymbol{\tau} \in \mathbf{L}^2(\mathcal{O})$, where $\mathbf{div}$ stands for the usual divergence operator div acting on each row of the tensor, The Hilbert norms of $\mathbf{H}(\mathrm{div};\mathcal{O})$ and $\mathbb{H}(\mathbf{div};\mathcal{O})$ are denoted by $\|\cdot\|_{\mathrm{div};\mathcal{O}}$ and $\|\cdot\|_{\mathbf{div};\mathcal{O}}$, respectively. Finally, we employ $\mathbf{0}$ to denote a generic null vector (including the null functional and operator), and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2. The Fluid-solid Interaction Problem

2.1. The model problem

We consider the two-dimensional fluid-solid interaction problem whose a priori error analysis was provided recently in [14] (see also [17] for a previous analysis of this problem). In other words, given an incident acoustic wave upon a bounded elastic body (obstacle) fully surrounded by a fluid, we are interested in determining both the response of the body and the scattered wave. The obstacle is supposed to be a long cylinder parallel to the x_3-axis whose cross-section is Ω_s. The boundary of Ω_s is denoted by Σ. The incident wave and the volume force acting on the body are assumed to exhibit a time-harmonic behaviour with $e^{-\imath\,\omega\,t}$ ansatz and phasors p_i and $\mathbf{f}$, respectively, so that p_i satisfies the Helmholtz equation in $\mathbb{R}^2\backslash\Omega_s$. Hence, since the phenomenon is supposed to be invariant under a translation in the x_3-direction, we may consider a bidimensional interaction problem posed in the frequency domain. In this way, and since we employ mixed formulations in both domains (solid and fluid), the main unknowns of our interaction problem are given by $\boldsymbol{\sigma}_s : \Omega_s \to \mathbb{C}^{2\times2}$, $\mathbf{u} : \Omega_s \to \mathbb{C}^2$, $p : \mathbb{R}^2\backslash\Omega_s \to \mathbb{C}$, and $\boldsymbol{\sigma}_f : \mathbb{R}^2\backslash\Omega_s \to \mathbb{C}^2$, corresponding to the amplitudes of the Cauchy stress tensor, the displacement field, the total (incident + scattered) pressure, and the gradient of p, respectively.

The fluid is assumed to be perfect, compressible, and homogeneous, with density ρ_f and wave number $\kappa_f := \dfrac{\omega}{v_0}$, where v_0 is the speed of sound in the linearized fluid, whereas the solid is supposed to be isotropic and linearly elastic with density ρ_s and Lamé constants μ and λ. The latter means, in particular, that the corresponding constitutive equation is given by Hooke's law, that is

$$\boldsymbol{\sigma}_s = \mathcal{C}\,\boldsymbol{\varepsilon}(\mathbf{u})\,, \qquad \text{where} \qquad \mathcal{C}\,\boldsymbol{\tau} := \lambda\,\mathrm{tr}\,\boldsymbol{\tau}\,\mathbf{I} + 2\,\mu\,\boldsymbol{\tau} \qquad \forall\,\boldsymbol{\tau}\in\mathbb{L}^2(\Omega)\,, \tag{2.1}$$

$\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}\,(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathtt{t}})$ is the strain tensor of small deformations, and ∇ is the gradient tensor. Consequently, under the hypotheses of small oscillations, both in the solid and the fluid, the unknowns $\boldsymbol{\sigma}_s$, $\mathbf{u}$, $\boldsymbol{\sigma}_f$, and p satisfy the elastodynamic and acoustic equations in time-harmonic regime, that is:

$$\begin{aligned} \mathbf{div}\,\boldsymbol{\sigma}_s + \kappa_s^2\,\mathbf{u} &= -\,\mathbf{f} && \text{in} \quad \Omega_s\,, \\ \mathrm{div}\,\boldsymbol{\sigma}_f + \kappa_f^2\,p &= 0 && \text{in} \quad \mathbb{R}^2\backslash\Omega_s\,, \end{aligned}$$

where κ_s is defined by $\sqrt{\rho_s}\,\omega$, together with the transmission conditions:

$$\begin{aligned} \boldsymbol{\sigma}_s\,\boldsymbol{\nu} &= -\,p\,\boldsymbol{\nu} && \text{on} \quad \Sigma\,, \\ \boldsymbol{\sigma}_f\cdot\boldsymbol{\nu} &= \rho_f\,\omega^2\,\mathbf{u}\cdot\boldsymbol{\nu} && \text{on} \quad \Sigma\,. \end{aligned} \tag{2.2}$$

and the Sommerfeld radiation condition

$$\frac{\partial(p-p_i)}{\partial\mathtt{r}} - \imath\,\kappa_f\,(p-p_i) = o(\mathtt{r}^{-1})\,, \tag{2.3}$$

as $\mathtt{r} := \|\mathbf{x}\| \to +\infty$, uniformly for all directions $\dfrac{\mathbf{x}}{\|\mathbf{x}\|}$. Hereafter, $\|\mathbf{x}\|$ is the euclidean norm of a vector $\mathbf{x} := (x_1, x_2)^{\mathtt{t}} \in \mathbb{R}^2$, and $\boldsymbol{\nu}$ denotes the unit outward normal on Σ, that is pointing toward $\mathbb{R}^2 \backslash \Omega_s$.

Next, according to the condition at infinity given by (2.3), which basically says that the outgoing waves are absorbed by the far field, and in order to obtain a convenient simplification of our model, we now proceed as in [14] and [17] and introduce a sufficiently large polyhedral surface Γ approximating a sphere centered at the origin, whose interior contains Ω_s. Then, we define Ω_f as the annular region bounded by Σ and Γ, and consider the Robin boundary condition:

$$\boldsymbol{\sigma}_f \cdot \boldsymbol{\nu} - \imath \kappa_f \, p = g := \nabla p_i \cdot \boldsymbol{\nu} - \imath \kappa_f \, p_i \quad \text{on} \quad \Gamma , \tag{2.4}$$

where $\boldsymbol{\nu}$ denotes the unit outward normal on Γ as well. Therefore, given $\mathbf{f} \in \mathbf{L}^2(\Omega_s)$ and $g \in H^{-1/2}(\Gamma)$, we are now interested in the following fluid-solid interaction problem: Find $\boldsymbol{\sigma}_s \in \mathbb{H}(\mathbf{div}; \Omega_s)$, $\mathbf{u} \in \mathbf{H}^1(\Omega_s)$, $\boldsymbol{\sigma}_f \in \mathbf{H}(\mathrm{div}; \Omega_f)$, and $p \in H^1(\Omega_f)$, such that there hold in the distributional sense:

$$\begin{aligned}
\boldsymbol{\sigma}_s &= \mathcal{C}\, \boldsymbol{\varepsilon}(\mathbf{u}) && \text{in} \quad \Omega_s \,,\\
\mathbf{div}\, \boldsymbol{\sigma}_s + \kappa_s^2\, \mathbf{u} &= -\mathbf{f} && \text{in} \quad \Omega_s \,,\\
\boldsymbol{\sigma}_f &= \nabla p && \text{in} \quad \Omega_f \,,\\
\mathrm{div}\, \boldsymbol{\sigma}_f + \kappa_f^2\, p &= 0 && \text{in} \quad \Omega_f \,,\\
\boldsymbol{\sigma}_s \boldsymbol{\nu} &= -p\, \boldsymbol{\nu} && \text{on} \quad \Sigma \,,\\
\boldsymbol{\sigma}_f \cdot \boldsymbol{\nu} &= \rho_f\, \omega^2\, \mathbf{u} \cdot \boldsymbol{\nu} && \text{on} \quad \Sigma \,,\\
\boldsymbol{\sigma}_f \cdot \boldsymbol{\nu} - \imath \kappa_f \, p &= g && \text{on} \quad \Gamma \,.
\end{aligned} \tag{2.5}$$

2.2. The fully-mixed variational formulation

In order to recall from [14] the fully-mixed variational formulation of (2.5), we need to introduce the auxiliary unknowns given by the trace of the displacement

$$\boldsymbol{\varphi}_s := \mathbf{u}|_{\Sigma} \in \mathbf{H}^{1/2}(\Sigma) \,,$$

the traces of the pressure

$$\boldsymbol{\varphi}_f = (\varphi_\Sigma, \varphi_\Gamma) := (p|_\Sigma, p|_\Gamma) \in H^{1/2}(\Sigma) \times H^{1/2}(\Gamma) \,,$$

and the rotation

$$\boldsymbol{\gamma} := \frac{1}{2}(\nabla \mathbf{u} - (\nabla \mathbf{u})^{\mathtt{t}}) \in \mathbb{L}^2_{\mathtt{skew}}(\Omega_s),$$

where $\mathbb{L}^2_{\mathtt{skew}}(\Omega_s)$ denotes the space of skew-symmetric tensors with entries in $L^2(\Omega_s)$. In addition, we let

$$\mathbf{H} := \mathbb{H}(\mathbf{div}; \Omega_s) \times \mathbf{H}(\mathrm{div}; \Omega_f) \quad \text{and} \quad \mathbf{Q} := \mathbb{L}^2_{\mathtt{skew}}(\Omega_s) \times \mathbf{H}^{1/2}(\Sigma) \times H^{1/2}(\partial\Omega_f)$$

endowed with the usual product norms. Hereafter, given $t \in \mathbb{R}$, we make the identification

$$H^t(\partial\Omega_f) \equiv H^t(\Sigma) \times H^t(\Gamma)$$

with the norm

$$\|\boldsymbol{\psi}_f\|_{t,\partial\Omega_f} := \|\psi_\Sigma\|_{t,\Sigma} + \|\psi_\Gamma\|_{t,\Gamma}$$

for each $\boldsymbol{\psi}_f := (\psi_\Sigma, \psi_\Gamma) \in H^t(\partial\Omega_f)$.

Next, as explained in [14], we employ a dual-mixed approach in the solid Ω_s as well as in the fluid Ω_f, and observe that both transmission conditions (cf. (2.2)) and the Robin boundary condition (2.4) become now essential. In addition, we use the elastodynamic and Helmholtz equations (cf. second and fourth equation of (2.5)), respectively, to eliminate $\mathbf{u}$ and p according to the formulae

$$\mathbf{u} = -\frac{1}{\kappa_s^2}(\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_s) \quad \text{in} \quad \Omega_s \qquad \text{and} \qquad p = -\frac{1}{\kappa_f^2}\operatorname{div}\boldsymbol{\sigma}_f \quad \text{in} \quad \Omega_f\,. \tag{2.6}$$

In this way, we arrive at the following fully-mixed variational formulation of (2.5): Find $\widehat{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}_s, \boldsymbol{\sigma}_f) \in \mathbf{H}$ and $\widehat{\boldsymbol{\gamma}} := (\boldsymbol{\gamma}, \boldsymbol{\varphi}_s, \boldsymbol{\varphi}_f) \in \mathbf{Q}$ such that

$$\begin{aligned} A(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\tau}}) + B(\widehat{\boldsymbol{\tau}}, \widehat{\boldsymbol{\gamma}}) &= F(\widehat{\boldsymbol{\tau}}) \qquad && \forall\, \widehat{\boldsymbol{\tau}} := (\boldsymbol{\tau}_s, \boldsymbol{\tau}_f) \in \mathbf{H}\,, \\ B(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\eta}}) + K(\widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\eta}}) &= G(\widehat{\boldsymbol{\eta}}) \qquad && \forall\, \widehat{\boldsymbol{\eta}} := (\boldsymbol{\eta}, \boldsymbol{\psi}_s, \boldsymbol{\psi}_f) \in \mathbf{Q}\,, \end{aligned} \tag{2.7}$$

where $F : \mathbf{H} \to \mathbb{C}$ and $G : \mathbf{Q} \to \mathbb{C}$ are the linear functionals

$$\begin{aligned} F(\widehat{\boldsymbol{\tau}}) &:= \frac{1}{\kappa_s^2}\int_{\Omega_s} \mathbf{f}\cdot\mathbf{div}\,\boldsymbol{\tau}_s \quad \forall\, \widehat{\boldsymbol{\tau}} := (\boldsymbol{\tau}_s, \boldsymbol{\tau}_f) \in \mathbf{H}\,, \\ G(\widehat{\boldsymbol{\eta}}) &:= -\langle g\,, \psi_\Gamma \rangle_\Gamma \quad \forall\, \widehat{\boldsymbol{\eta}} := (\boldsymbol{\eta}, \boldsymbol{\psi}_s, \boldsymbol{\psi}_f) := (\boldsymbol{\eta}, \boldsymbol{\psi}_s, (\psi_\Sigma, \psi_\Gamma)) \in \mathbf{Q}\,, \end{aligned} \tag{2.8}$$

and $A : \mathbf{H} \times \mathbf{H} \to \mathbb{C}$, $B : \mathbf{H} \times \mathbf{Q} \to \mathbb{C}$, and $K : \mathbf{Q} \times \mathbf{Q} \to \mathbb{C}$ are the bilinear forms defined by

$$A(\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\tau}}) := \int_{\Omega_s} \mathcal{C}^{-1}\boldsymbol{\zeta}_s : \boldsymbol{\tau}_s - \frac{1}{\kappa_s^2}\int_{\Omega_s} \mathbf{div}\,\boldsymbol{\zeta}_s \cdot \mathbf{div}\,\boldsymbol{\tau}_s + \int_{\Omega_f} \boldsymbol{\zeta}_f \cdot \boldsymbol{\tau}_f - \frac{1}{\kappa_f^2}\int_{\Omega_f} \operatorname{div}\boldsymbol{\zeta}_f \operatorname{div}\boldsymbol{\tau}_f$$

$$\forall\, (\widehat{\boldsymbol{\zeta}}, \widehat{\boldsymbol{\tau}}) := ((\boldsymbol{\zeta}_s, \boldsymbol{\zeta}_f), (\boldsymbol{\tau}_s, \boldsymbol{\tau}_f)) \in \mathbf{H} \times \mathbf{H}\,,$$

$$B(\widehat{\boldsymbol{\tau}}, \widehat{\boldsymbol{\eta}}) := B_s(\boldsymbol{\tau}_s, (\boldsymbol{\eta}, \boldsymbol{\psi}_s)) + B_f(\boldsymbol{\tau}_f, \boldsymbol{\psi}_f) \qquad \forall\, (\widehat{\boldsymbol{\tau}}, \widehat{\boldsymbol{\eta}}) := ((\boldsymbol{\tau}_s, \boldsymbol{\tau}_f), (\boldsymbol{\eta}, \boldsymbol{\psi}_s, \boldsymbol{\psi}_f)) \in \mathbf{H} \times \mathbf{Q}\,,$$

with

$$\begin{aligned} B_s(\boldsymbol{\tau}_s, (\boldsymbol{\eta}, \boldsymbol{\psi}_s)) &:= \int_{\Omega_s} \boldsymbol{\tau}_s : \boldsymbol{\eta} - \langle \boldsymbol{\tau}_s \boldsymbol{\nu}, \boldsymbol{\psi}_s \rangle_\Sigma\,, \\ B_f(\boldsymbol{\tau}_f, \boldsymbol{\psi}_f) &:= \langle \boldsymbol{\tau}_f \cdot \boldsymbol{\nu}, \psi_\Sigma \rangle_\Sigma - \langle \boldsymbol{\tau}_f \cdot \boldsymbol{\nu}, \psi_\Gamma \rangle_\Gamma\,, \end{aligned}$$

and

$$K(\widehat{\boldsymbol{\chi}}, \widehat{\boldsymbol{\eta}}) := -\langle \xi_\Sigma\,\boldsymbol{\nu}, \boldsymbol{\psi}_s \rangle_\Sigma - \rho_f\,\omega^2\,\langle \boldsymbol{\xi}_s \cdot \boldsymbol{\nu}, \psi_\Sigma \rangle_\Sigma + \imath\,\kappa_f\,\langle \xi_\Gamma, \psi_\Gamma \rangle_\Gamma$$

$$\forall\, \widehat{\boldsymbol{\chi}} := (\boldsymbol{\chi}, \boldsymbol{\xi}_s, \boldsymbol{\xi}_f) := (\boldsymbol{\chi}, \boldsymbol{\xi}_s, (\xi_\Sigma, \xi_\Gamma)) \in \mathbf{Q}\,,$$

$$\forall\, \widehat{\boldsymbol{\eta}} := (\boldsymbol{\eta}, \boldsymbol{\psi}_s, \boldsymbol{\psi}_f) := (\boldsymbol{\eta}, \boldsymbol{\psi}_s, (\psi_\Sigma, \psi_\Gamma)) \in \mathbf{Q}\,.$$

At this point we recall from [15, Section 2.4.3] that the inversion of the Hooke operator $\mathcal{C}$ (cf. (2.1)) yields

$$\mathcal{C}^{-1}\boldsymbol{\tau} = \frac{1}{2\mu}\boldsymbol{\tau} - \frac{\lambda}{4\mu(\lambda+\mu)}\operatorname{tr}\boldsymbol{\tau}\,\mathbf{I} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{L}^2(\Omega)\,,$$

from which it is easy to see that

$$\|\mathcal{C}^{-1}(\boldsymbol{\tau})\|_{0,\Omega} \le \frac{1}{\mu}\,\|\boldsymbol{\tau}\|_{0,\Omega} \quad \forall\, \boldsymbol{\tau} \in \mathbb{L}^2(\Omega).$$

The main result concerning the solvability analysis of (2.7) is stated as follows. To this respect, notice that irrespective of the particular functionals defined in (2.8), the following result is actually valid for any pair $(F, G) \in \mathbf{H}' \times \mathbf{Q}'$.

Theorem 2.1. *Assume that the homogeneous problem associated to* (2.7) *has only the trivial solution. Then, given $F \in \mathbf{H}'$ and $G \in \mathbf{Q}'$, there exists a unique $(\widehat{\boldsymbol{\sigma}}, \widehat{\gamma}) \in \mathbf{H} \times \mathbf{Q}$ solution to* (2.7). *In addition, there exists $C_{\mathtt{cd}} > 0$ such that*

$$\|(\widehat{\boldsymbol{\sigma}}, \widehat{\gamma})\|_{\mathbf{H}\times\mathbf{Q}} \;\le\; C_{\mathtt{cd}} \Big\{ \|F\|_{\mathbf{H}'} \,+\, \|G\|_{\mathbf{Q}'} \Big\}. \tag{2.9}$$

Proof. The proof basically consists of showing that the left hand side of (2.7) constitutes a Fredholm operator of index zero. We omit further details and refer to the whole analysis developed in [14, Section 3]. □

We end this section with the converse of the derivation of (2.7). Indeed, the following theorem establishes that the unique solution of (2.7) together with $\mathbf{u}$ and p given in (2.6), solves the original fluid-solid interaction problem (2.5). This result will be used later on in Section 3.3 to prove the efficiency of the a posteriori error estimator. Note that no extra regularity assumptions on the data, but only $\mathbf{f} \in \mathbf{L}^2(\Omega_s)$ and $g \in H^{-1/2}(\Gamma)$, are needed here.

Theorem 2.2. *Let $((\boldsymbol{\sigma}_s, \boldsymbol{\sigma}_f), (\gamma, \boldsymbol{\varphi}_s, \boldsymbol{\varphi}_f)) \in \mathbf{H} \times \mathbf{Q}$ be the unique solution of* (2.7), *where $\boldsymbol{\varphi}_f := (\varphi_{\Sigma}, \varphi_{\Gamma}) \in H^{1/2}(\Sigma) \times H^{1/2}(\Gamma)$, and let $\mathbf{u} \in \mathbf{L}^2(\Omega_s)$ and $p \in L^2(\Omega_f)$ be defined according to* (2.6). *Then $\nabla \mathbf{u} = \mathcal{C}^{-1}\boldsymbol{\sigma}_s + \boldsymbol{\gamma}$ in Ω_s (which yields $\mathbf{u} \in \mathbf{H}^1(\Omega_s)$), $\mathbf{u} = \boldsymbol{\varphi}_s$ on the interface Σ, $\boldsymbol{\sigma}_s = \boldsymbol{\sigma}_s^{\mathtt{t}}$ in Ω_s, and $\boldsymbol{\gamma} = \frac{1}{2}\left(\nabla \mathbf{u} - (\nabla \mathbf{u})^{\mathtt{t}}\right)$ in Ω_s (which yields $\boldsymbol{\sigma}_s = \mathcal{C}\,\boldsymbol{\varepsilon}(\mathbf{u})$). In addition, there hold $\boldsymbol{\sigma}_f = \nabla p$ in Ω_f (which yields $p \in H^1(\Omega_f)$), $\operatorname{div} \boldsymbol{\sigma}_f + \kappa_f^2\, p = 0$ in Ω_f, $\varphi_{\Sigma} = p|_{\Sigma}$ on Σ, $\varphi_{\Gamma} = p|_{\Gamma}$ on Γ, and hence $\boldsymbol{\sigma}_s \boldsymbol{\nu} = -\varphi_{\Sigma}\, \boldsymbol{\nu} = -p\, \boldsymbol{\nu}$ on Σ, $\boldsymbol{\sigma}_f \cdot \boldsymbol{\nu} = \rho_f\, \omega^2\, \boldsymbol{\varphi}_s \cdot \boldsymbol{\nu} = \rho_f\, \omega^2\, \mathbf{u} \cdot \boldsymbol{\nu}$ on Σ, and $\boldsymbol{\sigma}_f \cdot \boldsymbol{\nu} - \imath\, \kappa_f\, \varphi_{\Gamma} = \boldsymbol{\sigma}_f \cdot \boldsymbol{\nu} - \imath\, \kappa_f\, p = g$ on Γ.*

Proof. It basically follows by applying integration by parts backwardly in (2.7) and using suitable test functions. We omit further details. □

