Journal of Computational Mathematics
Vol.32, No.2, 2014, 205–214.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1401-m3837

SUPERCONVERGENCE ANALYSIS OF THE STABLE CONFORMING RECTANGULAR MIXED FINITE ELEMENTS FOR THE LINEAR ELASTICITY PROBLEM*

Dongyang Shi
School of Mathematics and Statistics, Zhengzhou University, Zhengzhou 450001, China
Email: shi_dy@zzu.edu.cn
Minghao Li
School of Aerospace Engineering and Applied Mechanics, Tongji University, Shanghai 200092, China
Email: lyminghao@126.com

Abstract

In this paper, we consider the linear elasticity problem based on the Hellinger-Reissner variational principle. An $\mathcal{O}(h^2)$ order superclose property for the stress and displacement and a global superconvergence result of the displacement are established by employing a Clément interpolation, an integral identity and appropriate postprocessing techniques.

Mathematics subject classification: 65N15, 65N30.
Key words: Elasticity, Supercloseness, Global superconvergence.

1. Introduction

1.1. Introduction

In this paper, we consider the mixed finite element (for short MFE) approximation of a stress-displacement system derived from the Hellinger-Reissner variational principle for the linear elasticity problem. As is known to all, the MFE methods require that the pair of finite element spaces satisfying the B-B condition. Although there are a number of well-known stable MFEs for the analogous problems involving vector fields and scalar fields [1], the combination of the symmetry and continuity conditions of the stress field is a substantial additional difficulty. On the other hand, a lot of efforts, dating back four decades, have been devoted to develop stable MFEs for the linear elasticity problem, but no stable MFE scheme with polynomial shape functions are yielded. Not until the year 2002, were there some development in this direction. In [2], a sufficient condition was given and then a family of stable MFEs were constructed with respect to arbitrary triangular meshes, with 24 stress and 6 displacement degrees of freedom for the lowest order element, and an optimal order error estimate was obtained. An analogous family of conforming MFEs based on rectangular meshes were proposed in [3], involving 45 stress and 12 displacement degrees of freedom for the lowest order element. Two nonconforming triangular elements were presented in [4] with 12 degrees of freedom for the stress and 3 degrees of freedom for the displacement.

Although many stable elements have been constructed for this problem, they involve too much degrees of freedom. Recently, some more simple elements have been developed. In [5], a group of