2.3. The Galerkin scheme

In this section we recall from [14] the definition of the Galerkin approximation of (2.7). To this end, we first let $\left\{\mathcal{T}_h^s\right\}_{h>0}$ and $\left\{\mathcal{T}_h^f\right\}_{h>0}$ be regular families of triangulations of the polygonal regions $\bar{\Omega}_s$ and $\bar{\Omega}_f$, respectively, by triangles T of diameter h_T, with global mesh sizes

$$h_s := \max\left\{ h_T : T \in \mathcal{T}_h^s \right\}, \quad h_f := \max\left\{ h_T : T \in \mathcal{T}_h^f \right\}, \quad \text{and} \quad h := \max\left\{ h_s,\, h_f \right\},$$

such that they are quasi-uniform around Σ and Γ, and so that their vertices coincide on Σ. In what follows, given an integer $\ell \ge 0$ and a subset S of $\mathbb{R}^2$, $P_\ell(S)$ denotes the space of polynomials defined in S of total degree $\le \ell$. According to the notation convention given in the introduction, we denote $\mathbf{P}_\ell(S) := [P_\ell(S)]^2$. Furthermore, given $T \in \mathcal{T}_h^s \cup \mathcal{T}_h^f$ and $\mathbf{x} := (x_1, x_2)^{\mathtt{t}}$ a generic vector of $\mathbb{R}^2$, we let

$$\mathrm{RT}_0(T) := \operatorname{span}\Big\{ (1,0), (0,1), (x_1, x_2) \Big\}$$

be the local Raviart-Thomas space of order 0 (cf. [8], [27]), and let $\mathbf{curl}^{\mathtt{t}}\, b_T := \left(\frac{\partial b_T}{\partial x_2}, -\frac{\partial b_T}{\partial x_1}\right)$, where b_T is the usual cubic bubble function on T. Then we define

$$\mathbf{H}_h^s := \Big\{ \mathbf{v}_{s,h} \in \mathbf{H}(\mathrm{div}; \Omega_s) : \quad \mathbf{v}_{s,h}|_T \in \mathrm{RT}_0(T) \oplus P_0(T)\, \mathbf{curl}^{\mathtt{t}}\, b_T \quad \forall\, T \in \mathcal{T}_h^s \Big\},$$

$$\mathbb{H}_h^s := \Big\{ \boldsymbol{\tau}_{s,h} \in \mathbb{H}(\mathbf{div}; \Omega_s) : \quad \mathbf{c}^{\mathtt{t}}\, \boldsymbol{\tau}_{s,h} \in \mathbf{H}_h^s \quad \forall\, \mathbf{c} \in \mathbb{R}^2 \Big\}, \tag{2.10}$$

$$\mathbf{H}_h^f := \Big\{ \boldsymbol{\tau}_{f,h} \in \mathbf{H}(\mathrm{div}; \Omega_f) : \quad \boldsymbol{\tau}_{f,h}|_T \in \mathrm{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^f \Big\}, \tag{2.11}$$

$$\mathbb{Q}_h^s := \left\{ \boldsymbol{\eta}_h := \begin{pmatrix} 0 & \eta_h \\ -\eta_h & 0 \end{pmatrix} : \quad \eta_h \in C(\bar{\Omega}_s), \quad \eta_h|_T \in P_1(T) \quad \forall T \in \mathcal{T}_h^s \right\}. \tag{2.12}$$

Next, in order to set the finite dimensional subspaces on the boundaries of the domains, we let Σ_h and Γ_h be the partitions of Σ and Γ, respectively, inherited from the triangulations, and suppose, without loss generality, that the numbers of edges of Σ_h and Γ_h are both even. The case of an odd number of edges is easily reduced to the even case (see [21]). Then, we let Σ_{2h} (resp. Γ_{2h}) be the partition of Σ (resp. Γ) arising by joining pairs of adjacent edges of Σ_h (resp. Γ_h). Because of the assumptions on the triangulations, Σ_h and Γ_h are automatically of bounded variation, and, therefore, so are Σ_{2h} and Γ_{2h}. Hence, we now define

$$\Lambda_h(\Sigma) := \left\{ \psi_h \in C(\Sigma) : \quad \psi_h|_e \in P_1(e) \quad \forall e \text{ edge of } \Sigma_{2h} \right\}, \tag{2.13}$$

$$\Lambda_h(\Gamma) := \left\{ \psi_h \in C(\Gamma) : \quad \psi_h|_e \in P_1(e) \quad \forall e \text{ edge of } \Gamma_{2h} \right\}, \tag{2.14}$$

$$\mathbf{Q}_h^s := \Lambda_h(\Sigma) \times \Lambda_h(\Sigma), \tag{2.15}$$

$$\mathbf{Q}_h^f := \Lambda_h(\Sigma) \times \Lambda_h(\Gamma), \tag{2.16}$$

and introduce the global finite element spaces

$$\mathbf{H}_h := \mathbb{H}_h^s \times \mathbf{H}_h^f \qquad \text{and} \qquad \mathbf{Q}_h := \mathbb{Q}_h^s \times \mathbf{Q}_h^s \times \mathbf{Q}_h^f. \tag{2.17}$$

In addition, our analysis below will also require the subspaces

$$\mathbf{U}_h^s := \left\{ \mathbf{v}_h \in \mathbf{L}^2(\Omega_s) : \quad \mathbf{v}_h|_T \in \mathbf{P}_0(T) \quad \forall T \in \mathcal{T}_h^s \right\}, \tag{2.18}$$

$$U_h^f := \left\{ v_h \in L^2(\Omega_f) : \quad v_h|_T \in P_0(T) \quad \forall T \in \mathcal{T}_h^f \right\}. \tag{2.19}$$

Notice here that $\mathbb{H}_h^s \times \mathbf{U}_h^s \times \mathbb{Q}_h^s$ constitutes the well known PEERS space introduced in [4] for a mixed finite element aproximation of the linear elasticity problem in the plane. In turn, $\mathbf{H}_h^f \times U_h^f$ is the lowest order Raviart-Thomas mixed finite element approximation of the Poisson problem for the Laplace equation (see, e.g., [8, 27]).

According to the above, the Galerkin scheme associated with our continuous problem (2.7) reduces to: Find $\widehat{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_{s,h}, \boldsymbol{\sigma}_{f,h}) \in \mathbf{H}_h$ and $\widehat{\boldsymbol{\gamma}}_h := (\gamma_h, \boldsymbol{\varphi}_{s,h}, \boldsymbol{\varphi}_{f,h}) \in \mathbf{Q}_h$ such that

$$\begin{aligned} A(\widehat{\boldsymbol{\sigma}}_h, \widehat{\boldsymbol{\tau}}_h) + B(\widehat{\boldsymbol{\tau}}_h, \widehat{\boldsymbol{\gamma}}_h) &= F(\widehat{\boldsymbol{\tau}}_h) \qquad && \forall\, \widehat{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{s,h}, \boldsymbol{\tau}_{f,h}) \in \mathbf{H}_h, \\ B(\widehat{\boldsymbol{\sigma}}_h, \widehat{\boldsymbol{\eta}}_h) + K(\widehat{\boldsymbol{\gamma}}_h, \widehat{\boldsymbol{\eta}}_h) &= G(\widehat{\boldsymbol{\eta}}_h) \qquad && \forall\, \widehat{\boldsymbol{\eta}}_h := (\boldsymbol{\eta}_h, \boldsymbol{\psi}_{s,h}, \boldsymbol{\psi}_{f,h}) \in \mathbf{Q}_h. \end{aligned} \tag{2.20}$$

The following theorem establishes the well-posedness and convergence of the discrete scheme (2.20).

Theorem 2.3. *Assume that the homogeneous problem associated to* (2.7) *has only the trivial solution, and let $h_0 > 0$ be the constant provided by* [14, Lemma 4.10]. *Then there exists $h_1 \in (0, h_0]$ such that for each $h \in (0, h_1]$, the fully-mixed finite element scheme* (2.20) *has a unique solution $(\widehat{\boldsymbol{\sigma}}_h, \widehat{\boldsymbol{\gamma}}_h) := ((\boldsymbol{\sigma}_{s,h}, \boldsymbol{\sigma}_{f,h}), (\gamma_h, \boldsymbol{\varphi}_{s,h}, \boldsymbol{\varphi}_{f,h})) \in \mathbf{H}_h \times \mathbf{Q}_h$, with $\boldsymbol{\varphi}_{f,h} := (\varphi_{\Sigma,h}, \varphi_{\Gamma,h}) \in \Lambda_h(\Sigma) \times \Lambda_h(\Gamma)$. In addition, there exist $C_1, C_2 > 0$, independent of h, such that for each $h \in (0, h_1]$ there hold*

$$\|(\widehat{\boldsymbol{\sigma}}_h, \widehat{\boldsymbol{\gamma}}_h)\|_{\mathbf{H}\times\mathbf{Q}}$$

$$\leq C_1 \left\{ \sup_{\widehat{\boldsymbol{\tau}}_h \in \mathbf{H}_h \setminus \{\mathbf{0}\}} \frac{|F(\widehat{\boldsymbol{\tau}}_h)|}{\|\widehat{\boldsymbol{\tau}}_h\|_{\mathbf{H}}} + \sup_{\widehat{\boldsymbol{\eta}}_h \in \mathbf{Q}_h \setminus \{\mathbf{0}\}} \frac{|G(\widehat{\boldsymbol{\eta}}_h)|}{\|\widehat{\boldsymbol{\eta}}_h\|_{\mathbf{Q}}} \right\} \leq C_1 \left\{ \|\mathbf{f}\|_{0,\Omega_s} + \|g\|_{-1/2,\Gamma} \right\}$$

and

$$\|(\widehat{\boldsymbol{\sigma}},\widehat{\gamma}) - (\widehat{\boldsymbol{\sigma}}_h,\widehat{\gamma}_h)\|_{\mathbf{H}\times\mathbf{Q}} \le C_2 \inf_{(\widehat{\boldsymbol{\tau}}_h,\widehat{\boldsymbol{\eta}}_h)\in\mathbf{H}_h\times\mathbf{Q}_h} \|(\widehat{\boldsymbol{\sigma}},\widehat{\gamma}) - (\widehat{\boldsymbol{\tau}}_h,\widehat{\boldsymbol{\eta}}_h)\|_{\mathbf{H}\times\mathbf{Q}},$$

where $(\widehat{\boldsymbol{\sigma}},\widehat{\gamma}) := ((\boldsymbol{\sigma}_s,\boldsymbol{\sigma}_f),(\boldsymbol{\gamma},\boldsymbol{\varphi}_s,\boldsymbol{\varphi}_f)) \in \mathbf{H}\times\mathbf{Q}$ *is the unique solution of* (2.7). *Furthermore, if there exists* $\delta \in (0,1]$ *such that* $\boldsymbol{\sigma}_s \in \mathbb{H}^\delta(\Omega_s)$, $\mathbf{div}\,\boldsymbol{\sigma}_s \in \mathbf{H}^\delta(\Omega_s)$, $\boldsymbol{\sigma}_f \in \mathbf{H}^\delta(\Omega_f)$, $\operatorname{div}\boldsymbol{\sigma}_f \in H^\delta(\Omega_f)$, $\boldsymbol{\gamma} \in \mathbb{H}^\delta(\Omega_s)$, $\boldsymbol{\varphi}_s \in \mathbf{H}^{1/2+\delta}(\Sigma)$, *and* $\boldsymbol{\varphi}_f \in H^{1/2+\delta}(\partial\Omega_f)$, *then there exists* $C_3 > 0$, *independent of* h, *such that for each* $h \in (0,h_1]$ *there holds*

$$\begin{aligned}\|(\widehat{\boldsymbol{\sigma}},\widehat{\gamma}) - (\widehat{\boldsymbol{\sigma}}_h,\widehat{\gamma}_h)\|_{\mathbf{H}\times\mathbf{Q}} \le C_3\, h^\delta \Big\{ &\|\boldsymbol{\sigma}_s\|_{\delta,\Omega_s} + \|\mathbf{div}\,\boldsymbol{\sigma}_s\|_{\delta,\Omega_s} + \|\boldsymbol{\sigma}_f\|_{\delta,\Omega_f} + \|\operatorname{div}\boldsymbol{\sigma}_f\|_{\delta,\Omega_f} \\ &+ \|\boldsymbol{\gamma}\|_{\delta,\Omega_s} + \|\boldsymbol{\varphi}_s\|_{1/2+\delta,\Sigma} + \|\boldsymbol{\varphi}_f\|_{1/2+\delta,\partial\Omega_f} \Big\}.\end{aligned}$$

Proof. See [14, Theorem 4.1] and the whole analysis in [14, Section 4] for full details. □

3. A Residual-based a Posteriori Error Estimator

In this section we derive a reliable and efficient residual based *a posteriori* error estimator for (2.20).

3.1. The main result

We begin by introducing further notations. Given $T \in \mathcal{T}_h^s \cup \mathcal{T}_h^f$, we let $\mathcal{E}(T)$ be the set of edges of T, and denote by $\mathcal{E}_h$ be the set of all edges of $\mathcal{T}_h^s \cup \mathcal{T}_h^f$. Then we can write

$$\mathcal{E}_h = \mathcal{E}_h(\Omega_s) \cup \mathcal{E}_h(\Sigma) \cup \mathcal{E}_h(\Omega_f) \cup \mathcal{E}_h(\Gamma), \tag{3.1}$$

where $\mathcal{E}_h(\Omega_s) := \{e \in \mathcal{E}_h : \ e \subseteq \Omega_s\}$, $\mathcal{E}_h(\Sigma) := \{e \in \mathcal{E}_h : \ e \subseteq \Sigma\}$, and similarly for $\mathcal{E}_h(\Omega_f)$ and $\mathcal{E}_h(\Gamma)$. In what follows, h_e stands for the length of the edge $e \in \mathcal{E}_h$. Also, for each edge $e \in \mathcal{E}_h$ we fix a unit normal vector $\boldsymbol{\nu} := (\nu_1,\nu_2)^{\mathtt{t}}$, and let $\mathbf{s} := (-\nu_2,\nu_1)^{\mathtt{t}}$ be the corresponding fixed unit tangential vector along e. Now, let $\mathbf{w}_s \in \mathbf{L}^2(\Omega_s)$ such that $\mathbf{w}_s|_T \in \mathbf{C}(T)$ for each $T \in \mathcal{T}_h^s$. Then, given $T \in \mathcal{T}_h^s$ and $e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega_s)$, we denote by $[\mathbf{w}_s]$ the jump of $\mathbf{w}_s$ across e, that is $[\mathbf{w}_s] := (\mathbf{w}_s|_T)|_e - (\mathbf{w}_s|_{T'})|_e$, where T and T' are the triangles of $\mathcal{T}_h^s$ having e as a common edge. Also, given $e \in \mathcal{E}_h(\Omega_s)$ and $\boldsymbol{\tau}_s \in \mathbb{L}(\Omega_s)$ such that $\boldsymbol{\tau}_s|_T \in \mathbb{C}(T)$ on each $T \in \mathcal{T}_h^s$, we let $[\boldsymbol{\tau}_s\,\mathbf{s}] := (\boldsymbol{\tau}_s|_T - \boldsymbol{\tau}_s|_{T'})|_e\,\mathbf{s}$. Similar definitions hold for $\mathbf{v}_f \in \mathbf{L}^2(\Omega_f)$ such that $\mathbf{v}_f|_T \in \mathbf{C}(T)$ for each $T \in \mathcal{T}_h^f$. In fact, given $e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega_f)$, we define

$$[\mathbf{v}_f \cdot \boldsymbol{\nu}] := \big((\mathbf{v}_f|_T)|_e - (\mathbf{v}_f|_{T'})|_e\big)|_e \cdot \boldsymbol{\nu}.$$

Finally, given a scalar function q, a vector $\boldsymbol{\chi} := (\boldsymbol{\chi}_1,\boldsymbol{\chi}_2)$ and a tensor $\boldsymbol{\tau} := (\tau_{ij})$, we let

$$\mathbf{curl}(q) := \begin{pmatrix} \frac{\partial q}{\partial x_2} \\ -\frac{\partial q}{\partial x_1} \end{pmatrix}, \quad \underline{\mathbf{curl}}(\boldsymbol{\chi}) := \begin{pmatrix} \frac{\partial \boldsymbol{\chi}_1}{\partial x_2} & -\frac{\partial \boldsymbol{\chi}_1}{\partial x_1} \\ \frac{\partial \boldsymbol{\chi}_2}{\partial x_2} & -\frac{\partial \boldsymbol{\chi}_2}{\partial x_1} \end{pmatrix}, \tag{3.2}$$

$$\operatorname{rot}\boldsymbol{\chi} := \frac{\partial \boldsymbol{\chi}_2}{\partial x_1} - \frac{\partial \boldsymbol{\chi}_1}{\partial x_2} \quad \text{and} \quad \operatorname{curl}(\boldsymbol{\tau}) := \begin{pmatrix} \frac{\partial \tau_{12}}{\partial x_1} - \frac{\partial \tau_{11}}{\partial x_2} \\ \frac{\partial \tau_{22}}{\partial x_1} - \frac{\partial \tau_{21}}{\partial x_2} \end{pmatrix}. \tag{3.3}$$

In addition, given a domain $\mathcal{O}$ with boundary $\partial\mathcal{O}$, we introduce the tangential derivatives

$$\frac{d\varphi}{d\mathbf{s}} := \nabla\varphi\cdot\mathbf{s} \in H^{-1/2}(\partial\mathcal{O}) \quad \forall\,\varphi\in H^1(\mathcal{O}),$$
$$\frac{d\boldsymbol{\varphi}}{d\mathbf{s}} := \nabla\boldsymbol{\varphi}\,\mathbf{s} \in \mathbf{H}^{-1/2}(\partial\mathcal{O}) \quad \forall\,\boldsymbol{\varphi}\in \mathbf{H}^1(\mathcal{O})\,.$$

Next, letting $(\widehat{\boldsymbol{\sigma}}_h,\widehat{\boldsymbol{\gamma}}_h) := ((\boldsymbol{\sigma}_{s,h},\boldsymbol{\sigma}_{f,h}),(\boldsymbol{\gamma}_h,\boldsymbol{\varphi}_{s,h},\boldsymbol{\varphi}_{f,h})) \in \mathbf{H}_h\times\mathbf{Q}_h$ be the unique solution of (2.20), with $\boldsymbol{\varphi}_{f,h} := (\varphi_{\Sigma,h},\varphi_{\Gamma,h}) \in \Lambda_h(\Sigma)\times\Lambda_h(\Gamma)$, and denoting by $\mathcal{P}_h^s$ the $\mathbf{L}^2(\Omega_s)$-orthogonal projector onto $\mathbf{U}_h^s$ (cf. (2.18)), we define for each $T\in\mathcal{T}_h^s$, and for each $T\in\mathcal{T}_h^f$, respectively, the *a posteriori* error indicators:

$$\begin{aligned}\theta_{T,s}^2 := {}& \|\boldsymbol{\sigma}_{s,h}-\boldsymbol{\sigma}_{s,h}^{\mathtt{t}}\|_{0,T}^2 + \|(\mathbf{I}-\mathcal{P}_h^s)\,\mathbf{f}\,\|_{0,T}^2 + h_T^2\,\|\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h}+\boldsymbol{\gamma}_h\|_{0,T}^2\\ &+ h_T^2\,\|\mathrm{curl}(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h}+\boldsymbol{\gamma}_h)\|_{0,T}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_s)} h_e\,\|[(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h}+\boldsymbol{\gamma}_h)\,\mathbf{s}]\|_{0,e}^2\,, \end{aligned} \tag{3.4}$$

$$\theta_{T,f}^2 := h_T^2\,\|\boldsymbol{\sigma}_{f,h}\|_{0,T}^2 + h_T^2\,\|\mathrm{rot}(\boldsymbol{\sigma}_{f,h})\|_{0,T}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_f)} h_e\,\|[\boldsymbol{\sigma}_{f,h}\cdot\mathbf{s}]\|_{0,e}^2\,. \tag{3.5}$$

Similarly, for each $e\in\mathcal{E}_h(\Sigma)$ we define

$$\begin{aligned}\theta_{e,\Sigma}^2 := {}& h_e\,\|\boldsymbol{\varphi}_{s,h}-\mathbf{u}_h\|_{0,e}^2 + h_e\,\|\boldsymbol{\sigma}_{f,h}\cdot\boldsymbol{\nu}-\rho_f\omega^2\boldsymbol{\varphi}_{s,h}\cdot\boldsymbol{\nu}\|_{0,e}^2\\ &+ h_e\,\|\boldsymbol{\sigma}_{s,h}\boldsymbol{\nu}+\varphi_{\Sigma,h}\,\boldsymbol{\nu}\|_{0,e}^2 + h_e\left\|(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h}+\boldsymbol{\gamma}_h)\,\mathbf{s}-\frac{d\boldsymbol{\varphi}_{s,h}}{d\mathbf{s}}\right\|_{0,e}^2\\ &+ h_e\left\|\boldsymbol{\sigma}_{f,h}\cdot\mathbf{s}-\frac{d\,\varphi_{\Sigma,h}}{d\mathbf{s}}\right\|_{0,e}^2 + h_e\,\|\varphi_{\Sigma,h}-p_h\|_{0,e}^2\,,\end{aligned} \tag{3.6}$$

where, resembling (2.6) (see also [14]), we set

$$\mathbf{u}_h := -\frac{1}{\kappa_s^2}\Big(\mathcal{P}_h^s(\mathbf{f})+\mathbf{div}\,\boldsymbol{\sigma}_{s,h}\Big) \quad\text{in}\quad \Omega_s \quad\text{and}\quad p_h := -\frac{1}{\kappa_f^2}\mathrm{div}\,\boldsymbol{\sigma}_{f,h} \quad\text{in}\quad \Omega_f\,. \tag{3.7}$$

In addition, assuming that the Robin datum $g\in L^2(\Gamma)$, we set for each $e\in\mathcal{E}_h(\Gamma)$

$$\theta_{e,\Gamma}^2 := h_e\left\|\boldsymbol{\sigma}_{f,h}\cdot\mathbf{s}-\frac{d\,\varphi_{\Gamma,h}}{d\mathbf{s}}\right\|_{0,e}^2 + h_e\,\|\varphi_{\Gamma,h}-p_h\|_{0,e}^2 + h_e\,\|\boldsymbol{\sigma}_{f,h}\cdot\boldsymbol{\nu}-\imath\,\kappa_f\,\varphi_{\Gamma,h}-g\|_{0,e}^2\,. \tag{3.8}$$

Therefore, we introduce the global *a posteriori* error estimator

$$\boldsymbol{\theta} := \left\{\sum_{T\in\mathcal{T}_h^s}\theta_{T,s}^2 + \sum_{T\in\mathcal{T}_h^f}\theta_{T,f}^2 + \sum_{e\in\mathcal{E}_h(\Sigma)}\theta_{e,\Sigma}^2 + \sum_{e\in\mathcal{E}_h(\Gamma)}\theta_{e,\Gamma}^2\right\}^{1/2}, \tag{3.9}$$

and state the main result of this section as follows.

Theorem 3.1. *Assume that the homogeneous problem associated to* (2.7) *has only the trivial solution, and let* $(\widehat{\boldsymbol{\sigma}},\widehat{\boldsymbol{\gamma}}) := \big((\boldsymbol{\sigma}_s,\boldsymbol{\sigma}_f),(\boldsymbol{\gamma},\boldsymbol{\varphi}_s,\boldsymbol{\varphi}_f)\big) \in \mathbf{H}\times\mathbf{Q}$ *and*

$$(\widehat{\boldsymbol{\sigma}}_h,\widehat{\boldsymbol{\gamma}}_h) := \big((\boldsymbol{\sigma}_{s,h},\boldsymbol{\sigma}_{f,h}),(\boldsymbol{\gamma}_h,\boldsymbol{\varphi}_{s,h},\boldsymbol{\varphi}_{f,h})\big) \in \mathbf{H}_h\times\mathbf{Q}_h$$

be the unique solutions of (2.7) *and* (2.20)*, respectively. In addition, let* $\mathbf{u}\in\mathbf{L}^2(\Omega_s)$ *and* $p\in L^2(\Omega_f)$ *be defined according to* (2.6)*, that is* $\mathbf{u} := -\frac{1}{\kappa_s^2}\big(\mathbf{f}+\mathbf{div}\,\boldsymbol{\sigma}_s\big)$ *and* $p = -\frac{1}{\kappa_f^2}\mathrm{div}\,\boldsymbol{\sigma}_f$*, and assume that the Robin datum g belongs to* $L^2(\Gamma)$*. Then, there exist* C_{eff}, $C_{\mathrm{rel}}>0$ *independent of h, such that*

$$C_{\mathrm{eff}}\,\boldsymbol{\theta} \le \|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega_s} + \|p-p_h\|_{0,\Omega_f} + \|\widehat{\boldsymbol{\sigma}}-\widehat{\boldsymbol{\sigma}}_h\|_{\mathbf{H}} + \|\widehat{\boldsymbol{\gamma}}-\widehat{\boldsymbol{\gamma}}_h\|_{\mathbf{Q}} \le C_{\mathrm{rel}}\,\boldsymbol{\theta}\,. \tag{3.10}$$

The lower and upper estimates given by (3.10) constitute what we call the efficiency and reliability of $\boldsymbol{\theta}$, respectively.

3.2. Reliability of the a posteriori error estimator

We begin with the upper bounds for $\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega_s}$ and $\|p-p_h\|_{0,\Omega_f}$. In fact, according to the definitions of $\mathbf{u}$ and p (cf. (2.6)), $\mathbf{u}_h$ and p_h (cf. (3.7)), we easily find that

$$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega_s} \le \frac{1}{\kappa_s^2}\Big\{\|(I-\mathcal{P}_h^s)\mathbf{f}\|_{0,\Omega_s} + \|\boldsymbol{\sigma}_s-\boldsymbol{\sigma}_{s,h}\|_{\mathbf{div};\Omega_s}\Big\} \tag{3.11}$$

and

$$\|p-p_h\|_{0,\Omega_f} \le \frac{1}{\kappa_f^2}\,\|\boldsymbol{\sigma}_f-\boldsymbol{\sigma}_{f,h}\|_{\mathrm{div};\Omega_f}\,. \tag{3.12}$$

We continue our analysis by recalling that the continuous dependence result given by (2.9) (cf. Theorem 2.1) is equivalent to the global inf-sup condition for the continuous formulation (2.7) with the constant $\alpha = \frac{1}{2\,C_{\mathrm{cd}}} > 0$. Then, by applying this estimate to the error $(\widehat{\boldsymbol{\sigma}},\widehat{\boldsymbol{\gamma}}) - (\widehat{\boldsymbol{\sigma}}_h,\widehat{\boldsymbol{\gamma}}_h) \in \mathbf{H}\times\mathbf{Q}$, we obtain

$$\alpha\,\|(\widehat{\boldsymbol{\sigma}},\widehat{\boldsymbol{\gamma}}) - (\widehat{\boldsymbol{\sigma}}_h,\widehat{\boldsymbol{\gamma}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \;\le\; \sup_{(\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\eta}})\in\mathbf{H}\times\mathbf{Q}\setminus\{\mathbf{0}\}} \frac{|E(\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\eta}})|}{\|(\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\eta}})\|_{\mathbf{H}\times\mathbf{Q}}}\,,$$

where

$$E(\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\eta}}) \;:=\; A(\widehat{\boldsymbol{\sigma}}-\widehat{\boldsymbol{\sigma}}_h,\widehat{\boldsymbol{\tau}}) \,+\, B(\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\gamma}}-\widehat{\boldsymbol{\gamma}}_h) \,+\, B(\widehat{\boldsymbol{\sigma}}-\widehat{\boldsymbol{\sigma}}_h,\widehat{\boldsymbol{\eta}}) \,+\, K(\widehat{\boldsymbol{\gamma}}-\widehat{\boldsymbol{\gamma}}_h,\widehat{\boldsymbol{\eta}})\,,$$

for all

$$(\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\eta}}) \;:=\; ((\boldsymbol{\tau}_s,\boldsymbol{\tau}_f),(\boldsymbol{\eta},\boldsymbol{\psi}_s,\boldsymbol{\psi}_f)) \;\in\; \mathbf{H}\times\mathbf{Q},$$

with

$$\boldsymbol{\psi}_f \;=\; (\psi_{\Sigma},\psi_{\Gamma}) \in H^{1/2}(\Sigma)\times H^{1/2}(\Gamma).$$

More precisely, thanks to the equations of the continuous variational formulation (2.7), we deduce that

$$E(\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\eta}}) \;=\; E_1(\boldsymbol{\tau}_s) \,+\, E_2(\boldsymbol{\tau}_f) \,+\, E_3(\boldsymbol{\eta}) \,+\, E_4(\boldsymbol{\psi}_s) \,+\, E_5(\psi_{\Sigma}) \,+\, E_6(\psi_{\Gamma})\,, \tag{3.13}$$

where E_1 up to E_6 are the linear functionals defined by

$$\begin{aligned}
E_1(\boldsymbol{\tau}_s) \;:=&\; \frac{1}{\kappa_s^2}\int_{\Omega_s}\big\{\mathbf{f} \,+\, \mathbf{div}\,\boldsymbol{\sigma}_{s,h}\big\}\cdot\mathbf{div}\,\boldsymbol{\tau}_s \\
&\quad - \int_{\Omega_s}\big\{\mathcal{C}^{-1}\,\boldsymbol{\sigma}_{s,h} \,+\, \boldsymbol{\gamma}_h\big\} : \boldsymbol{\tau}_s + \langle\boldsymbol{\tau}_s\,\boldsymbol{\nu},\boldsymbol{\varphi}_{s,h}\rangle_{\Sigma}\,,
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
E_2(\boldsymbol{\tau}_f) \;:=&\; \frac{1}{\kappa_f^2}\int_{\Omega_f}\mathrm{div}\,\boldsymbol{\sigma}_{f,h}\,\mathrm{div}\,\boldsymbol{\tau}_f \,-\, \int_{\Omega_f}\boldsymbol{\sigma}_{f,h}\cdot\boldsymbol{\tau}_f \\
&\quad - \,\langle\boldsymbol{\tau}_f\cdot\boldsymbol{\nu},\varphi_{\Sigma,h}\rangle_{\Sigma} \,+\, \langle\boldsymbol{\tau}_f\cdot\boldsymbol{\nu},\varphi_{\Gamma,h}\rangle_{\Gamma}\,,
\end{aligned} \tag{3.15}$$

$$E_3(\boldsymbol{\eta}) \;:=\; -\int_{\Omega_s}\boldsymbol{\sigma}_{s,h} : \boldsymbol{\eta}\,,$$

$$E_4(\boldsymbol{\psi}_s) \;:=\; \langle\boldsymbol{\sigma}_{s,h}\,\boldsymbol{\nu} + \varphi_{\Sigma,h}\,\boldsymbol{\nu},\boldsymbol{\psi}_s\rangle_{\Sigma}\,,$$

$$E_5(\psi_{\Sigma}) \;:=\; -\,\langle\boldsymbol{\sigma}_{f,h}\cdot\boldsymbol{\nu} \,-\, \rho_f\,\omega^2\,\boldsymbol{\varphi}_{s,h}\cdot\boldsymbol{\nu},\psi_{\Sigma}\rangle_{\Sigma}\,,$$

$$E_6(\psi_{\Gamma}) \;:=\; \langle\boldsymbol{\sigma}_{f,h}\cdot\boldsymbol{\nu} \,-\, i\,\kappa_f\,\varphi_{\Gamma,h} - g,\psi_{\Gamma}\rangle_{\Gamma}\,.$$

In addition, it is not difficult to see that

$$\sup_{(\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\eta}})\in\mathbf{H}\times\mathbf{Q}\backslash\{\mathbf{0}\}} \frac{|E(\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\eta}})|}{\|(\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\eta}})\|_{\mathbf{H}\times\mathbf{Q}}}$$

$$\leq \sup_{\boldsymbol{\tau}_s\in\mathbb{H}(\mathbf{div};\Omega_s)\backslash\{\mathbf{0}\}} \frac{|E_1(\boldsymbol{\tau}_s)|}{\|\boldsymbol{\tau}_s\|_{\mathbf{div};\Omega_s}} + \sup_{\boldsymbol{\tau}_f\in\mathbf{H}(\mathrm{div};\Omega_f)\backslash\{\mathbf{0}\}} \frac{|E_2(\boldsymbol{\tau}_f)|}{\|\boldsymbol{\tau}_f\|_{\mathrm{div};\Omega_f}}$$

$$+ \sup_{\boldsymbol{\eta}\in\mathbb{L}^2_{\mathtt{skew}}(\Omega_s)\backslash\{\mathbf{0}\}} \frac{|E_3(\boldsymbol{\eta})|}{\|\boldsymbol{\eta}\|_{0,\Omega_s}} + \sup_{\boldsymbol{\psi}_s\in\mathbf{H}^{1/2}(\Sigma)\backslash\{\mathbf{0}\}} \frac{|E_4(\boldsymbol{\psi}_s)|}{\|\boldsymbol{\psi}_s\|_{1/2,\Sigma}}$$

$$+ \sup_{\psi_\Sigma\in H^{1/2}(\Sigma)\backslash\{0\}} \frac{|E_5(\psi_\Sigma)|}{\|\psi_\Sigma\|_{1/2,\Sigma}} + \sup_{\psi_\Gamma\in H^{1/2}(\Gamma)\backslash\{0\}} \frac{|E_6(\psi_\Gamma)|}{\|\psi_\Gamma\|_{1/2,\Gamma}}\,. \tag{3.16}$$

Furthermore, the "*Galerkin orthogonality condition*" arising from (2.7) and (2.20) establishes that

$$E(\widehat{\boldsymbol{\tau}}_h,\widehat{\boldsymbol{\eta}}_h) \,=\, 0 \qquad \forall\,(\widehat{\boldsymbol{\tau}}_h,\widehat{\boldsymbol{\eta}}_h) \,\in\, \mathbf{H}_h\times\mathbf{Q}_h\,,$$

and hence, in order to estimate the above norms of the six functionals defining $E(\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\eta}})$, we could replace $(\boldsymbol{\tau}_s,\boldsymbol{\tau}_f,\boldsymbol{\eta},\boldsymbol{\psi}_s,\psi_\Sigma,\psi_\Gamma)$ by $(\boldsymbol{\tau}_s-\boldsymbol{\tau}_{s,h},\boldsymbol{\tau}_f-\boldsymbol{\tau}_{f,h},\boldsymbol{\eta}-\boldsymbol{\eta}_h,\boldsymbol{\psi}_s-\boldsymbol{\psi}_{s,h},\psi_\Sigma-\psi_{\Sigma,h},\psi_\Gamma-\psi_{\Gamma,h})$ with any suitable choice of $\widehat{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{s,h},\boldsymbol{\tau}_{f,h}) \in \mathbf{H}_h$ and $\widehat{\boldsymbol{\eta}}_h := (\boldsymbol{\eta}_h,\boldsymbol{\psi}_{s,h},(\psi_{\Sigma,h},\psi_{\Gamma,h})) \in \mathbf{Q}_h$, whenever it is necessary. However, this procedure is applied in what follows only to estimate the first two suprema on the right hand side of (3.16).

We begin the estimates of all these suprema with the last four of them.

Lemma 3.1. *There holds*

$$\|E_3\| \,:=\, \sup_{\boldsymbol{\eta}\in\mathbb{L}^2_{\mathtt{skew}}(\Omega_s)\backslash\{\mathbf{0}\}} \frac{|E_3(\boldsymbol{\eta})|}{\|\boldsymbol{\eta}\|_{0,\Omega_s}} \,\leq\, \frac{1}{2}\,\|\boldsymbol{\sigma}_{s,h}-\boldsymbol{\sigma}^{\mathtt{t}}_{s,h}\|^2_{0,\Omega_s}\,. \tag{3.17}$$

Proof. It suffices to see that $\boldsymbol{\sigma}_{s,h} \,=\, \frac{1}{2}(\boldsymbol{\sigma}_{s,h}+\boldsymbol{\sigma}^{\mathtt{t}}_{s,h}) + \frac{1}{2}(\boldsymbol{\sigma}_{s,h}-\boldsymbol{\sigma}^{\mathtt{t}}_{s,h})$, which yields

$$\int_{\Omega_s}\boldsymbol{\sigma}_{s,h}:\boldsymbol{\eta} \,=\, \frac{1}{2}\int_{\Omega_s}\big(\boldsymbol{\sigma}_{s,h}-\boldsymbol{\sigma}^{\mathtt{t}}_{s,h}\big):\boldsymbol{\eta} \qquad \forall\,\boldsymbol{\eta}\in\mathbb{L}^2_{\mathtt{skew}}(\Omega_s)\,,$$

and hence the Cauchy-Schwarz inequality completes the proof. □

The upper bounds for the norms of E_4, E_5, and E_6, being all consequence of the same arguments, are collected in the following lemma.

Lemma 3.2. *There exist C_4, C_5, $C_6 \geq 0$, independent of h, such that*

$$\|E_4\| \,:=\, \sup_{\boldsymbol{\psi}_s\in\mathbf{H}^{1/2}(\Sigma)\backslash\{\mathbf{0}\}} \frac{|E_4(\boldsymbol{\psi}_s)|}{\|\boldsymbol{\psi}_s\|_{1/2,\Sigma}} \,\leq\, C_4\left\{\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\,\|\boldsymbol{\sigma}_{s,h}\,\boldsymbol{\nu}+\varphi_{\Sigma,h}\,\boldsymbol{\nu}\|^2_{0,e}\right\}^{1/2}, \tag{3.18}$$

$$\|E_5\| \,:=\, \sup_{\psi_\Sigma\in H^{1/2}(\Sigma)\backslash\{0\}} \frac{|E_5(\psi_\Sigma)|}{\|\psi_\Sigma\|_{1/2,\Sigma}} \,\leq\, C_5\left\{\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\,\|\boldsymbol{\sigma}_{f,h}\cdot\boldsymbol{\nu}-\rho_f\,\omega^2\,\boldsymbol{\varphi}_{s,h}\cdot\boldsymbol{\nu}\|^2_{0,e}\right\}^{1/2}, \tag{3.19}$$

and

$$\|E_6\| := \sup_{\psi_\Gamma \in H^{1/2}(\Gamma)\setminus\{0\}} \frac{|E_6(\psi_\Gamma)|}{\|\psi_\Gamma\|_{1/2,\Gamma}} \leq C_6 \left\{ \sum_{e\in\mathcal{E}_h(\Gamma)} h_e \, \|\boldsymbol{\sigma}_{f,h}\cdot\boldsymbol{\nu} - i\,\kappa_f\,\varphi_{\Gamma,h} - g\|_{0,e}^2 \right\}^{1/2} . \quad (3.20)$$

Proof. It follows easily from the definitions of the functionals involved that

$$\begin{aligned} \|E_4\| &= \|\boldsymbol{\sigma}_{s,h}\,\boldsymbol{\nu} + \varphi_{\Sigma,h}\,\boldsymbol{\nu}\|_{-1/2,\Sigma}\,, \\ \|E_5\| &= \|\boldsymbol{\sigma}_{f,h}\cdot\boldsymbol{\nu} \,-\, \rho_f\,\omega^2\,\boldsymbol{\varphi}_{s,h}\cdot\boldsymbol{\nu}\|_{-1/2,\Sigma}\,, \\ \|E_6\| &= \|\boldsymbol{\sigma}_{f,h}\cdot\boldsymbol{\nu} \,-\, i\,\kappa_f\,\varphi_{\Gamma,h} - g\|_{-1/2,\Gamma}\,. \end{aligned}$$

Next, we observe from the equations forming the Galerkin scheme (2.20), that the discrete versions of the transmission and Robin boundary conditions become, respectively,

$$\begin{aligned} &\langle \boldsymbol{\sigma}_{s,h}\,\boldsymbol{\nu} + \varphi_{\Sigma,h}\,\boldsymbol{\nu}, \boldsymbol{\psi}_{s,h}\rangle_\Sigma = 0 && \forall\,\boldsymbol{\psi}_{s,h} \in \Lambda_h(\Sigma)\times\Lambda_h(\Sigma)\,, \\ &\langle \boldsymbol{\sigma}_{f,h}\cdot\boldsymbol{\nu} \,-\, \rho_f\,\omega^2\,\boldsymbol{\varphi}_{s,h}\cdot\boldsymbol{\nu}, \psi_{\Sigma,h}\rangle_\Sigma && \forall\,\psi_{\Sigma,h} \in \Lambda_h(\Sigma)\,, \\ &\langle \boldsymbol{\sigma}_{f,h}\cdot\boldsymbol{\nu} \,-\, i\,\kappa_f\,\varphi_{\Gamma,h} - g, \psi_{\Gamma,h}\rangle_\Gamma && \forall\,\psi_{\Gamma,h} \in \Lambda_h(\Gamma)\,, \end{aligned}$$

which say, equivalently, that each expression on the left hand side of the above dualities is orthogonal to the corresponding finite element subspace indicated at the end of each equation. In particular, $\boldsymbol{\sigma}_{s,h}\,\boldsymbol{\nu} + \varphi_{\Sigma,h}\,\boldsymbol{\nu}$ is $\mathbf{L}^2(\Sigma)$-orthogonal to $\Lambda_h(\Sigma)\times\Lambda_h(\Sigma)$, and therefore, a straightforward application of [9, Theorem 2] and the fact that Σ_h and Σ_{2h} are of bounded variation, yield the existence of a constant $C_4 > 0$, independent of h, such that, denoting by $\mathcal{E}_{2h}(\Sigma)$ the set of edges of Σ_{2h}, there holds

$$\begin{aligned} &\|\boldsymbol{\sigma}_{s,h}\,\boldsymbol{\nu} + \varphi_{\Sigma,h}\,\boldsymbol{\nu}\|_{-1/2,\Sigma} \\ &\leq C \sum_{e\in\mathcal{E}_{2h}(\Sigma)} h_e \|\boldsymbol{\sigma}_{s,h}\,\boldsymbol{\nu} + \varphi_{\Sigma,h}\,\boldsymbol{\nu}\|_{0,e}^2 \,\leq\, C_4 \sum_{e\in\mathcal{E}_h(\Sigma)} h_e \|\boldsymbol{\sigma}_{s,h}\,\boldsymbol{\nu} + \varphi_{\Sigma,h}\,\boldsymbol{\nu}\|_{0,e}^2\,, \end{aligned}$$

which shows (3.18). The proofs of (3.19) and (3.20), being also based on [9, Theorem 2] and the above mentioned properties of Σ_h and Σ_{2h}, are derived similarly. We omit further details.□

We now aim to establish the upper bounds of $\|E_1\|$ and $\|E_2\|$, for which, as announced before, we plan to use that

$$E_1(\boldsymbol{\tau}_s) \,=\, E_1(\boldsymbol{\tau}_s - \boldsymbol{\tau}_{s,h}) \quad \text{and} \quad E_2(\boldsymbol{\tau}_f) \,=\, E_2(\boldsymbol{\tau}_f - \boldsymbol{\tau}_{f,h}) \qquad \forall\,\widehat{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{s,h}, \boldsymbol{\tau}_{f,h}) \in \mathbf{H}_h\,. \quad (3.21)$$

To this end, we also need the auxiliary results given in the following section.

3.2.1. Auxiliary results

We first consider the space of pure Raviart-Thomas tensors of order 0, that is

$$\mathbb{RT}_h^s := \left\{ \boldsymbol{\tau}_{s,h} \in \mathbb{H}(\mathbf{div};\Omega_s) : \quad \mathbf{c}^{\mathrm{t}}\,\boldsymbol{\tau}_{s,h}|_T \,\in\, \mathrm{RT}_0(T) \quad \forall\,T\in\mathcal{T}_h^s\,, \quad \forall\,\mathbf{c}\in\mathbb{R}^2 \right\},$$

which is clearly contained in $\mathbb{H}_h^s$ (cf. (2.10)). Then, we let $\Pi_h^s : \mathbb{H}^1(\Omega_s) \to \mathbb{RT}_h^s$ and $\Pi_h^f : \mathbf{H}^1(\Omega_f) \to \mathbf{H}_h^f$ be the usual Raviart–Thomas interpolation operators, which are characterized by the identities

$$\int_e \Pi_h^s(\boldsymbol{\zeta}_s)\,\boldsymbol{\nu} = \int_e \boldsymbol{\zeta}_s\,\boldsymbol{\nu} \qquad \forall\,e\in\mathcal{T}_h^s\,, \quad \forall\,\boldsymbol{\zeta}_s \in \mathbb{H}^1(\Omega_s)\,, \quad (3.22)$$

and

$$\int_e \Pi_h^f(\boldsymbol{\zeta}_f)\cdot\boldsymbol{\nu} = \int_e \boldsymbol{\zeta}_f\cdot\boldsymbol{\nu} \qquad \forall\, e\in\mathcal{T}_h^f\,, \quad \forall\,\boldsymbol{\zeta}_f\in\mathbf{H}^1(\Omega_f)\,. \tag{3.23}$$

It is easy to show, using (3.22) and (3.23), that

$$\mathbf{div}(\Pi_h^s(\boldsymbol{\zeta}_s)) \;=\; \mathcal{P}_h^s(\mathbf{div}\,\boldsymbol{\zeta}_s) \quad\text{and}\quad \mathrm{div}(\Pi_h^f(\boldsymbol{\zeta}_f)) \;=\; \mathcal{P}_h^f(\mathrm{div}\,\boldsymbol{\zeta}_f)\,, \tag{3.24}$$

where, as said before, $\mathcal{P}_h^s$ is the $\mathbf{L}^2(\Omega_s)$-orthogonal projector onto $\mathbf{U}_h^s$ (cf. (2.18)), and $\mathcal{P}_h^f$ is the $L^2(\Omega_f)$-orthogonal projector onto U_h^f (cf. (2.10)). In addition, it is well known (see, e.g. [8], [27], and [20, Theorem 4.5]) that Π_h^s and Π_h^f satisfy the following approximation properties:

$$\|\boldsymbol{\zeta}_s - \Pi_h^s(\boldsymbol{\zeta}_s)\|_{0,T} \le C\,h_T\|\boldsymbol{\zeta}_s\|_{1,T} \qquad \forall\,T\in\mathcal{T}_h^s, \quad \forall\,\boldsymbol{\zeta}_s\in\mathbb{H}^1(\Omega_s)\,, \tag{3.25}$$

$$\|(\boldsymbol{\zeta}_s - \Pi_h^s(\boldsymbol{\zeta}_s))\,\boldsymbol{\nu}\|_{0,e} \le C\,h_e^{1/2}\|\boldsymbol{\zeta}_s\|_{1,T_e} \qquad \forall\,e\in\mathcal{T}_h^s, \quad \forall\,\boldsymbol{\zeta}_s\in\mathbb{H}^1(\Omega_s)\,, \tag{3.26}$$

$$\|\boldsymbol{\zeta}_f - \Pi_h^f(\boldsymbol{\zeta}_f)\|_{0,T} \le C\,h_T\|\boldsymbol{\zeta}_f\|_{1,T} \qquad \forall\,T\in\mathcal{T}_h^f, \quad \forall\,\boldsymbol{\zeta}_f\in\mathbf{H}^1(\Omega_f)\,, \tag{3.27}$$

$$\|(\boldsymbol{\zeta}_f - \Pi_h^f(\boldsymbol{\zeta}_f))\cdot\boldsymbol{\nu}\|_{0,e} \le C\,h_e^{1/2}\|\boldsymbol{\zeta}_f\|_{1,T_e} \quad \forall\,e\in\mathcal{T}_h^f, \quad \forall\,\boldsymbol{\zeta}_f\in\mathbf{H}^1(\Omega_f)\,, \tag{3.28}$$

where T_e in (3.26) (resp. in (3.28)) is a triangle of $\mathcal{T}_h^s$ (resp. $\mathcal{T}_h^f$) containing e on its boundary.

We now let $I_{s,h} : H^1(\Omega_s) \to X_{s,h}$ and $I_{f,h} : H^1(\Omega_f) \to X_{f,h}$ be the usual Clément interpolation operators (cf. [12]), where

$$X_{s,h} := \Big\{ v\in C(\bar\Omega_s): \quad v|_T\in P_1(T), \quad \forall\,T\in\mathcal{T}_h^s\Big\}\,,$$

$$X_{f,h} := \Big\{ v\in C(\bar\Omega_f): \quad v|_T\in P_1(T), \quad \forall\,T\in\mathcal{T}_h^f\Big\}\,.$$

A vectorial version of $I_{s,h}$, say $\mathbf{I}_{s,h} : \mathbf{H}^1(\Omega_s) \to \mathbf{X}_{s,h} := X_{s,h}\times X_{s,h}$, which is defined componentwise by $I_{s,h}$, is also required. The following lemma provides the local approximation properties of $I_{s,h}$. Analogue estimates hold for the operator $I_{f,h}$.

Lemma 3.3. *There exist constants c_1, $c_2>0$, independent of h_s, such that for all $v\in H^1(\Omega_s)$ there holds*

$$\begin{aligned}
\|v - I_{s,h}(v)\|_{0,T} \;&\le\; c_1\; h_T\,\|v\|_{1,\Delta(T)} \qquad \forall\,T\in\mathcal{T}_h^s\\
\|v - I_{s,h}(v)\|_{0,e} \;&\le\; c_2\; h_e^{1/2}\,\|v\|_{1,\Delta(e)} \qquad \forall\,e\in\mathcal{E}_h(\Omega_s)\cup\mathcal{E}_h(\Sigma)\,,
\end{aligned}$$

where

$$\Delta(T) := \cup\{T'\in\mathcal{T}_h^s : T'\cap T\neq\emptyset\} and \Delta(e) := \cup\{T'\in\mathcal{T}_h^s : T'\cap e\neq\emptyset\}.$$

Proof. See [12]. $\square$

Next, in order to define a suitable $\widehat{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{s,h}, \boldsymbol{\tau}_{f,h}) \in \mathbf{H}_h$ to be employed in (3.21), we first demonstrate the existence of continuous Helmholtz decompositions of the spaces $\mathbb{H}(\mathbf{div};\Omega_s)$ and $\mathbf{H}(\mathrm{div};\Omega_f)$. More precisely, we adapt the analysis from [16, Section 3.2.2] to establish the following result.

Lemma 3.4. *For each* $\boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s)$ *there exist* $\boldsymbol{\zeta}_s \in \mathbb{H}^1(\Omega_s)$ *and* $\boldsymbol{\chi}_s := (\chi_1,\chi_2)^{\mathtt{t}} \in \mathbf{H}^1(\Omega_s)$, *with* $\int_{\Omega_s}\chi_1 = \int_{\Omega_s}\chi_2 = 0$, *such that* $\boldsymbol{\tau}_s \,=\, \boldsymbol{\zeta}_s \,+\, \underline{\mathbf{curl}}\,\boldsymbol{\chi}_s$ *in* Ω_s *and*

$$\|\boldsymbol{\zeta}_s\|_{1,\Omega_s} + \|\boldsymbol{\chi}_s\|_{1,\Omega_s} \,\le\, C_s\,\|\boldsymbol{\tau}_s\|_{\mathbf{div};\Omega_s}\,, \tag{3.29}$$

where C_s *is a positive constant independent of* $\boldsymbol{\tau}_s$. *In turn, for each* $\boldsymbol{\tau}_f \in \mathbf{H}(\mathrm{div};\Omega_f)$ *there exist* $\mathbf{w}_f \in \mathbf{H}^1(\Omega_f)$ *and* $\phi_f \in H^1(\Omega_f)$, *such that* $\boldsymbol{\tau}_f \,=\, \mathbf{w}_f \,+\, \mathbf{curl}\,\phi_f$ *in* Ω_f *and*

$$\|\mathbf{w}_f\|_{1,\Omega_f} + \|\phi_f\|_{1,\Omega_f} \,\le\, C_f\,\|\boldsymbol{\tau}_f\|_{\mathrm{div};\Omega_f}\,, \tag{3.30}$$

where C_f *is a positive constant independent of* $\boldsymbol{\tau}_f$.

Proof. We proceed as in [16, Section 3.2.2] by considering first a convex domain $\widetilde{\Omega}$ containing Ω_s. Then, given $\boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s)$, we define the auxiliary function $\mathbf{q} \in \mathbf{L}^2(\widetilde{\Omega})$ by

$$\mathbf{q} \,:=\, \begin{cases} \mathbf{div}\,\boldsymbol{\tau}_s & \text{in} \quad \Omega_s \\ \mathbf{0} & \text{in} \quad \widetilde{\Omega}\backslash\bar{\Omega}_s \end{cases},$$

and let $\mathbf{z} \in \mathbf{H}^1_0(\widetilde{\Omega})$ be the unique weak solution of the boundary value problem:

$$\Delta\mathbf{z} \,=\, \mathbf{q} \quad \text{in} \quad \widetilde{\Omega}\,, \qquad \mathbf{z} = 0 \quad \text{on} \quad \partial\widetilde{\Omega}\,.$$

The elliptic regularity result for the above problem guarantees that $\mathbf{z} \in \mathbf{H}^2(\widetilde{\Omega})$ and

$$\|\mathbf{z}\|_{2,\widetilde{\Omega}} \,\le\, C\,\|\mathbf{q}\|_{0,\widetilde{\Omega}} \,=\, \|\mathbf{div}\,\boldsymbol{\tau}_s\|_{0,\Omega_s}\,.$$

It follows that $\boldsymbol{\zeta}_s \,:=\, \nabla\mathbf{z}|_{\Omega_s}$ belongs to $\mathbb{H}^1(\Omega_s)$,

$$\mathbf{div}\,\boldsymbol{\zeta}_s \,=\, \mathbf{div}\,\boldsymbol{\tau}_s \quad \text{in} \quad \Omega_s\,, \tag{3.31}$$

$$\|\boldsymbol{\zeta}_s\|_{1,\Omega_s} \,\le\, C\,\|\mathbf{z}\|_{2,\Omega_s} \,\le\, C\,\|\mathbf{div}\,\boldsymbol{\tau}_s\|_{0,\Omega_s}\,. \tag{3.32}$$

In this way, since $\mathbf{div}(\boldsymbol{\tau}_s - \boldsymbol{\zeta}_s) = 0$ in Ω_s, and Ω_s is connected, there exist $\boldsymbol{\chi}_s \,:=\, (\chi_1,\chi_2)^{\mathtt{t}} \in \mathbf{H}^1(\Omega_s)$, with $\int_{\Omega_s}\chi_1 \,=\, \int_{\Omega_s}\chi_2 \,=\, 0$, such that $\boldsymbol{\tau}_s - \boldsymbol{\zeta}_s \,=\, \underline{\mathbf{curl}}\,\boldsymbol{\chi}_s$. Note that this identity, the generalized Poincaré inequality, and (3.32) imply that

$$\begin{aligned} \|\boldsymbol{\chi}_s\|_{1,\Omega_s} \,&\le C\,|\boldsymbol{\chi}_s|_{1,\Omega_s} \,=\, C\,\|\boldsymbol{\tau}_s - \boldsymbol{\zeta}_s\|_{0,\Omega_s} \\ &\le C\,\big\{\|\boldsymbol{\tau}_s\|_{0,\Omega_s} \,+\, \|\boldsymbol{\zeta}_s\|_{0,\Omega_s}\big\} \,\le\, C\,\|\boldsymbol{\tau}_s\|_{\mathbf{div};\Omega_s}\,, \end{aligned}$$

which, together with (3.32) again, yields (3.29).

In turn, given $\boldsymbol{\tau}_f \,\in\, \mathbf{H}(\mathrm{div};\Omega_f)$, and since Ω_f is not connected, we first need to perform a suitable extension of $\boldsymbol{\tau}_f$ to the domain $\Omega \,:=\, \Omega_s \cup \Sigma \cap \Omega_f$. To this end, we now let $v \in H^1(\Omega_s)$ be the unique solution of the Neumann problem:

$$\Delta\,v \,=\, -\,\frac{\langle \boldsymbol{\tau}_f \cdot \boldsymbol{\nu}, 1\rangle_\Sigma}{|\Omega_s|} \quad \text{in} \quad \Omega_s\,, \quad \frac{\partial v}{\partial \boldsymbol{\nu}} \,=\, \boldsymbol{\tau}_f \cdot \boldsymbol{\nu} \quad \text{on} \quad \Sigma\,, \quad \int_{\Omega_s} v \,=\, 0\,.$$

The unique solvability of the above problem is guaranteed by the Lax-Milgram Lemma, whose corresponding continuous dependence result establishes that

$$\|v\|_{1,\Omega_s} \,\le\, c\,\|\boldsymbol{\tau}_f \cdot \boldsymbol{\nu}\|_{-1/2,\Sigma}\,. \tag{3.33}$$

Then we define

$$\widetilde{\boldsymbol{\tau}} := \begin{cases} \boldsymbol{\tau}_f & \text{in} \quad \Omega_f\,, \\ \nabla v & \text{in} \quad \Omega_s\,, \end{cases}$$

which clearly belongs to $\mathbf{H}(\mathrm{div};\Omega)$, and observe, using (3.33), that

$$\|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{div};\Omega} \,\leq\, \|\boldsymbol{\tau}_f\|_{\mathrm{div};\Omega_f} \,+\, \|\nabla v\|_{\mathrm{div};\Omega_s} \,\leq\, \|\boldsymbol{\tau}_f\|_{\mathrm{div};\Omega_f} \,+\, \widetilde{c}\,\|\boldsymbol{\tau}_f\cdot\boldsymbol{\nu}\|_{-1/2,\Sigma} \,\leq\, C\,\|\boldsymbol{\tau}_f\|_{\mathrm{div};\Omega_f}\,.$$

In this way, proceeding as in the first part of the present proof, but now applied to $\widetilde{\boldsymbol{\tau}} \in \mathbf{H}(\mathrm{div};\Omega)$, we deduce the existence of $\widetilde{\mathbf{w}} \in \mathbf{H}^1(\Omega)$ and $\widetilde{\phi} \in H^1(\Omega)$, with $\int_\Omega \tilde{\phi} = 0$, such that $\widetilde{\boldsymbol{\tau}} = \widetilde{\mathbf{w}} + \mathbf{curl}(\widetilde{\phi})$ in Ω and

$$\|\widetilde{\mathbf{w}}\|_{1,\Omega} \,+\, \|\widetilde{\phi}\|_{1,\Omega} \,\leq\, C\,\|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{div};\Omega} \,\leq\, C\,\|\boldsymbol{\tau}_f\|_{\mathrm{div};\Omega_f}\,.$$

The proof is completed by defining $\mathbf{w}_f := \widetilde{\mathbf{w}}|_{\Omega_f}$ and $\phi_f := \widetilde{\phi}|_{\Omega_f}$. $\square$

Finally, the following lemma provides a couple of identities involving the differential operators from (3.2) - (3.3), which will be employed below.

Lemma 3.5. *Let $\mathcal{O}$ be a bounded domain with Lipschitz-continuous boundary $\partial\mathcal{O}$. Then there hold*

$$\langle \mathbf{curl}(\chi)\cdot\boldsymbol{\nu}, \varphi\rangle_{\partial\mathcal{O}} \,=\, -\left\langle \frac{d\varphi}{d\mathbf{s}}, \chi\right\rangle_{\partial\mathcal{O}} \qquad \forall\,\chi,\,\varphi \,\in\, H^1(\mathcal{O})\,, \tag{3.34}$$

and

$$\langle \underline{\mathbf{curl}}\,\boldsymbol{\chi}\,\boldsymbol{\nu}, \boldsymbol{\varphi}\rangle_{\partial\mathcal{O}} \,=\, -\left\langle \frac{d\boldsymbol{\varphi}}{d\mathbf{s}}, \boldsymbol{\chi}\right\rangle_{\partial\mathcal{O}} \qquad \forall\,\boldsymbol{\chi},\,\boldsymbol{\varphi} \,\in\, \mathbf{H}^1(\mathcal{O})\,. \tag{3.35}$$

Proof. We first recall from [23, eq. (2.17) and Theorem 2.11] that the Green formulae in $\mathbf{H}(\mathrm{div};\mathcal{O})$ and $\mathbf{H}(\mathrm{rot};\mathcal{O})$ establish, respectively, that

$$\int_{\mathcal{O}} \phi\,\mathrm{div}\,\boldsymbol{\tau} \,+\, \int_{\mathcal{O}} \boldsymbol{\tau}\cdot\nabla\phi \,=\, \langle\boldsymbol{\tau}\cdot\boldsymbol{\nu}, \phi\rangle_{\partial\mathcal{O}} \qquad \forall\,\boldsymbol{\tau}\in\mathbf{H}(\mathrm{div};\mathcal{O}),\ \forall\,\phi\in H^1(\mathcal{O})\,, \tag{3.36}$$

$$\int_{\mathcal{O}} \phi\,\mathrm{rot}\,\boldsymbol{\tau} \,-\, \int_{\mathcal{O}} \boldsymbol{\tau}\cdot\mathbf{curl}(\phi) \,=\, \langle\boldsymbol{\tau}\cdot\mathbf{s}, \phi\rangle_{\partial\mathcal{O}} \qquad \forall\,\boldsymbol{\tau}\in\mathbf{H}(\mathrm{rot};\mathcal{O}),\ \forall\,\phi\in H^1(\mathcal{O})\,, \tag{3.37}$$

where $\mathbf{H}(\mathrm{div};\mathcal{O})$ is given by (1.1) and

$$\mathbf{H}(\mathrm{rot};\mathcal{O}) \,:=\, \big\{\boldsymbol{\tau}\in\mathbf{L}^2(\mathcal{O}):\quad \mathrm{rot}\,\boldsymbol{\tau}\in L^2(\mathcal{O})\big\}\,. \tag{3.38}$$

Then, given now $\chi,\,\varphi \,\in\, H^1(\mathcal{O})$, we first apply (3.36) with $\boldsymbol{\tau} = \mathbf{curl}(\chi) \in \mathbf{H}(\mathrm{div};\mathcal{O})$ and $\phi = \varphi \in H^1(\Omega)$, and then employ (3.37) with $\boldsymbol{\tau} = \nabla\varphi \in \mathbf{H}(\mathrm{rot};\mathcal{O})$ and $\phi = \chi \in H^1(\Omega)$, to obtain

$$\begin{aligned}\langle \mathbf{curl}(\chi)\cdot\boldsymbol{\nu}, \varphi\rangle_{\partial\mathcal{O}} &= \int_{\mathcal{O}} \mathbf{curl}(\chi)\cdot\nabla\varphi = \int_{\mathcal{O}} \chi\,\mathrm{rot}(\nabla\varphi) \,-\, \langle\nabla\varphi\cdot\mathbf{s}, \chi\rangle_{\partial\mathcal{O}} \\ &= -\,\langle\nabla\varphi\cdot\mathbf{s}, \chi\rangle_{\partial\mathcal{O}} = -\left\langle\frac{d\varphi}{d\mathbf{s}}, \chi\right\rangle_{\partial\mathcal{O}}\,,\end{aligned}$$

which proves (3.34). The proof of (3.35) uses (3.36) and (3.37) along rows, and hence, being similar to (3.34), is omitted. $\square$

3.2.2. Estimating $\|E_1\|$

Given $\boldsymbol{\tau}_s \in \mathbb{H}(\mathbf{div};\Omega_s)$, we use (3.21) to estimate $E_1(\boldsymbol{\tau}_s) = E_1\big(\boldsymbol{\tau}_s - \boldsymbol{\tau}_{s,h}\big)$ with a suitable chosen $\boldsymbol{\tau}_{s,h} \in \mathbb{H}_h^s$. More precisely, as suggested by the Helmholtz decomposition for $\boldsymbol{\tau}_s$ provided by Lemma 3.4, that is $\boldsymbol{\tau}_s = \boldsymbol{\zeta}_s + \underline{\mathbf{curl}}(\boldsymbol{\chi}_s)$, with $\boldsymbol{\zeta}_s \in \mathbb{H}^1(\Omega_s)$ and $\boldsymbol{\chi}_s \in \mathbf{H}^1(\Omega_s)$, we consider in what follows

$$\boldsymbol{\chi}_{s,h} := \mathbf{I}_{s,h}(\boldsymbol{\chi}_s) \in \mathbf{X}_{s,h} \quad \text{and} \quad \boldsymbol{\tau}_{s,h} := \Pi_h^s(\boldsymbol{\zeta}_s) + \underline{\mathbf{curl}}(\boldsymbol{\chi}_{s,h}) \in \mathbb{RT}_h^s \subseteq \mathbb{H}_h^s,$$

which yields

$$\boldsymbol{\tau}_s - \boldsymbol{\tau}_{s,h} = \boldsymbol{\zeta}_s - \Pi_h^s(\boldsymbol{\zeta}_s) + \underline{\mathbf{curl}}\big(\boldsymbol{\chi}_s - \boldsymbol{\chi}_{s,h}\big)\,.$$

In particular, using (3.24) and (3.31) we find from the above identity that

$$\mathbf{div}\big(\boldsymbol{\tau}_s - \boldsymbol{\tau}_{s,h}\big) = \big(\mathbf{I} - \mathcal{P}_h^s\big)(\mathbf{div}\,\boldsymbol{\zeta}_s) = \big(\mathbf{I} - \mathcal{P}_h^s\big)(\mathbf{div}\,\boldsymbol{\tau}_s)\,,$$

and hence, according to the definition of E_1 (cf. (3.14)), we find that

$$E_1(\boldsymbol{\tau}_s - \boldsymbol{\tau}_{s,h}) = E_{11}(\boldsymbol{\tau}_s) + E_{12}(\boldsymbol{\zeta}_s) + E_{13}(\boldsymbol{\chi}_s)\,,$$

where

$$E_{11}(\boldsymbol{\tau}_s) = \frac{1}{\kappa_s^2}\int_{\Omega_s}\big\{\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_{s,h}\big\}(\mathbf{I} - \mathcal{P}_h^s)(\mathbf{div}\,\boldsymbol{\tau}_s) = \frac{1}{\kappa_s^2}\int_{\Omega_s}(\mathbf{I} - \mathcal{P}_h^s)(\mathbf{f})\cdot(\mathbf{div}\,\boldsymbol{\tau}_s)\,,$$

$$E_{12}(\boldsymbol{\zeta}_s) = -\int_{\Omega_s}\big\{\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h\big\} : (\boldsymbol{\zeta}_s - \Pi_h^s(\boldsymbol{\zeta}_s)) + \langle(\boldsymbol{\zeta}_s - \Pi_h^s(\boldsymbol{\zeta}_s))\boldsymbol{\nu}, \boldsymbol{\varphi}_{s,h}\rangle_\Sigma\,,$$

$$E_{13}(\boldsymbol{\chi}_s) = -\int_{\Omega_s}\big\{\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h\big\} : \underline{\mathbf{curl}}(\boldsymbol{\chi}_s - \boldsymbol{\chi}_{s,h}) + \langle\underline{\mathbf{curl}}(\boldsymbol{\chi}_s - \boldsymbol{\chi}_{s,h})\boldsymbol{\nu}, \boldsymbol{\varphi}_{s,h}\rangle_\Sigma\,.$$

Note that the second expression defining $E_{11}(\boldsymbol{\tau}_s)$ follows from the fact that $\mathcal{P}_h^s$ is self-adjoint and that, according to the definitions of $\mathbb{H}_h^s$ (cf. (2.10)) and $\mathbf{U}_h^s$ (cf. (2.18)), there holds $\mathbf{div}\big(\mathbb{H}_h^s\big) \subseteq \mathbf{U}_h^s$, whence $\big(\mathbf{I} - \mathcal{P}_h^s\big)(\mathbf{div}\,\boldsymbol{\sigma}_{s,h}) = \mathbf{0}$.

The following three lemmata provide the upper bounds for $E_{11}(\boldsymbol{\tau}_s)$, $E_{12}(\boldsymbol{\zeta}_s)$, and $E_{13}(\boldsymbol{\chi}_s)$.

Lemma 3.6. *There holds*

$$|E_{11}(\boldsymbol{\tau}_s)| \le \frac{1}{\kappa_s^2}\left\{\sum_{T\in\mathcal{T}_h^s}\|(\mathbf{I} - \mathcal{P}_h^s)\,\mathbf{f}\,\|_{0,T}^2\right\}^{1/2}\|\mathbf{div}\,\boldsymbol{\tau}_s\|_{0,\Omega_s}\,.$$

Proof. It follows from a straightforward application of the Cauchy-Schwarz inequality. □

Lemma 3.7. *There exists $C > 0$, independent of μ, λ, and κ_s, such that*

$$|E_{12}(\boldsymbol{\zeta}_s)| \le C\left\{\sum_{T\in\mathcal{T}_h^s} h_T^2\,\|\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h(\Sigma)} h_e\,\|\boldsymbol{\varphi}_{s,h} - \mathbf{u}_h\|_{0,e}^2\right\}^{1/2}\|\mathbf{div}\,\boldsymbol{\tau}_s\|_{0,\Omega_s}\,.$$

Proof. The present estimate was actually proved in [16, Lemma 5]. For sake of completeness we provide here the main aspects of the corresponding proof. We first observe, thanks to the fact that $\boldsymbol{\zeta}_s$ belongs to $\mathbb{H}^1(\Omega_s)$, that $(\boldsymbol{\zeta}_s - \Pi_h^s(\boldsymbol{\zeta}_s))\,\boldsymbol{\nu}|_\Sigma \in \mathbf{L}^2(\Sigma)$, and hence

$$\langle(\boldsymbol{\zeta}_s - \Pi_h^s(\boldsymbol{\zeta}_s))\boldsymbol{\nu}, \boldsymbol{\varphi}_{s,h}\rangle_\Sigma = \sum_{e\in\mathcal{E}_h(\Sigma)}\int_e \boldsymbol{\varphi}_{s,h}\cdot(\boldsymbol{\zeta}_s - \Pi_h^s(\boldsymbol{\zeta}_s))\,\boldsymbol{\nu}\,. \tag{3.39}$$

Next, it is clear from (3.7) that $\mathbf{u}_h \in \mathbf{U}_h^s$, which means, in particular, that for each $e \in \mathcal{E}_h(\Sigma)$ there holds $\mathbf{u}_h|_e \in \mathbf{P}_0(e)$, and therefore the identity (3.22) yields

$$\sum_{e\in\mathcal{E}_h(\Sigma)} \int_e \mathbf{u}_h \cdot (\boldsymbol{\zeta}_s - \Pi_h^s(\boldsymbol{\zeta}_s))\,\boldsymbol{\nu} \,=\, 0\,.$$

Thus, by introducing the above null expression in the right hand side of (3.39), and then re-incorporating the resulting equation in the definition of E_{12}, we find that

$$\begin{aligned} E_{12}(\boldsymbol{\zeta}_s) \,=\, &- \sum_{T\in\mathcal{T}_h^s} \int_T \left\{\mathcal{C}^{-1}\,\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h\right\} : (\boldsymbol{\zeta}_s - \Pi_h^s(\boldsymbol{\zeta}_s)) \\ &+ \sum_{e\in\mathcal{E}_h(\Sigma)} \int_e \left(\boldsymbol{\varphi}_{s,h} - \mathbf{u}_h\right) \cdot (\boldsymbol{\zeta}_s - \Pi_h^s(\boldsymbol{\zeta}_s))\,\boldsymbol{\nu}\,, \end{aligned}$$

where we have replaced the original integration $\displaystyle\int_{\Omega_s}$ by $\displaystyle\sum_{T\in\mathcal{T}_h^s}\int_T$. In this way, the rest of the proof reduces to apply the Cauchy-Schwarz inequality, the approximation properties (3.25) and (3.26), and finally the upper bound given by (3.32). We omit further details. □

Lemma 3.8. *There exists $C > 0$, independent of μ, λ and κ_s, such that*

$$\begin{aligned} |E_{13}(\boldsymbol{\chi}_s)| \,\le\, C\, \Bigg\{ &\sum_{T\in\mathcal{T}_h^s} h_T^2\,\|\mathrm{curl}(\mathcal{C}^{-1}\,\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h)\|_{0,T}^2 \,+ \sum_{e\in\mathcal{E}_h(\Omega_s)} h_e\,\|[(\mathcal{C}^{-1}\,\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h)\,\mathbf{s}]\|_{0,e}^2 \\ &+ \sum_{e\in\mathcal{E}_h(\Sigma)} h_e\,\left\|(\mathcal{C}^{-1}\,\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h)\,\mathbf{s} - \frac{d\boldsymbol{\varphi}_{s,h}}{d\mathbf{s}}\right\|_{0,e}^2 \Bigg\}^{1/2} \|\boldsymbol{\tau}_s\|_{\mathbf{div};\Omega_s}\,. \end{aligned}$$

Proof. While this result is also available in several places (see, e.g. [16, Lemma 6]), here we proceed similarly as for the previous lemma and provide an sketch of its proof. Indeed, we first note that $\boldsymbol{\varphi}_{s,h}$, being a continuous piecewise linear function on Σ, is the trace of a function in $\mathbf{H}^1(\Omega_s)$, say denoted again by $\boldsymbol{\varphi}_{s,h}$. Then, a straightforward application of (3.35) yields

$$\langle \underline{\mathbf{curl}}(\boldsymbol{\chi}_s - \boldsymbol{\chi}_{s,h})\,\boldsymbol{\nu}, \boldsymbol{\varphi}_{s,h}\rangle_\Sigma \,=\, -\left\langle \frac{d\boldsymbol{\varphi}_{s,h}}{d\mathbf{s}}, \boldsymbol{\chi}_s - \boldsymbol{\chi}_{s,h}\right\rangle_\Sigma \,=\, \int_\Sigma \frac{d\boldsymbol{\varphi}_{s,h}}{d\mathbf{s}} \cdot (\boldsymbol{\chi}_s - \boldsymbol{\chi}_{s,h})\,, \qquad (3.40)$$

where the last equality makes use of the fact that $\frac{d\boldsymbol{\varphi}_{s,h}}{d\mathbf{s}} \in \mathbf{L}^2(\Sigma)$.

In turn, integrating by parts on each $T \in \mathcal{T}_h^s$, we obtain that

$$\begin{aligned} &- \int_{\Omega_s} \left\{\mathcal{C}^{-1}\,\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h\right\} : \underline{\mathbf{curl}}(\boldsymbol{\chi}_s - \boldsymbol{\chi}_{s,h}) \\ &= - \sum_{T\in\mathcal{T}_h^s} \int_T \left\{\mathcal{C}^{-1}\,\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h\right\} : \underline{\mathbf{curl}}(\boldsymbol{\chi}_s - \boldsymbol{\chi}_{s,h}) \end{aligned}$$

$$= \sum_{T\in\mathcal{T}_h^s} \left\{ - \int_T \operatorname{curl}\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h\big)\cdot(\boldsymbol{\chi}_s - \boldsymbol{\chi}_{s,h}) + \int_{\partial T} \big(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h\big)\,\mathbf{s}\cdot(\boldsymbol{\chi}_s - \boldsymbol{\chi}_{s,h}) \right\}$$

$$= - \sum_{T\in\mathcal{T}_h^s} \int_T \operatorname{curl}\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h\big)\cdot(\boldsymbol{\chi}_s - \boldsymbol{\chi}_{s,h})$$

$$+ \sum_{e\in\mathcal{E}_h(\Omega_s)} \int_e \big[\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h\big)\,\mathbf{s}\big]\cdot(\boldsymbol{\chi}_s - \boldsymbol{\chi}_{s,h})$$

$$+ \sum_{e\in\mathcal{E}_h(\Sigma)} \int_e \big(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h\big)\,\mathbf{s}\cdot(\boldsymbol{\chi}_s - \boldsymbol{\chi}_{s,h}),$$

which, together with (3.40), yields

$$\begin{aligned} E_{13}(\boldsymbol{\chi}_s) = & - \sum_{T\in\mathcal{T}_h^s} \int_T \operatorname{curl}\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h\big)\cdot(\boldsymbol{\chi}_s - \boldsymbol{\chi}_{s,h}) \\ & + \sum_{e\in\mathcal{E}_h(\Omega_s)} \int_e \big[\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h\big)\,\mathbf{s}\big]\cdot(\boldsymbol{\chi}_s - \boldsymbol{\chi}_{s,h}) \\ & + \sum_{e\in\mathcal{E}_h(\Sigma)} \int_e \left\{ \big(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h\big)\,\mathbf{s} - \frac{d\boldsymbol{\varphi}_{s,h}}{d\mathbf{s}} \right\}\cdot(\boldsymbol{\chi}_s - \boldsymbol{\chi}_{s,h}). \end{aligned}$$

In this way, and recalling that $\boldsymbol{\chi}_{s,h} = \mathbf{I}_{s,h}(\boldsymbol{\chi}_s)$, the rest of the proof follows from obvious applications of the Cauchy-Schwarz inequality and the approximation properties of the Clément interpolation operator $\mathbf{I}_{s,h}$ (cf. Lemma 3.3), taking into account as well that the number of elements in $\Delta(T)$ and $\Delta(e)$ are bounded and that $\|\boldsymbol{\chi}_s\|_{1,\Omega_s} \le C_s\,\|\boldsymbol{\tau}_s\|_{\mathbf{div};\Omega_s}$ (cf. (3.29)). Further details are omitted. $\square$

As a direct consequence of Lemmata 3.6, 3.7, and 3.8, the norm of the functional E_1 (cf. (3.14)) is estimated as follows.

Lemma 3.9. *There exists $C > 0$, independent of μ, λ and κ_s, such that*

$$\begin{aligned} \|E_1\| \le C \Bigg\{ & \frac{1}{\kappa_s^4} \sum_{T\in\mathcal{T}_h^s} \|(\mathbf{I} - \mathcal{P}_h^s)\,\mathbf{f}\,\|_{0,T}^2 + \sum_{T\in\mathcal{T}_h^s} h_T^2\,\|\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h\|_{0,T}^2 \\ & + \sum_{e\in\mathcal{E}_h(\Sigma)} h_e\,\|\boldsymbol{\varphi}_{s,h} - \mathbf{u}_h\|_{0,e}^2 + \sum_{T\in\mathcal{T}_h^s} h_T^2\,\|\operatorname{curl}(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h)\|_{0,T}^2 \\ & + \sum_{e\in\mathcal{E}_h(\Omega_s)} h_e\,\|[(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h)\,\mathbf{s}]\|_{0,e}^2 + \sum_{e\in\mathcal{E}_h(\Sigma)} h_e\,\left\|\big(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h\big)\,\mathbf{s} - \frac{d\boldsymbol{\varphi}_{s,h}}{d\mathbf{s}}\right\|_{0,e}^2 \Bigg\}^{1/2}. \end{aligned}$$

3.2.3. Estimating $\|E_2\|$

We proceed analogously to the case of $\|E_1\|$. This means that, given $\boldsymbol{\tau}_f \in \mathbf{H}(\mathrm{div};\Omega_f)$, we consider from Lemma 3.4 its Helmholtz decomposition $\boldsymbol{\tau}_f = \mathbf{w}_f + \mathbf{curl}\,\phi_f$ in Ω_f, with $\mathbf{w}_f \in \mathbf{H}^1(\Omega_f)$ and $\phi_f \in H^1(\Omega_f)$, and define

$$\phi_{f,h} := I_{f,h}(\phi_f) \quad \text{and} \quad \boldsymbol{\tau}_{f,h} := \Pi_h^f(\mathbf{w}_f) + \mathbf{curl}(\phi_{f,h}),$$

so that, using the second equality in (3.21), we can write $E_2(\boldsymbol{\tau}_f) = E_2\big(\boldsymbol{\tau}_f - \boldsymbol{\tau}_{f,h}\big)$. It follows that

$$\boldsymbol{\tau}_f - \boldsymbol{\tau}_{f,h} = \mathbf{w}_f - \Pi_h^f(\mathbf{w}_f) + \mathbf{curl}(\phi_f - \phi_{f,h}),$$

from which, employing the second identity in (3.24), and noting from the definitions (2.11) and (2.19) that $\operatorname{div}\boldsymbol{\sigma}_{f,h} \in U_h^f$, we find that

$$\int_{\Omega_f} \operatorname{div}\boldsymbol{\sigma}_{f,h} \operatorname{div}\big(\boldsymbol{\tau}_f - \boldsymbol{\tau}_{f,h}\big) = \int_{\Omega_f} \operatorname{div}\boldsymbol{\sigma}_{f,h}\, \big(\mathbf{I} - \mathcal{P}_h^f\big)(\operatorname{div}\mathbf{w}_f) = 0.$$

Hence, according to (3.15) and the above computation, we get

$$E_2(\boldsymbol{\tau}_f - \boldsymbol{\tau}_{f,h}) = E_{21}(\mathbf{w}_f) + E_{22}(\phi_f),$$

where

$$\begin{aligned} E_{21}(\mathbf{w}_f) &:= -\int_{\Omega_f} \boldsymbol{\sigma}_{f,h} \cdot \big(\mathbf{w}_f - \Pi_h^f(\mathbf{w}_f)\big) - \langle\big(\mathbf{w}_f - \Pi_h^f(\mathbf{w}_f)\big)\cdot\boldsymbol{\nu}, \varphi_{\Sigma,h}\rangle_\Sigma \\ &\quad + \langle\big(\mathbf{w}_f - \Pi_h^f(\mathbf{w}_f)\big)\cdot\boldsymbol{\nu}, \varphi_{\Gamma,h}\rangle_\Gamma \\ E_{22}(\phi_f) &:= -\int_{\Omega_f} \boldsymbol{\sigma}_{f,h} \cdot \mathbf{curl}(\phi_f - \phi_{f,h}) - \langle \mathbf{curl}(\phi_f - \phi_{f,h})\cdot\boldsymbol{\nu}, \varphi_{\Sigma,h}\rangle_\Sigma \\ &\quad + \langle \mathbf{curl}(\phi_f - \phi_{f,h})\cdot\boldsymbol{\nu}, \varphi_{\Gamma,h}\rangle_\Gamma. \end{aligned} \tag{3.41}$$

The following two lemmata establish the upper bounds for $|E_{21}(\mathbf{w}_f)|$ and $|E_{22}(\phi_f)|$.

Lemma 3.10. *There exists $C > 0$, independent of κ_f and h, such that*

$$\begin{aligned} |E_{21}(\mathbf{w}_f)| \le C \Bigg\{ &\sum_{T\in\mathcal{T}_h^f} h_T^2 \|\boldsymbol{\sigma}_{f,h}\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h(\Sigma)} h_e \|\varphi_{\Sigma,h} - p_h\|_{0,e}^2 \\ &+ \sum_{e\in\mathcal{E}_h(\Gamma)} h_e \|\varphi_{\Gamma,h} - p_h\|_{0,e}^2 \Bigg\} \|\boldsymbol{\tau}_f\|_{\operatorname{div};\Omega_f}. \end{aligned}$$

Proof. We proceed as in the proof of Lemma 3.7. Indeed, since $\mathbf{w}_f \in \mathbf{H}^1(\Omega_f)$ it is clear that

$$\big(\mathbf{w}_f - \Pi_h^f(\mathbf{w}_f)\big)\cdot\boldsymbol{\nu}|_\Sigma \in L^2(\Sigma) \quad \text{and} \quad \big(\mathbf{w}_f - \Pi_h^f(\mathbf{w}_f)\big)\cdot\boldsymbol{\nu}|_\Gamma \in L^2(\Gamma),$$

which, together with the fact that $p_h|_e \in P_0(e)\ \ \forall\, e \in \mathcal{E}_h(\Sigma)\cup\mathcal{E}_h(\Gamma)$ (cf. (3.7) and (2.11)), and thanks to the characterization property (3.23), allow to show that

$$\langle\big(\mathbf{w}_f - \Pi_h^f(\mathbf{w}_f)\big)\cdot\boldsymbol{\nu}, \varphi_{\Sigma,h}\rangle_\Sigma = \sum_{e\in\mathcal{E}_h(\Sigma)} \int_e (\varphi_{\Sigma,h} - p_h)\,\big(\mathbf{w}_f - \Pi_h^f(\mathbf{w}_f)\big)\cdot\boldsymbol{\nu}$$

and

$$\langle\big(\mathbf{w}_f - \Pi_h^f(\mathbf{w}_f)\big)\cdot\boldsymbol{\nu}, \varphi_{\Gamma,h}\rangle_\Gamma = \sum_{e\in\mathcal{E}_h(\Gamma)} \int_e (\varphi_{\Gamma,h} - p_h)\,\big(\mathbf{w}_f - \Pi_h^f(\mathbf{w}_f)\big)\cdot\boldsymbol{\nu}.$$

In this way, we find that

$$\begin{aligned} E_{21}(\mathbf{w}_f) := &-\sum_{T\in\mathcal{T}_h^f} \int_T \boldsymbol{\sigma}_{f,h}\cdot\big(\mathbf{w}_f - \Pi_h^f(\mathbf{w}_f)\big) - \sum_{e\in\mathcal{E}_h(\Sigma)} \int_e (\varphi_{\Sigma,h} - p_h)\,\big(\mathbf{w}_f - \Pi_h^f(\mathbf{w}_f)\big)\cdot\boldsymbol{\nu} \\ &+ \sum_{e\in\mathcal{E}_h(\Gamma)} \int_e (\varphi_{\Gamma,h} - p_h)\,\big(\mathbf{w}_f - \Pi_h^f(\mathbf{w}_f)\big)\cdot\boldsymbol{\nu}, \end{aligned}$$

and hence, the proof is completed by applying the Cauchy-Schwarz inequality, the approximation properties (3.27) and (3.28), and the fact that $\|\mathbf{w}_f\|_{1,\Omega_f} \leq C_f \,\|\boldsymbol{\tau}_f\|_{\mathrm{div};\Omega_f}$ (cf. (3.30)). We omit further details. $\square$

Lemma 3.11. *There exists $C > 0$, independent of κ_f and h, such that*

$$|E_{22}(\phi_f)| \,\leq\, C \left\{ \sum_{T\in\mathcal{T}_h^f} h_T^2 \,\|\mathrm{rot}(\boldsymbol{\sigma}_{f,h})\|_{0,T}^2 \,+ \sum_{e\in\mathcal{E}_h(\Omega_f)} h_e \,\|[\boldsymbol{\sigma}_{f,h}\cdot\mathbf{s}]\|_{0,e}^2 \right.$$
$$\left. + \sum_{e\in\mathcal{E}_h(\Sigma)} h_e \left\|\boldsymbol{\sigma}_{f,h}\cdot\mathbf{s} - \frac{d\,\varphi_{\Sigma,h}}{d\mathbf{s}}\right\|_{0,e}^2 + \sum_{e\in\mathcal{E}_h(\Gamma)} h_e \left\|\boldsymbol{\sigma}_{f,h}\cdot\mathbf{s} - \frac{d\,\varphi_{\Gamma,h}}{d\mathbf{s}}\right\|_{0,e}^2 \right\}^{1/2} \|\boldsymbol{\tau}_f\|_{\mathrm{div};\Omega_f}.$$

Proof. The analysis here is analogous to the proof of Lemma 3.8. In fact, by applying now the identity (3.34) to the boundary terms defining E_{22} (cf. (3.41), that is to $\langle \mathbf{curl}(\phi_f - \phi_{f,h})\cdot \boldsymbol{\nu}, \varphi_{\Sigma,h}\rangle_\Sigma$ and $\langle \mathbf{curl}(\phi_f - \phi_{f,h})\cdot\boldsymbol{\nu}, \varphi_{\Gamma,h}\rangle_\Gamma$, and integrating by parts on each $T\in\mathcal{T}_h^f$, we find that

$$E_{22}(\phi_f) \,=\, -\sum_{T\in\mathcal{T}_h^f}\int_T \mathrm{rot}(\boldsymbol{\sigma}_{f,h})\,(\phi_f - \phi_{f,h}) \,+ \sum_{e\in\mathcal{E}_h(\Omega_f)}\int_e [\boldsymbol{\sigma}_{f,h}\cdot\mathbf{s}](\phi_f - \phi_{f,h})$$
$$- \sum_{e\in\mathcal{E}_h(\Sigma)}\int_e \Big(\boldsymbol{\sigma}_{f,h}\cdot\mathbf{s} - \frac{d\,\varphi_{\Sigma,h}}{d\mathbf{s}}\Big)\,(\phi_f - \phi_{f,h}) \,+ \sum_{e\in\mathcal{E}_h(\Gamma)}\int_e \Big(-\boldsymbol{\sigma}_{f,h}\cdot\mathbf{s} + \frac{d\,\varphi_{\Gamma,h}}{d\mathbf{s}}\Big)(\phi_f - \phi_{f,h})\,,$$

where we have also employed that $\frac{d\varphi_{\Sigma,h}}{d\mathbf{s}} \in L^2(\Sigma)$ and $\frac{d\varphi_{\Gamma,h}}{d\mathbf{s}} \in L^2(\Gamma)$. Consequently, and similarly as for Lemma 3.8, the rest of the proof follows from straightforward applications of the Cauchy-Schwarz inequality, the approximation properties of the Clément interpolator $\phi_{f,h} := I_{f,h}(\phi_f)$ (cf. Lemma 3.3), the fact that the cardinalities of $\Delta(T)$ and $\Delta(e)$ are bounded, and the upper bound $\|\phi_f\|_{1,\Omega_f} \leq C_f\,\|\boldsymbol{\tau}_f\|_{\mathrm{div};\Omega_f}$ (cf. (3.30)). We omit further details. $\square$

The norm of E_2 (cf. (3.15) is bounded now as a consequence of Lemmata 3.10 and 3.11.

Lemma 3.12. *There exists $C > 0$, independent of κ_f and h, such that*

$$\|E_2\| \,\leq\, C \left\{ \sum_{T\in\mathcal{T}_h^f} h_T^2\,\|\boldsymbol{\sigma}_{f,h}\|_{0,T}^2 \,+ \sum_{e\in\mathcal{E}_h(\Sigma)} h_e\,\|\varphi_{\Sigma,h} - p_h\|_{0,e}^2 \right.$$
$$+ \sum_{e\in\mathcal{E}_h(\Gamma)} h_e\,\|\varphi_{\Gamma,h} - p_h\|_{0,e}^2 \,+ \sum_{T\in\mathcal{T}_h^f} h_T^2\,\|\mathrm{rot}(\boldsymbol{\sigma}_{f,h})\|_{0,T}^2 \,+ \sum_{e\in\mathcal{E}_h(\Omega_f)} h_e\,\|[\boldsymbol{\sigma}_{f,h}\cdot\mathbf{s}]\|_{0,e}^2$$
$$\left. + \sum_{e\in\mathcal{E}_h(\Sigma)} h_e \left\|\boldsymbol{\sigma}_{f,h}\cdot\mathbf{s} - \frac{d\,\varphi_{\Sigma,h}}{d\mathbf{s}}\right\|_{0,e}^2 + \sum_{e\in\mathcal{E}_h(\Gamma)} h_e \left\|\boldsymbol{\sigma}_{f,h}\cdot\mathbf{s} - \frac{d\,\varphi_{\Gamma,h}}{d\mathbf{s}}\right. \right\}^{1/2}.$$

We end this section by observing that the reliability estimate (cf. Theorem 3.1) is a direct consequence of (3.11) and (3.12), together with Lemmata 3.1, 3.2, 3.9, and 3.12.

3.3. Efficiency of the a posteriori error estimator

In this section we prove the efficiency of our a posteriori error estimator $\boldsymbol{\theta}$ (lower bound in (3.10)). We begin with the first two terms defining $\theta_{T,s}^2$ (cf. (3.4)). In fact, since $\boldsymbol{\sigma}_s$ is symmetric in Ω_s, we easily notice, adding and substracting $\boldsymbol{\sigma}_s$, that there holds

$$\|\boldsymbol{\sigma}_{s,h} - \boldsymbol{\sigma}_{s,h}^{\mathtt{t}}\|_{0,T}^2 \,\leq\, 4\,\|\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{s,h}\|_{0,T}^2\,. \tag{3.42}$$

Next, according to the definitions of $\mathbf{u}$ (cf. (2.6)) and $\mathbf{u}_h$ (cf. (3.7)), we find that

$$\|\big(\mathbf{I} - \mathcal{P}_h^s\big)\mathbf{f}\|_{0,T}^2 \,\leq\, 2\,\kappa_s^4\,\|\mathbf{u} - \mathbf{u}_h\|_{0,T}^2 \,+\, 2\,\|\mathbf{div}(\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{s,h})\|_{0,T}^2\,. \tag{3.43}$$

Throughout the rest of the section we provide the corresponding upper bounds for the terms in (3.4), (3.5), (3.6), and (3.8) that involve the mesh parameters h_T and h_e. Actually, most of these estimates are already available in the literature (see, e.g. [10], [11], [16], and [19]), but for sake of completeness we sketch here some of their proofs, which employ the localization technique based on triangle-bubble and edge-bubble functions, together with extension operators, discrete trace and inverse inequalities, and certainly the original identities recovered by Theorem 2.2. To this end, we now introduce further notations and preliminary results. Given $T \in \mathcal{T}_h^s \cup \mathcal{T}_h^f$ and $e \in \mathcal{E}(T)$, we let ψ_T and ψ_e be the usual triangle-bubble and edge-bubble functions, respectively (see [29, eqs. (1.5) and (1.6)]), which satisfy:

i) $\psi_T \in P_3(T)$, $\psi_T = 0$ on ∂T, $\operatorname{supp}(\psi_T) \subseteq T$, and $0 \leq \psi_T \leq 1$ in T.

ii) $\psi_e|_T \in P_2(T)$, $\psi_e = 0$ on $\partial T \setminus e$, $\operatorname{supp}(\psi_e) \subseteq w_e := \cup\{T' \in \mathcal{T}_h^s \cup \mathcal{T}_h^f : \quad e \in \mathcal{E}(T')\}$, and $0 \leq \psi_e \leq 1$ in w_e.

We also recall from [28] that, given $k \in \mathbb{N}\cup\{0\}$, there exists an extension operator $L : C(e) \to C(T)$ that satisfies $L(p) \in P_k(T)$ and $L(p)|_e = p$ for all $p \in P_k(e)$. Additional properties of ψ_T, ψ_e and L are collected in the following lemma.

Lemma 3.13. *Given $k \in \mathbb{N} \cup \{0\}$, there exist positive constants c_1, c_2 and c_3, depending only on k and the shape regularity of the triangulations (minimun angle condition), such that for each $T \in \mathcal{T}_h^s \cup \mathcal{T}_h^f$ and $e \in \mathcal{E}(T)$, there hold*

$$\|q\|_{0,T}^2 \leq c_1 \|\psi_T^{1/2} q\|_{0,T}^2 \qquad \forall q \in P_k(T), \tag{3.44}$$

$$\|p\|_{0,e}^2 \leq c_2 \|\psi_e^{1/2} p\|_{0,e}^2 \qquad \forall p \in P_k(e), \tag{3.45}$$

$$\|\psi_e^{1/2} L(p)\|_{0,T}^2 \leq c_3\, h_e \|p\|_{0,e}^2 \qquad \forall p \in P_k(e). \tag{3.46}$$

Proof. See [28, Lemma 1.3]. □

The following inverse and discrete trace inequalities will also be used.

Lemma 3.14. *Let k, l, $m \in \mathbb{N} \cup \{0\}$ such that $l \leq m$. Then there exists $c > 0$, depending only on k, l, m and the shape regularity of the triangulations, such that for each $T \in \mathcal{T}_h^s \cup \mathcal{T}_h^f$ there holds*

$$|q|_{m,T} \,\leq\, c\, h_T^{l-m}\, |q|_{l,T} \qquad \forall\, q \in P_k(T)\,. \tag{3.47}$$

Proof. See [13, Theorem 3.2.6]. □

Lemma 3.15. *There exists $C > 0$, depending only on the shape regularity of the triangulations, such that for each $T \in \mathcal{T}_h^s \cup \mathcal{T}_h^f$ and $e \in \mathcal{E}(T)$, there holds*

$$\|v\|_{0,e}^2 \,\leq\, C\,\big\{h_e^{-1}\,\|v\|_{0,T}^2 \,+\, h_e\,|v|_{1,T}^2\big\} \quad \forall\, v \in H^1(T)\,. \tag{3.48}$$

Proof. See [1, Theorem 3.10] or [3, eq. (2.4)]. $\square$

The following three lemmas, whose proofs make use of the techniques and results described above, provide the upper bounds for the remaining terms defining $\theta^2_{T,s}$ (cf. (3.4)).

Lemma 3.16. *There exists $C > 0$, independent of h and λ, such that for each $T \in \mathcal{T}_h^s$ there holds*

$$h_T^2 \, \|\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h\|^2_{0,T} \,\leq\, C \left\{ \|\mathbf{u} - \mathbf{u}_h\|^2_{0,T} \,+\, h_T^2 \, \|\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{s,h}\|^2_{0,T} \,+\, h_T^2 \, \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|^2_{0,T} \right\}.$$

Proof. See [11, Lemma 6.6]. $\square$

Lemma 3.17. *There exists $C > 0$, independent of h and λ, such that for each $T \in \mathcal{T}_h^s$ there holds*

$$h_T^2 \, \|\mathrm{curl}\,(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h)\|^2_{0,T} \,\leq\, C \left\{ \|\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{s,h}\|^2_{0,T} \,+\, \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|^2_{0,T} \right\}.$$

Proof. See [11, Lemma 6.3] or [6, Lemma 4.7]. $\square$

Lemma 3.18. *There exists $C > 0$, independent of h and λ, such that for each $e \in \mathcal{E}_h(\Omega_s)$ there holds*

$$h_e \, \|[(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h)\mathbf{s}]\|^2_{0,e} \,\leq\, C \sum_{T \subseteq \omega_e} \left\{ \|\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{s,h}\|^2_{0,T} \,+\, \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|^2_{0,T} \right\},$$

where $\omega_e := \cup\big\{T' \in \mathcal{T}_h^s : e \in \mathcal{E}(T')\big\}$.

Proof. See [11, Lemma 6.4]. $\square$

At this point we remark that in the case of efficiency estimates involving jumps on an interior edge e, such as the ones provided by Lemmas 3.18 (above) and 3.21 (below), the extension operator $L : C(e) \to C(T)$ is applied from e to each one of the two triangles T forming ω_e. Equivalently, one defines an extension operator $L : C(e) \to C(\omega_e)$ for which there holds (3.46) with ω_e instead of T.

The analogue of the above three lemmas for the terms defining $\theta^2_{T,f}$ (cf. (3.5)) are stated next. Since it will be used in some of the forthcoming results, we now recall from the notations introduced in Section 1 that $\overline{z}$ denotes the conjugate of a given complex number z.

Lemma 3.19. *There exists $C > 0$, independent of h, such that for each $T \in \mathcal{T}_h^f$ there holds*

$$h_T^2 \, \|\boldsymbol{\sigma}_{f,h}\|^2_{0,T} \,\leq\, C \left\{ h_T^2 \, \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|^2_{0,T} \,+\, \|p - p_h\|^2_{0,T} \right\}.$$

Proof. It is a slight modification of [10, Lemma 6.3] (see also [19, Lemma 4.13]). In fact, given $T \in \mathcal{T}_h^f$, we apply (3.44), use that $\boldsymbol{\sigma}_f = \nabla p$ in Ω_f and $\nabla p_h = 0$ in T (which follows from the fact that p_h is piecewise constant in virtue of (2.11) and (3.7)), and then integrate by parts. In this way, we find that

$$\begin{aligned} \|\boldsymbol{\sigma}_{f,h}\|^2_{0,T} \,&\leq\, C \, \|\psi_T^{1/2} \, \boldsymbol{\sigma}_{f,h}\|^2_{0,T} \,=\, C \int_T \psi_T \, \overline{\boldsymbol{\sigma}_{f,h}} \cdot \Big\{ (\boldsymbol{\sigma}_{f,h} - \boldsymbol{\sigma}_f) - \nabla(p_h - p) \Big\} \\ &=\, C \left\{ \int_T \psi_T \, \overline{\boldsymbol{\sigma}_{f,h}} \cdot (\boldsymbol{\sigma}_{f,h} - \boldsymbol{\sigma}_f) \,+\, \int_T \mathrm{div}(\psi_T \, \overline{\boldsymbol{\sigma}_{f,h}}) \, (p - p_h) \right\}. \end{aligned}$$

Then, employing the Cauchy- Schwarz inequality, the inverse estimate (3.47) (cf. Lemma 3.14), and the fact that $0 \le \psi_T \le 1$, we get

$$\|\boldsymbol{\sigma}_{f,h}\|_{0,T} \,\le\, C\left\{\|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T}^2 \,+\, h_T^{-1}\,\|p - p_h\|_{0,T}^2\right\},$$

which implies the required bound and completes the proof. □

Lemma 3.20. *There exists $C > 0$, independent of h, such that for each $T \in \mathcal{T}_h^f$ there holds*

$$h_T^2\,\|\mathrm{rot}\,\boldsymbol{\sigma}_{f,h}\|_{0,T}^2 \,\le\, C\,\|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T}^2\,.$$

Proof. It basically follows from the general estimate provided by [6, Lemma 4.3]. Indeed, a row-wise interpretation of this result allows to show that, given a piecewise polynomial $\rho_h \in \mathbf{L}^2(\Omega_f)$ of degree $k \ge 0$ on each $T \in \mathcal{T}_h^f$, and $\rho \in \mathbf{L}^2(\Omega_f)$ such that $\mathrm{rot}\,\rho = 0$ in Ω_f, there exists $c > 0$, independent of h, such that

$$h_T\,\|\mathrm{rot}\,\rho_h\|_{0,T} \,\le\, c\,\|\rho - \rho_h\|_{0,T} \qquad \forall\, T \in \mathcal{T}_h^f\,. \tag{3.49}$$

Hence, since $\mathrm{rot}\,\boldsymbol{\sigma}_f = \mathrm{rot}(\nabla p) = 0$, it suffices to apply (3.49) to $\rho_h = \boldsymbol{\sigma}_{f,h}$ and $\rho = \boldsymbol{\sigma}_f$.

Lemma 3.21. *There exists $C > 0$, independent of h, such that for each $e \in \mathcal{E}_h(\Omega_f)$ there holds*

$$h_e\,\|[\boldsymbol{\sigma}_{f,h}\cdot\mathbf{s}]\|_{0,e}^2 \,\le\, C\,\|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,\omega_e}^2\,,$$

where $\omega_e := \cup\big\{T' \in \mathcal{T}_h^f : \quad e \in \mathcal{E}(T')\big\}$.

Proof. We first observe that a slight modification of the proof of [6, Lemma 4.4] allows to show that, under the same hypotheses leading to (3.49), that is given a piecewise polynomial $\rho_h \in \mathbf{L}^2(\Omega_f)$ of degree $k \ge 0$ on each $T \in \mathcal{T}_h^f$, and $\rho \in \mathbf{L}^2(\Omega_f)$ such that $\mathrm{rot}\,\rho = 0$ in Ω_f, there exists $c > 0$, independent of h, such that for each $e \in \mathcal{E}_h(\Omega_f)$ there holds

$$h_e\,\|[\rho_h\cdot\mathbf{s}]\|_{0,e}^2 \,\le\, c\,\|\rho - \rho_h\|_{0,\omega_e}^2\,. \tag{3.50}$$

Hence, the present proof is a straightforward application of (3.50) to $\rho_h = \boldsymbol{\sigma}_{f,h}$ and $\rho = \boldsymbol{\sigma}_f = \nabla p$.

We now aim to bound the first three terms defining $\theta_{e,\Sigma}^2$ (cf. (3.6)).

Lemma 3.22. *There exists $C > 0$, independent of h and λ, such that for each $e \in \mathcal{E}_h(\Sigma)$ there holds*

$$h_e\,\|\boldsymbol{\varphi}_{s,h} - \mathbf{u}_h\|_{0,e}^2 \,\le\, C\left\{\|\mathbf{u} - \mathbf{u}_h\|_{0,T}^2 + h_T^2\,\|\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{s,h}\|_{0,T}^2 + h_T^2\,\|\gamma - \gamma_h\|_{0,T}^2 + h_e\,\|\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}\|_{0,e}^2\right\},$$

where T is the triangle of $\mathcal{T}_h^s$ having e as an edge.

Proof. It is based mainly on the discrete trace inequality (3.48), the fact that $\nabla\mathbf{u} = \mathcal{C}^{-1}\boldsymbol{\sigma}_s + \boldsymbol{\gamma}$ in Ω_s, and the upper bound for $h_T^2\,\|\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h\|_{0,T}^2$ provided by Lemma 3.16. We omit further details and refer to [16, Lemma 22]. □

Lemma 3.23. *There exists $C > 0$, independent of h and λ, such that for each $e \in \mathcal{E}_h(\Sigma)$ there holds*

$$h_e \, \|\boldsymbol{\sigma}_{f,h} \cdot \boldsymbol{\nu} - \rho_f \, \omega^2 \, \boldsymbol{\varphi}_{s,h} \cdot \boldsymbol{\nu}\|_{0,e}^2$$

$$\leq C \left\{ \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T}^2 \, + \, h_T^2 \, \|\mathrm{div}(\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h})\|_{0,T}^2 \, + \, h_e \, \|\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}\|_{0,e}^2 \right\},$$

where T is the triangle of $\mathcal{T}_h^f$ having e as an edge.

Proof. We proceed similarly as in [5, Lemma 4.7] (see also [22, Lemma 3.15]). Indeed, given $e \in \mathcal{E}_h(\Sigma)$, we let T be the triangle of $\mathcal{T}_h^f$ having e as an edge, define $v_e := \boldsymbol{\sigma}_{f,h} \cdot \boldsymbol{\nu} - \rho_f \, \omega^2 \, \boldsymbol{\varphi}_{s,h} \cdot \boldsymbol{\nu}$ on e, and consider the extension operator $L : C(e) \to C(T)$. Then, applying (3.45), recalling that $\psi_e = 0$ on $\partial T \backslash e$, extending $\psi_e \, \overline{L(v_e)}$ by zero in $\Omega_f \backslash T$ so that the resulting function belongs to $H^1(\Omega_f)$, and adding and substracting $\boldsymbol{\sigma}_f \cdot \boldsymbol{\nu} = \rho_f \, \omega^2 \, \boldsymbol{\varphi}_s \cdot \boldsymbol{\nu}$ on Σ, we get

$$\begin{aligned}
\|v_e\|_{0,e}^2 \, &\leq c_2 \, \|\psi_e^{1/2} \, v_e\|_{0,e}^2 \, = \, c_2 \int_e \psi_e \, \overline{v_e} \, (\boldsymbol{\sigma}_{f,h} \cdot \boldsymbol{\nu} - \rho_f \, \omega^2 \, \boldsymbol{\varphi}_{s,h} \cdot \boldsymbol{\nu}) \\
&= c_2 \, \langle \boldsymbol{\sigma}_{f,h} \cdot \boldsymbol{\nu} - \rho_f \, \omega^2 \, \boldsymbol{\varphi}_{s,h} \cdot \boldsymbol{\nu}, \psi_e \, \overline{L(v_e)} \rangle_\Sigma \\
&= c_2 \left\{ - \, \langle (\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}) \cdot \boldsymbol{\nu}, \psi_e \, \overline{L(v_e)} \rangle_\Sigma \, + \, \rho_f \, \omega^2 \, \langle (\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}) \cdot \boldsymbol{\nu}, \psi_e \, \overline{L(v_e)} \rangle_\Sigma \right\}, \qquad (3.51)
\end{aligned}$$

where, as indicated in Section 1, $\langle \cdot, \cdot \rangle_\Sigma$ stands here for the duality pairing between $H^{-1/2}(\Sigma)$ and $H^{1/2}(\Sigma)$. Next, integrating by parts in Ω_f, and then employing the Cauchy-Schwarz inequality, the inverse estimate (3.47) (cf. Lemma 3.14), and (3.46), we find that

$$\begin{aligned}
&\langle (\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}) \cdot \boldsymbol{\nu}, \psi_e \overline{L(v_e)} \rangle_\Sigma \\
&= \int_T \nabla(\psi_e \, \overline{L(v_e)}) \cdot (\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}) \, + \, \int_T \psi_e \, \overline{L(v_e)} \, \mathrm{div}(\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}) \\
&\leq |\psi_e \, L(v_e)|_{1,T} \, \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T} \, + \, \|\psi_e \, L(v_e)\|_{0,T} \, \|\mathrm{div}(\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h})\|_{0,T} \\
&\leq C \left\{ h_T^{-1} \, h_e^{1/2} \, \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T} \, + \, h_e^{1/2} \, \|\mathrm{div}(\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h})\|_{0,T} \right\} \|v_e\|_{0,e} \, . \qquad (3.52)
\end{aligned}$$

In turn, noting that $(\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}) \cdot \boldsymbol{\nu} \in L^2(\Sigma)$, recalling that $0 \leq \psi_e \leq 1$ in w_e, and applying again the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\langle (\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}) \cdot \boldsymbol{\nu}, \psi_e \, \overline{L(v_e)} \rangle_\Sigma \, &= \, \int_e (\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}) \cdot \boldsymbol{\nu} \, \psi_e \, \overline{v_e} \\
&\leq \|(\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}) \cdot \boldsymbol{\nu}\|_{0,e} \, \|\psi_e \, v_e\|_{0,e} \, \leq \, \|\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}\|_{0,e} \, \|v_e\|_{0,e} \, . \qquad (3.53)
\end{aligned}$$

Finally, inserting the estimates (3.52) and (3.53) into (3.51), and using that $h_e \leq h_T$, we get after minor simplifications the required upper bound for $h_e \, \|\boldsymbol{\sigma}_{f,h} \cdot \boldsymbol{\nu} - \rho_f \, \omega^2 \, \boldsymbol{\varphi}_{s,h} \cdot \boldsymbol{\nu}\|_{0,e}^2$. □

Lemma 3.24. *There exists $C > 0$, independent of h and λ, such that for each $e \in \mathcal{E}_h(\Sigma)$ there holds*

$$\begin{aligned}
&h_e \, \|\boldsymbol{\sigma}_{s,h} \, \boldsymbol{\nu} \, + \, \varphi_{\Sigma,h} \, \boldsymbol{\nu}\|_{0,e}^2 \\
&\leq C \left\{ \|\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{s,h}\|_{0,T}^2 \, + \, h_T^2 \, \|\mathbf{div}(\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{s,h})\|_{0,T}^2 \, + \, h_e \, \|\varphi_\Sigma - \varphi_{\Sigma,h}\|_{0,e}^2 \right\},
\end{aligned}$$

where T is the triangle of $\mathcal{T}_h^s$ having e as an edge.

Proof. It proceeds similarly as for Lemma 3.23. This means that given $e \in \mathcal{E}_h(\Sigma)$, we now let T be the triangle of $\mathcal{T}_h^s$ having e as an edge, consider the vector version $\mathbf{L} : \mathbf{C}(e) \to \mathbf{C}(T)$ of the extension operator $L : C(e) \to C(T)$, define $\mathbf{v}_e := \boldsymbol{\sigma}_{s,h}\,\boldsymbol{\nu} + \varphi_{\Sigma,h}\,\boldsymbol{\nu}$ on e, and extend $\psi_e\,\mathbf{L}(\mathbf{v}_e)$ by zero in $\Omega_s\backslash T$ so that the resulting function belongs to $\mathbf{H}^1(\Omega_s)$. The rest of the proof follows basically by applying (3.45), using that $\boldsymbol{\sigma}_s\,\boldsymbol{\nu} = -\varphi_\Sigma\,\boldsymbol{\nu}$ on Σ, integrating by parts and applying Cauchy-Schwarz and inverse inequalities. We omit further details. □

The upper bounds for the terms of $\theta_{e,\Sigma}^2$ and $\theta_{e,\Gamma}^2$ involving tangential derivatives are given now.

Lemma 3.25. *There exists $C > 0$, independent of h and λ, such that*

$$\sum_{e\in\mathcal{E}_h(\Sigma)} h_e \left\| (\mathcal{C}^{-1}\,\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h)\,\mathbf{s} - \frac{d\,\boldsymbol{\varphi}_{s,h}}{d\mathbf{s}} \right\|_{0,e}^2$$

$$\leq C \left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} \left\{ \|\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{s,h}\|_{0,T_e}^2 + \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,T_e}^2 \right\} + \|\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}\|_{1/2,\Sigma}^2 \right\},$$

where, given $e \in \mathcal{E}_h(\Sigma)$, T_e is the triangle of $\mathcal{T}_h^s$ having e as an edge.

Proof. It makes use again of the extension operator $\mathbf{L} : \mathbf{C}(e) \to \mathbf{C}(T)$ (vector version of $L : C(e) \to C(T)$), the fact that $\nabla\mathbf{u} = \mathcal{C}^{-1}\,\boldsymbol{\sigma}_s + \boldsymbol{\gamma}$ in Ω_s, the boundedness of the tangential derivative $\frac{d}{d\mathbf{s}} : \mathbf{H}^{1/2}(\Sigma) \to \mathbf{H}^{-1/2}(\Sigma)$, the inverse and the Cauchy-Schwarz inequalities, and the upper bound for $h_{T_e}^2\,\|\mathrm{curl}\,(\mathcal{C}^{-1}\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h)\|_{0,T_e}^2$ (cf. Lemma 3.17). We omit further details and refer to [16, Lemma 20] where this result was established and proved. □

We remark that the upper bound provided by Lemma 3.25 is one of the three non-local estimates of the present efficiency analysis (see Lemma 3.27 below for the other two). However, the following lemma establishes that, under an additional regularity assumption on $\boldsymbol{\varphi}_s$, a corresponding local estimate can also be obtained.

Lemma 3.26. *Assume that $\boldsymbol{\varphi}_s|_e \in \mathbf{H}^1(e)$ for each $e \in \mathcal{E}_h(\Sigma)$. Then there exists $C > 0$, independent of h and λ, such that*

$$h_e \left\| (\mathcal{C}^{-1}\,\boldsymbol{\sigma}_{s,h} + \boldsymbol{\gamma}_h)\,\mathbf{s} - \frac{d\,\boldsymbol{\varphi}_{s,h}}{d\mathbf{s}} \right\|_{0,e}^2$$

$$\leq C \left\{ \|\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{s,h}\|_{0,T_e}^2 + \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,T_e}^2 + h_e \left\| \frac{d}{d\mathbf{s}}(\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}) \right\|_{0,e}^2 \right\},$$

where, given $e \in \mathcal{E}_h(\Sigma)$, T_e is the triangle of $\mathcal{T}_h^s$ having e as an edge.

Proof. See [16, Lemma 21]. □

Lemma 3.27. *There exist $C_1, C_2 > 0$, independent of h, such that*

$$\sum_{e\in\mathcal{E}_h(\Sigma)} h_e \left\| \boldsymbol{\sigma}_{f,h}\cdot\mathbf{s} - \frac{d\,\varphi_{\Sigma,h}}{d\mathbf{s}} \right\|_{0,e}^2 \leq C_1 \left\{ \sum_{e\in\mathcal{E}_h(\Sigma)} \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T_e}^2 + \|\varphi_\Sigma - \varphi_{\Sigma,h}\|_{1/2,\Sigma}^2 \right\},$$

$$\sum_{e\in\mathcal{E}_h(\Gamma)} h_e \left\| \boldsymbol{\sigma}_{f,h}\cdot\mathbf{s} - \frac{d\,\varphi_{\Gamma,h}}{d\mathbf{s}} \right\|_{0,e}^2 \leq C_2 \left\{ \sum_{e\in\mathcal{E}_h(\Gamma)} \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T_e}^2 + \|\varphi_\Gamma - \varphi_{\Gamma,h}\|_{1/2,\Gamma}^2 \right\},$$

where, given $e \in \mathcal{E}_h(\Sigma) \cup \mathcal{E}_h(\Gamma)$, T_e is the triangle of $\mathcal{T}_h^f$ having e as an edge.

Proof. Having the same structure of the estimate provided by Lemma 3.25, the present bounds follow from slight modifications of the proof of [16, Lemma 20]. □

Similarly as for Lemma 3.26, the following result establishes that, under additional regularity assumptions on φ_Σ and φ_Γ, corresponding local estimates can also be obtained.

Lemma 3.28. *Assume that $\varphi_\Sigma|_e \in H^1(e)$ for each $e \in \mathcal{E}_h(\Sigma)$ and $\varphi_\Gamma|_e \in H^1(e)$ for each $e \in \mathcal{E}_h(\Gamma)$. Then there exist $C_1, C_2 > 0$, independent of h, such that*

$$h_e \left\| \boldsymbol{\sigma}_{f,h} \cdot \mathbf{s} - \frac{d\,\varphi_{\Sigma,h}}{d\mathbf{s}} \right\|_{0,e}^2 \le C_1 \left\{ \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T_e}^2 + h_e \left\| \frac{d}{d\mathbf{s}} (\varphi_\Sigma - \varphi_{\Sigma,h}) \right\|_{0,e}^2 \right\},$$

$$h_e \left\| \boldsymbol{\sigma}_{f,h} \cdot \mathbf{s} - \frac{d\,\varphi_{\Gamma,h}}{d\mathbf{s}} \right\|_{0,e}^2 \le C_2 \left\{ \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T_e}^2 + h_e \left\| \frac{d}{d\mathbf{s}} (\varphi_\Gamma - \varphi_{\Gamma,h}) \right\|_{0,e}^2 \right\},$$

where, given $e \in \mathcal{E}_h(\Sigma) \cup \mathcal{E}_h(\Gamma)$, T_e is the triangle of $\mathcal{T}_h^f$ having e as an edge.

Proof. These bounds follow from slight modifications of the proof of [16, Lemma 21]. □

The remaining three terms defining $\theta_{e,\Sigma}^2$ and $\theta_{e,\Gamma}^2$ are bounded in what follows.

Lemma 3.29. *There exists $C > 0$, independent of h, such that for each $e \in \mathcal{E}_h(\Sigma)$ there holds*

$$h_e \|\varphi_{\Sigma,h} - p_h\|_{0,e}^2 \le C \left\{ h_T^2 \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T}^2 + \|p - p_h\|_{0,T}^2 + h_e \|\varphi_\Sigma - \varphi_{\Sigma,h}\|_{0,e}^2 \right\},$$

where T is the triangle of $\mathcal{T}_h^f$ having e as an edge.

Proof. Adding and substracting $\varphi_\Sigma = p$ on Σ, and then employing the discrete trace inequality (3.48) (cf. Lemma 3.15), we obtain for each $e \in \mathcal{E}_h(\Sigma)$

$$\begin{aligned} h_e \|\varphi_{\Sigma,h} - p_h\|_{0,e}^2 &\le 2\, h_e \left\{ \|\varphi_{\Sigma,h} - \varphi_\Sigma\|_{0,e}^2 + \|p - p_h\|_{0,e}^2 \right\} \\ &\le C \left\{ h_e \|\varphi_{\Sigma,h} - \varphi_\Sigma\|_{0,e}^2 + \|p - p_h\|_{0,T}^2 + h_T^2 |p - p_h|_{1,T}^2 \right\}, \end{aligned} \tag{3.54}$$

where the last term uses that $h_e \le h_T$. Then, recalling that p_h is piecewise constant (cf. (3.7)), using that $\boldsymbol{\sigma}_f = \nabla p$ in Ω_f, adding and substracting $\boldsymbol{\sigma}_{f,h}$, and employing the upper bound from Lemma 3.19, we find that

$$\begin{aligned} &h_T^2 |p - p_h|_{1,T}^2 = h_T^2 \|\nabla p\|_{0,T}^2 = h_T^2 \|\boldsymbol{\sigma}_f\|_{0,T}^2 \le 2\, h_T^2 \left\{ \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T}^2 + \|\boldsymbol{\sigma}_{f,h}\|_{0,T}^2 \right\} \\ &\le C \left\{ h_T^2 \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T}^2 + \|p - p_h\|_{0,T}^2 \right\}. \end{aligned} \tag{3.55}$$

Finally, (3.54) and (3.55) yield the required estimate and finish the proof. □

Lemma 3.30. *There exists $C > 0$, independent of h, such that for each $e \in \mathcal{E}_h(\Gamma)$ there holds*

$$h_e \|\varphi_{\Gamma,h} - p_h\|_{0,e}^2 \le C \left\{ h_T^2 \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T}^2 + \|p - p_h\|_{0,T}^2 + h_e \|\varphi_\Gamma - \varphi_{\Gamma,h}\|_{0,e}^2 \right\},$$

where T is the triangle of $\mathcal{T}_h^f$ having e as an edge.

Proof. It follows exactly as in the proof of Lemma 3.29 replacing Σ by Γ everywhere. $\square$

We complete the efficiency analysis of the a posteriori error estimator $\boldsymbol{\theta}$ with the upper bound for the term concerning the Robin boundary condition on Γ. To this end, and for simplicity, we assume that g is piecewise polynomial on Γ. Otherwise, one would proceed as in the proof of [16, Lemma 23] by adding and substracting a suitable projection of g onto a polynomial space.

Lemma 3.31. *There exists $C > 0$, independent of h, such that for each $e \in \mathcal{E}_h(\Gamma)$ there holds*

$$\begin{aligned} & h_e\, \|\boldsymbol{\sigma}_{f,h}\cdot\boldsymbol{\nu} \,-\, \imath\,\kappa_f\,\varphi_{\Gamma,h} - g\|_{0,e}^2 \\ & \le C\left\{ \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{0,T}^2 \,+\, h_T^2\,\|\mathrm{div}(\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h})\|_{0,T}^2 \,+\, h_e\,\|\varphi_\Gamma - \varphi_{\Gamma,h}\|_{0,e}^2 \right\}, \end{aligned}$$

where T is the triangle of $\mathcal{T}_h^f$ having e as an edge.

Proof. It proceeds analogously to the proofs of Lemmas 3.23 and 3.24. We omit further details here and refer to those lemmas. $\square$

We end this section by remarking that the efficiency of $\boldsymbol{\theta}$ follows straightforwardly from estimates (3.42) and (3.43), together with Lemmas 3.16 - 3.25, 3.27, 3.29 - 3.31, after summing up over triangles $T \in \mathcal{T}_h^s \cup \mathcal{T}_h^s$ and edges $e \in \mathcal{E}_h$ (cf. (3.1)), and using that the number of triangles on each domain ω_e is bounded by two. In particular, note that the global efficiency estimates induced by the terms of the form $h_e\,\|\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}\|_{0,e}^2$, $h_e\,\|\varphi_\Sigma - \varphi_{\Sigma,h}\|_{0,e}^2$, and $h_e\,\|\varphi_\Gamma - \varphi_{\Gamma,h}\|_{0,e}^2$ (cf. Lemmas 3.22, 3.23, 3.24, 3.29, and 3.30), follow easily from the fact that

$$\begin{aligned} \sum_{e\in\mathcal{E}_h(\Sigma)} h_e\,\|\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}\|_{0,e}^2 \,&\le\, h\,\|\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}\|_{0,\Sigma}^2 \,\le\, C\,h\,\|\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}\|_{1/2,\Sigma}^2\,, \\ \sum_{e\in\mathcal{E}_h(\Sigma)} h_e\,\|\varphi_\Sigma - \varphi_{\Sigma,h}\|_{0,e}^2 \,&\le\, h\,\|\varphi_\Sigma - \varphi_{\Sigma,h}\|_{0,\Sigma}^2 \,\le\, C\,h\,\|\varphi_\Sigma - \varphi_{\Sigma,h}\|_{1/2,\Sigma}^2\,, \\ \sum_{e\in\mathcal{E}_h(\Gamma)} h_e\,\|\varphi_\Gamma - \varphi_{\Gamma,h}\|_{0,e}^2 \,&\le\, h\,\|\varphi_\Gamma - \varphi_{\Gamma,h}\|_{0,\Gamma}^2 \,\le\, C\,h\,\|\varphi_\Gamma - \varphi_{\Gamma,h}\|_{1/2,\Sigma}^2\,. \end{aligned}$$

4. Numerical results

In this section we present some numerical results confirming the reliability and efficiency of the a posteriori error estimator $\boldsymbol{\theta}$ analyzed in Section 3. We begin by introducing additional notations. The variable N stands for the number of degrees of freedom defining the finite element subspaces $\mathbf{H}_h$ and $\mathbf{Q}_h$ (equivalently, the number of unknowns of (2.20)), and the individual and global errors are denoted by:

$$\mathsf{e}(\boldsymbol{\sigma}_s) \,:=\, \|\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{s,h}\|_{\mathbf{div};\Omega_s}\,, \quad \mathsf{e}(\boldsymbol{\sigma}_f) \,:=\, \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{\mathrm{div};\Omega_f}\,, \quad \mathsf{e}(\boldsymbol{\gamma}) \,:=\, \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,\Omega_s}\,,$$

$$\mathsf{e}(\boldsymbol{\varphi}_s) \,:=\, \|\boldsymbol{\varphi}_s - \boldsymbol{\varphi}_{s,h}\|_{1/2,\Sigma}\,, \quad \mathsf{e}(\varphi_\Sigma) \,:=\, \|\varphi_\Sigma - \varphi_{\Sigma,h}\|_{1/2,\Sigma}\,, \quad \mathsf{e}(\varphi_\Gamma) \,:=\, \|\varphi_\Gamma - \varphi_{\Gamma,h}\|_{1/2,\Gamma}\,,$$

$$\mathsf{e}(\widehat{\boldsymbol{\sigma}}) := \left\{ [\mathsf{e}(\boldsymbol{\sigma}_s)]^2 \,+\, [\mathsf{e}(\boldsymbol{\sigma}_f)]^2 \right\}^{1/2},\ \mathsf{e}(\widehat{\boldsymbol{\gamma}}) := \left\{ [\mathsf{e}(\boldsymbol{\gamma})]^2 + [\mathsf{e}(\boldsymbol{\varphi}_s)]^2 + [\mathsf{e}(\varphi_\Sigma)]^2 + [\mathsf{e}(\varphi_\Gamma)]^2 \right\}^{1/2},$$

$$\mathsf{e}(\mathbf{u}) \,:=\, \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_s}\,, \quad \mathsf{e}(p) \,:=\, \|p - p_h\|_{0,\Omega_f}\,,$$

and

$$\mathtt{e} := \left\{[\mathtt{e}(\widehat{\boldsymbol{\sigma}})]^2 + [\mathtt{e}(\widehat{\boldsymbol{\gamma}})]^2 + [\mathtt{e}(\mathbf{u})]^2 + [\mathtt{e}(p)]^2\right\}^{1/2},$$

where $\boldsymbol{\varphi}_f := (\varphi_\Sigma, \varphi_\Gamma) \in H^{1/2}(\Sigma) \times H^{1/2}(\Gamma)$ and

$$\boldsymbol{\varphi}_{f,h} := (\varphi_{\Sigma,h}, \varphi_{\Gamma,h}) \in \mathbf{Q}_h^f := \Lambda_h(\Sigma) \times \Lambda_h(\Gamma).$$

Bear in mind here that $\mathbf{u}_h$ and p_h are the postprocessed variables computed according to (3.7). Also, we define the effectivity index

$$\mathtt{eff}(\boldsymbol{\theta}) := \mathtt{e}/\boldsymbol{\theta}.$$

In turn, we let $r(\boldsymbol{\sigma}_s)$, $r(\boldsymbol{\sigma}_f)$, $r(\boldsymbol{\gamma})$, $r(\boldsymbol{\varphi}_s)$, $r(\varphi_\Sigma)$, $r(\varphi_\Gamma)$, $r(\mathbf{u})$, $r(p)$, and r be the experimental rates of convergence given by

$$r(\%) := \frac{\log\big(\mathtt{e}(\%)/\mathtt{e}'(\%)\big)}{\log(h/h')} \quad \forall \% \in \{\boldsymbol{\sigma}_s, \boldsymbol{\sigma}_f, \boldsymbol{\gamma}, \boldsymbol{\varphi}_s, \varphi_\Sigma, \varphi_\Gamma, \mathbf{u}, p\}, \quad \text{and} \quad r := \frac{\log\big(\mathtt{e}/\mathtt{e}'\big)}{\log(h/h')},$$

where h and h' denote two consecutive meshsizes with corresponding individual errors $\mathtt{e}(\%)$ and $\mathtt{e}'(\%)$, and global errors $\mathtt{e}$ and $\mathtt{e}'$, respectively. However, when the adaptive algorithm is applied (see details below), the expression $\log(h/h')$ is replaced by $-\frac{1}{2}\log(N/N')$, where N and N' denote the corresponding degrees of freedom of each triangulation.

In what follows we describe the examples to be considered. For Example 1 we consider $\Omega_s := (-0.2, 0.2) \times (-0.4, 0.4)$ and let the artificial boundary Γ be the ellipse centered at the origin with minor and major semiaxis given by 0.4 and 0.6, respectively, that is

$$\Omega_f := \left\{(x_1, x_2)^{\mathtt{t}} \in \mathbb{R}^2 : \quad \frac{x_1^2}{0.4^2} + \frac{x_2^2}{0.6^2} < 1\right\} \backslash \overline{\Omega}_s.$$

We take $\rho_s = \rho_f = \lambda = \mu = 1$, and the rest of parameters are given as follows: $v_0 = 1$, $\omega = 5$, $\kappa_s = 5$, $\kappa_f = 5$. Furthermore, let K_0, K_1 and K_2 be the modified Bessel functions of the second kind and order 0, 1, and 2, respectively, and let $H_0^{(1)}$ be the Hankel function of the first kind and order zero. Then, we choose the data in such a way that the exact solution of (2.5) (or (2.7)) is determined by

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} \dfrac{1}{2\pi}\,\psi(\mathbf{x}) - \dfrac{(x_1-1)^2}{r_1^2}\,\chi(\mathbf{x}) \\[2ex] -\dfrac{(x_1-1)\,x_2}{r_1^2}\,\chi(\mathbf{x}) \end{pmatrix} \quad \forall\,\mathbf{x} := (x_1, x_2)^{\mathtt{t}} \in \Omega_s,$$

$$p(\mathbf{x}) = H_0^{(1)}(\kappa_f\,|\mathbf{x}|) \qquad \forall\,\mathbf{x} \in \Omega_f,$$

where

$$r_1 := \sqrt{(x_1-1)^2 + x_2^2},$$

$$\psi(\mathbf{x}) := K_0(\imath\,\omega\,r_1) + \frac{1}{\imath\,\omega\,r_1}\left\{K_1(\imath\,\omega\,r_1) - \frac{1}{\sqrt{3}}\,K_1\left(\frac{\imath\,\omega\,r_1}{\sqrt{3}}\right)\right\},$$

$$\chi(\mathbf{x}) := K_2(\imath\,\omega\,r_1) - \frac{1}{3}\,K_2\left(\frac{\imath\,\omega\,r_1}{\sqrt{3}}\right).$$

Table 4.1: Convergence history for $\boldsymbol{\sigma}_s$, $\boldsymbol{\sigma}_f$, and $\boldsymbol{\gamma}$ (Example 1).

h	N	$\mathtt{e}(\boldsymbol{\sigma}_s)$	$r(\boldsymbol{\sigma}_s)$	$\mathtt{e}(\boldsymbol{\sigma}_f)$	$r(\boldsymbol{\sigma}_f)$	$\mathtt{e}(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$
$2\pi/64$	1117	6.150E−02	–	8.865E−01	–	6.642E−03	–
$2\pi/96$	2090	4.264E−02	0.903	5.996E−01	0.964	3.975E−03	1.266
$2\pi/128$	3686	3.112E−02	1.095	4.414E−01	1.065	2.570E−03	1.516
$2\pi/192$	7869	2.107E−02	0.962	3.044E−01	0.917	1.530E−03	1.279
$2\pi/256$	13666	1.586E−02	0.987	2.249E−01	1.053	1.018E−03	1.415
$2\pi/384$	31282	1.038E−02	1.046	1.489E−01	1.017	6.623E−04	1.061
$2\pi/512$	55438	7.784E−03	1.000	1.106E−01	1.035	4.324E−04	1.482
$2\pi/768$	125069	5.152E−03	1.017	7.397E−02	0.991	2.745E−04	1.121
$2\pi/1024$	221848	3.871E−03	0.994	5.540E−02	1.005	2.034E−04	1.041
$2\pi/1536$	498545	2.579E−03	1.001	3.670E−02	1.016	1.298E−04	1.109
$2\pi/2048$	887629	1.927E−03	1.014	2.770E−02	0.978	9.678E−05	1.019

Table 4.2: Convergence history for $\boldsymbol{\varphi}_s$, φ_Σ, and φ_Γ (Example 1).

N	$\mathtt{e}(\boldsymbol{\varphi}_s)$	$r(\boldsymbol{\varphi}_s)$	$\mathtt{e}(\varphi_\Sigma)$	$r(\varphi_\Sigma)$	$\mathtt{e}(\varphi_\Gamma)$	$r(\varphi_\Gamma)$
1117	9.684E−03	–	1.689E−01	–	4.819E−02	–
2090	4.899E−03	1.681	7.439E−02	2.022	2.030E−02	2.133
3686	2.727E−03	2.037	4.415E−02	1.813	1.226E−02	1.752
7869	1.427E−03	1.598	2.362E−02	1.542	5.610E−03	1.928
13666	8.446E−04	1.822	1.348E−02	1.951	3.850E−03	1.308
31282	4.023E−04	1.829	6.741E−03	1.708	1.834E−03	1.830
55438	2.521E−04	1.625	3.849E−03	1.948	1.187E−03	1.511
125069	1.266E−04	1.699	1.896E−03	1.746	6.280E−04	1.571
221848	8.236E−05	1.494	1.290E−03	1.339	4.437E−04	1.208
498545	4.112E−05	1.713	6.765E−04	1.592	2.231E−04	1.695
887629	2.633E−05	1.550	4.455E−04	1.452	1.533E−04	1.305

Table 4.3: Convergence history for $\mathbf{u}$, p, $\mathtt{e}$, and effectivity index (Example 1).

N	$\mathtt{e}(\mathbf{u})$	$r(\mathbf{u})$	$\mathtt{e}(p)$	$r(p)$	$\mathtt{e}$	r	$\mathtt{eff}(\boldsymbol{\theta})$
1117	2.207E−03	–	3.419E−02	–	9.065E−01	–	0.7495
2090	1.547E−03	0.877	2.317E−02	0.960	6.065E−01	0.991	0.7315
3686	1.131E−03	1.087	1.706E−02	1.064	4.452E−01	1.075	0.7424
7869	7.671E−04	0.958	1.177E−02	0.916	3.063E−01	0.922	0.7328
13666	5.781E−04	0.983	8.700E−03	1.050	2.260E−01	1.057	0.7437
31282	3.781E−04	1.044	5.760E−03	1.017	1.495E−01	1.019	0.7417
55438	2.840E−04	0.999	4.277E−03	1.035	1.110E−01	1.036	0.7377
125069	1.881E−04	1.018	2.863E−03	0.991	7.423E−02	0.992	0.7445
221848	1.413E−04	0.993	2.144E−03	1.005	5.559E−02	1.005	0.7413
498545	9.417E−05	1.001	1.420E−03	1.016	3.682E−02	1.016	0.7366
887629	7.036E−05	1.013	1.072E−03	0.978	2.779E−02	0.978	0.7360

Actually, $\mathbf{u}$ is the fundamental solution, centered at $(1,0)^{\mathtt{t}}$, of the elastodynamic equation, which yields $\mathbf{f} = \mathbf{0}$ in Ω_s, and p is the fundamental solution, centered at the origin, of the Helmholtz equation in Ω_f.

Then, for Example 2 we let Ω_s be the L-shaped domain $(-0.3,0.3)^2 \setminus (0,0.3)^2$ and consider

Table 4.4: Convergence history for $\boldsymbol{\sigma}_s$, $\boldsymbol{\sigma}_f$, and $\boldsymbol{\gamma}$ (quasi-uniform scheme, Example 2).

h	N	$\mathtt{e}(\boldsymbol{\sigma}_s)$	$r(\boldsymbol{\sigma}_s)$	$\mathtt{e}(\boldsymbol{\sigma}_f)$	$r(\boldsymbol{\sigma}_f)$	$\mathtt{e}(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$
$2\pi/64$	2215	9.127E−01	–	4.267E−01	–	3.210E−02	–
$2\pi/96$	4767	6.802E−01	0.725	1.896E−01	2.000	1.371E−02	2.098
$2\pi/128$	8495	5.408E−01	0.797	1.185E−01	1.634	9.156E−03	1.403
$2\pi/192$	19067	4.465E−01	0.472	6.492E−02	1.484	4.033E−03	2.022
$2\pi/256$	33331	3.898E−01	0.472	4.851E−02	1.013	2.828E−03	1.234
$2\pi/384$	75077	2.800E−01	0.816	3.053E−02	1.142	1.630E−03	1.359
$2\pi/512$	133497	2.351E−01	0.607	2.317E−02	0.960	1.049E−03	1.532
$2\pi/768$	299000	1.883E−01	0.547	1.528E−02	1.026	6.357E−04	1.235
$2\pi/1024$	534105	1.493E−01	0.807	1.139E−02	1.023	4.391E−04	1.286
$2\pi/1536$	1199275	1.109E−01	0.735	7.601E−03	0.997	2.663E−04	1.233

Table 4.5: Convergence history for $\mathbf{u}$, p, $\mathtt{e}$, and effectivity index (quasi-uniform scheme, Example 2).

N	$\mathtt{e}(\mathbf{u})$	$r(\mathbf{u})$	$\mathtt{e}(p)$	$r(p)$	$\mathtt{e}$	r	$\mathtt{eff}(\boldsymbol{\theta})$
2215	9.444E−03	–	5.476E−02	–	1.155E−00	–	0.6179
4767	5.899E−03	1.161	2.980E−02	1.501	7.360E−01	1.111	0.6313
8495	4.430E−03	0.996	2.024E−02	1.345	5.645E−01	0.922	0.6546
19067	2.942E−03	1.010	1.292E−02	1.107	4.529E−01	0.543	0.7241
33331	2.189E−03	1.028	9.722E−03	0.988	3.935E−01	0.488	0.7679
75077	1.459E−03	1.000	6.359E−03	1.047	2.819E−01	0.823	0.7943
133497	1.091E−03	1.009	4.801E−03	0.977	2.364E−01	0.612	0.8232
299000	7.360E−04	0.971	3.191E−03	1.008	1.890E−01	0.552	0.8679
534105	5.567E−04	0.971	2.388E−03	1.008	1.498E−01	0.809	0.8806
1199275	3.685E−04	1.018	1.594E−03	0.996	1.111E−01	0.736	0.9004

Γ as the boundary of the unit circle $B(\mathbf{0},1)$. In addition, we take $\rho_s = \rho_f = \lambda = \mu = 1$, $v_0 = 10$, and $\omega = 10$, so that $\kappa_s = 10$ and $\kappa_f = 1$. Then, we choose the data in such a way that the exact solution of (2.5) (or (2.7)) is given by

$$\mathbf{u}(\mathbf{r},\theta) \,:=\, \mathbf{r}^{5/3}\,\sin\big((2\,\theta-\pi)/3\big)\begin{pmatrix} 1+\imath \\ 1+\imath \end{pmatrix} \qquad \forall\,(\mathbf{r},\theta)\,\in\,\Omega_s\,,$$

in polar coordinates, and

$$p(\mathbf{x}) = H_0^{(1)}(\kappa_f\,|\mathbf{x}+(0.15,0)|) \qquad \forall\,\mathbf{x}\in\Omega_f\,,$$

Note that $\mathbf{u}$ becomes singular at the origin, the corner of the L. More precisely, it is not difficult to see that around this singularity $\mathbf{div}\,\boldsymbol{\sigma}_s$ behaves of order $\mathbf{r}^{-1/3}$. It follows that $\mathbf{div}\,\boldsymbol{\sigma}_s$ belongs to $\mathbf{H}^{2/3-\epsilon}(\Omega_s)$ for each $\epsilon > 0$, and hence, according to Theorem 2.3, we expect experimental rates of convergence, particularly $r(\boldsymbol{\sigma}_s)$, close to $2/3$. According to the preceding remarks, this example is utilized to illustrate the behavior of the adaptive algorithm associated with $\boldsymbol{\theta}$, which applies the usual procedure from [29] with the *blue-green* strategy for refinement. We just

Table 4.6: Convergence history for $\boldsymbol{\sigma}_s$, $\boldsymbol{\sigma}_f$, and $\boldsymbol{\gamma}$ (adaptive scheme, Example 2).

h	N	$\mathsf{e}(\boldsymbol{\sigma}_s)$	$r(\boldsymbol{\sigma}_s)$	$\mathsf{e}(\boldsymbol{\sigma}_f)$	$r(\boldsymbol{\sigma}_f)$	$\mathsf{e}(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$
0.1169	2215	9.127E−01	–	4.267E−01	–	3.210E−02	–
0.1169	2503	7.145E−01	4.006	2.996E−01	5.786	2.589E−02	3.520
0.1169	3471	5.377E−01	1.739	2.607E−01	0.851	2.394E−02	0.478
0.1169	4459	4.417E−01	1.570	1.713E−01	3.354	1.472E−02	3.883
0.1169	6355	3.477E−01	1.351	1.401E−01	1.134	1.299E−02	0.707
0.1169	9410	2.753E−01	1.189	1.088E−01	1.287	9.272E−03	1.717
0.1169	11985	2.411E−01	1.097	9.418E−02	1.196	8.363E−03	0.853
0.1169	19655	1.882E−01	1.002	7.556E−02	0.890	5.892E−03	1.416
0.0934	38391	1.406E−01	0.870	5.126E−02	1.159	4.545E−03	0.775
0.0832	65934	1.058E−01	1.051	4.117E−02	0.810	3.321E−03	1.161
0.0832	98472	9.131E−02	0.736	3.519E−02	0.783	3.022E−03	0.470
0.0622	125924	8.021E−02	1.055	3.056E−02	1.146	2.723E−03	0.847
0.0511	151119	7.225E−02	1.146	2.681E−02	1.436	2.257E−03	2.060
0.0493	196274	6.617E−02	0.673	2.456E−02	0.670	2.161E−03	0.331
0.0471	241916	6.067E−02	0.830	2.287E−02	0.684	2.065E−03	0.436
0.0467	282385	5.684E−02	0.843	2.144E−02	0.830	1.904E−03	1.051
0.0400	343470	4.852E−02	1.617	1.836E−02	1.586	1.581E−03	1.900
0.0298	570415	3.694E−02	1.075	1.382E−02	1.120	1.177E−03	1.162
0.0244	894088	3.037E−02	0.872	1.139E−02	0.861	9.605E−04	0.905
0.0240	1269053	2.654E−02	0.769	9.882E−03	0.811	8.686E−04	0.574
0.0234	1635325	2.360E−02	0.926	8.777E−03	0.935	7.831E−04	0.817

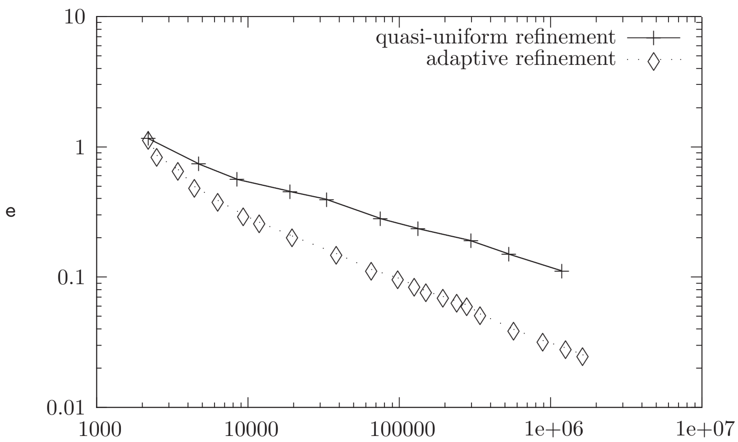

Fig. 4.1. EXAMPLE 2, total error e vs. N for the quasi-uniform and adaptive schemes.

mention that the error indicators θ_T on each triangle $T \in \mathcal{T}_h^s \cup \mathcal{T}_h^f$ are computed as follows:

$$\theta_T^2 := \begin{cases} \theta_{T,s}^2 + \dfrac{1}{2} \displaystyle\sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Sigma)} \theta_{e,\Sigma}^2 & \text{if} \quad T \in \mathcal{T}_h^s, \\ \theta_{T,f}^2 + \dfrac{1}{2} \displaystyle\sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Sigma)} \theta_{e,\Sigma}^2 + \displaystyle\sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Gamma)} \theta_{e,\Gamma}^2 & \text{if} \quad T \in \mathcal{T}_h^f. \end{cases}$$

The numerical results shown below were obtained using a MATLAB code. In Tables 4.1 up

Table 4.7: Convergence history for $\mathbf{u}$, p, $\mathtt{e}$, and effectivity index (adaptive scheme, Example 2).

N	$\mathtt{e}(\mathbf{u})$	$r(\mathbf{u})$	$\mathtt{e}(p)$	$r(p)$	$\mathtt{e}$	r	$\mathtt{eff}(\boldsymbol{\theta})$
2215	9.444E−03	–	5.476E−02	–	1.155E−00	–	0.6179
2503	8.923E−03	0.928	4.779E−02	2.229	8.576E−01	4.868	0.5530
3471	6.348E−03	2.083	4.289E−02	0.661	6.693E−01	1.516	0.5277
4459	5.179E−03	1.625	3.797E−02	0.974	4.949E−01	2.411	0.4727
6355	4.091E−03	1.332	3.583E−02	0.328	3.880E−01	1.374	0.4537
9410	3.008E−03	1.566	3.101E−02	0.735	3.014E−01	1.287	0.4249
11985	2.772E−03	0.678	2.814E−02	0.803	2.628E−01	1.133	0.4205
19655	2.196E−03	0.942	2.250E−02	0.904	2.059E−01	0.986	0.4089
38391	1.549E−03	1.042	1.499E−02	1.214	1.510E−01	0.927	0.4300
65934	1.215E−03	0.899	1.223E−02	0.752	1.146E−01	1.018	0.3973
98472	1.013E−03	0.908	1.045E−02	0.786	9.870E−02	0.747	0.4051
125924	9.152E−04	0.822	9.149E−03	1.077	8.653E−02	1.070	0.4050
151119	8.144E−04	1.280	7.918E−03	1.585	7.762E−02	1.192	0.4108
196274	7.452E−04	0.679	7.221E−03	0.704	7.109E−02	0.672	0.4082
241916	6.858E−04	0.795	6.727E−03	0.678	6.532E−02	0.809	0.3933
282385	6.388E−04	0.917	6.308E−03	0.832	6.121E−02	0.842	0.4030
343470	5.594E−04	1.356	5.398E−03	1.591	5.225E−02	1.616	0.4038
570415	4.196E−04	1.134	4.004E−03	1.178	3.969E−02	1.084	0.4075
894088	3.470E−04	0.846	3.315E−03	0.840	3.264E−02	0.871	0.4025
1269053	3.032E−04	0.770	2.886E−03	0.792	2.850E−02	0.773	0.3792
1635325	2.680E−04	0.972	2.565E−03	0.931	2.534E−02	0.928	0.4013

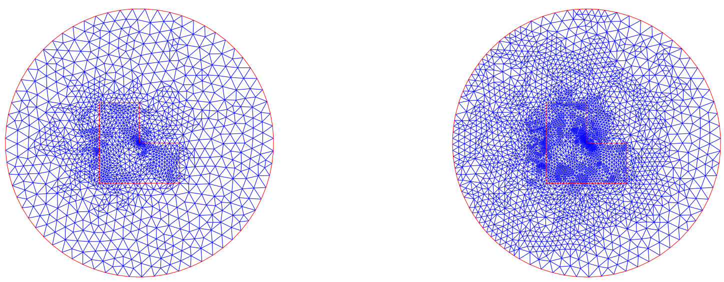

Fig. 4.2. EXAMPLE 2: adapted meshes for $N \in \{9410, 19655\}$.

to 4.3 we summarize the convergence history of our fully-mixed finite element scheme (2.20) as applied to Example 1 for a finite sequence of quasi-uniform triangulations of the computational domain $\overline{\Omega}_s \cup \overline{\Omega}_f$. While this example coincides with one presented in [14, Section 5], the novelty now is certainly the computation of the effectivity indexes. We observe in those tables, looking at the corresponding experimental rates of convergence, that the $O(h)$ predicted by Theorem 2.3 when $\delta = 1$ (see [14, Theorem 4.1]) is attained in all the unknowns for this example. In addition, we notice from the last column of Table 4.3 that the effectivity indexes $\mathtt{eff}(\boldsymbol{\theta})$ remain always in a neighborhood of 0.74, which illustrates the reliability and efficiency of $\boldsymbol{\theta}$ in the case of a regular solution.

Then, in Tables 4.4 up to 4.7 we provide most details on the convergence history of the quasi-uniform and adaptive refinements, as applied to Example 2. As already announced, we notice in the quasi-uniform case that $r(\boldsymbol{\sigma}_s)$ oscillates in fact around $2/3$, whereas the rates of convergence of the other unknowns are not affected by the lack of regularity of $\boldsymbol{\sigma}_s$. However,

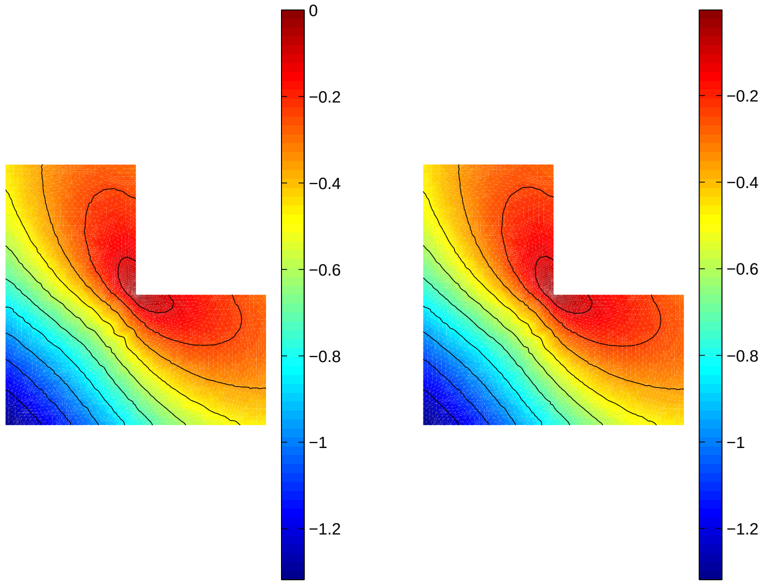

Fig. 4.3. Approximate and exact real part of $\boldsymbol{\sigma}_{s,21}$ (Example 2).

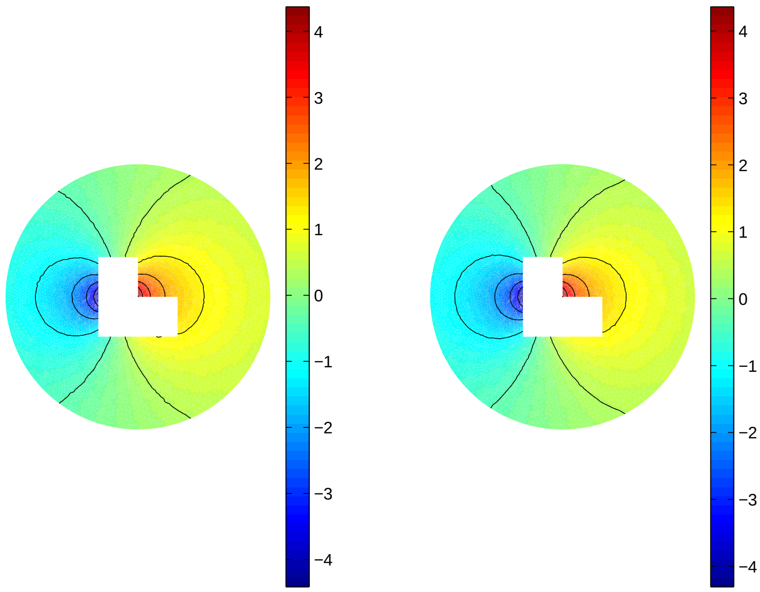

Fig. 4.4. Approximate and exact imaginary part of $\boldsymbol{\sigma}_{f,1}$ (Example 2).

since $\mathtt{e}(\boldsymbol{\sigma}_s)$ is the dominant component of the total error $\mathtt{e}$, the above feature is also reflected in the global rate of convergence r (see Table 4.5). Furthermore, it is clear from these tables that the total errors of the adaptive scheme decrease faster than those obtained by the quasi-uniform one, which is confirmed by the global experimental rates of convergence provided in Table 4.7. This fact is also illustrated by Fig. 4.1 where we display the total errors $\mathtt{e}$ vs. the number of degrees of freedom N for both refinements. Moreover, as shown by these values of r, the adaptive method is able to recover the quasi-optimal rate of convergence $O(h)$ for $\mathtt{e}$. On the other hand, the effectivity indexes remain bounded from above and below for both the quasi-uniform and adaptive schemes, which confirms the reliability and efficiency of $\boldsymbol{\theta}$ in the present case of a non-smooth solution. Intermediate meshes obtained with the adaptive refinement are displayed in Fig. 4.2. We remark from there that the method is able to recognize the origin as a singularity of the solution of this example. Finally, some components of the approximate (left) and exact (right) solutions of Example 2 are displayed in Figs. 4.3 and 4.4 for $N = 65934$. The fact that the approximate and exact solutions do not distinguish from each other in all the components shown illustrates the accurateness of the proposed fully-mixed method and the corresponding adaptive scheme.

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References

[1] S. Agmon, Lectures on Elliptic Boundary Value Problems, Van Nostrand, Princeton, New Jersey, 1965.

[2] A. Alonso, Error estimators for a mixed method, *Numer. Math.*, **74** (1996), 385-395.

[3] D.N. Arnold, An interior penalty finite element method with discontinuous elements, *SIAM J. Numer. Anal.*, **19** (1982), 4, 742-760.

[4] D.N. Arnold, F. Brezzi and J. Douglas, PEERS: A new mixed finite element method for plane elasticity, *Japan J. Appl. Math.*, **1** (1984), 2, 347-367.

[5] I. Babuška and G.N. Gatica, A residual-based a posteriori error estimator for the Stokes-Darcy coupled problem, *SIAM J. Numer. Anal.*, **48** (2010), 2, 498-523.

[6] T. Barrios, G.N. Gatica, M. González and N. Heuer, A residual based a posteriori error estimator for an augmented mixed finite element method in linear elasticity, *ESAIM: Math. Model. Numer. Anal.*, **40** (2006), 5, 843-869.

[7] Ph. Bouillard, Influence of the pollution on the admissible field error estimation for FE solutions of the Helmholtz equation, *International Journal for Numerical Methods in Engineering*, **45** (1999), 7, 783-800.

[8] F. Brezzi and M. Fortin, Mixed and Hybrid Finite Element Methods, Springer Verlag, 1991.

[9] C. Carstensen, An a posteriori error estimate for a first kind integral equation, *Math. Comput.*, **66** (1997), 217, 139-155.

[10] C. Carstensen, A posteriori error estimate for the mixed finite element method, *Math. Comput.*, **66** (1997), 218, 465-476.

[11] C. Carstensen and G. Dolzmann, A posteriori error estimates for mixed FEM in elasticity, *Numer. Math.*, **81** (1998), 1, 187-209.

[12] P. Clément, Approximation by finite element functions using local regularisation, *RAIRO Modélisation Mathématique et Analyse Numérique*, **9** (1975), 77-84.

[13] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, New York, Oxford, 1978.

[14] C. Domínguez, G.N. Gatica, S. Meddahi and R. Oyarzúa, A priori error analysis of a fully-mixed finite element method for a two-dimensional fluid-solid interaction problem, *ESAIM: Math. Model. Numer. Anal.*, **47** (2013), 2, 471-506.

[15] G.N. Gatica, A Simple Introduction to the Mixed Finite Element Method. Theory and Applications, SpringerBriefs in Mathematics, Springer, Cham Heidelberg New York Dordrecht London, 2014.

[16] G.N. Gatica, G.C. Hsiao and S. Meddahi, A residual-based a posteriori error estimator for a two-dimensional fluid-solid interaction problem, *Numerische Mathematik*, **114** (2009), 1, 63-106.

[17] G.N. Gatica, A. Márquez and S. Meddahi, Analysis of the coupling of primal and dual-mixed finite element methods for a two-dimensional fluid-solid interaction problem, *SIAM J. Numer. Anal.*, **45** (2007), 5, 2072-2097.

[18] G.N. Gatica, A. Márquez and S. Meddahi, Analysis of the coupling of BEM, FEM and mixed-FEM for a two-dimensional fluid-solid interaction problem, *Appl. Numer. Math.*, **59** (2009), 11, 2735-2750,

[19] G.N. Gatica, A. Márquez and M.A. Sánchez, Analysis of a velocity-pressure-pseudostress formulation for the stationary Stokes equations, *Computer Methods in Applied Mechanics and Engineering*, **199** (2010), 17-20, 1064-1079.

[20] G.N. Gatica and S. Meddahi, An a-posteriori error estimate for the coupling of BEM and mixed-FEM, *Numerical Functional Analysis and Optimization*, **20** (1999), 5-6, 449-472.

[21] G.N. Gatica, R. Oyarzúa and F.-J. Sayas, Analysis of fully-mixed finite element methods for the Stokes-Darcy coupled problem, *Math. Comput.*, **80** (2011), 276, 1911-1948.

[22] G.N. Gatica, R. Oyarzúa and F.-J. Sayas, A residual-based a posteriori error estimator for a fully mixed formulation of the Stokes-Darcy coupled problem, *Computer Methods in Applied Mechanics and Engineering*, **200** (2011), 21-22, 1877-1891.

[23] V. Girault and P.-A. Raviart, Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms, Springer Series in Computational Mathematics, vol. 5, Springer-Verlag, 1986.

[24] F. Ihlenburg and I. Babuška, Dispersion analysis and error estimation of Galerkin finite element methods for the Helmholtz equation, *International Journals for Numerical Methods in Engineering*, **38** (1995), 22, 3745-3774.

[25] S. Irimie and Ph. Boiullard, A reliable a posteriori error estimator for the finite element solution of the Helmholtz equation, *Computer Methods in Applied Mechanics and Engineering*, **190** (2001), 31, 4027-4042.

[26] M. Lonsing and R. Verfürth, A posteriori error estimators for mixed finite element methods in linear elasticity, *Numerische Mathematik*, **97** (2004), 4, 757-778.

[27] J.E. Roberts and J.M. Thomas, Mixed and Hybrid Methods. In: Handbook of Numerical Analysis, edited by P.G. Ciarlet and J.L. Lions, vol. II, Finite Element Methods (Part 1), 1991, North-Holland, Amsterdam.

[28] R. Verfürth, A posteriori error estimation and adaptive mesh-refinement techniques, *Journal of Computational and Applied Mathematics*, **50** (1994), 1-3, 67-83.

[29] R. Verfürth, A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques, Wiley-Teubner (Chichester), 1996.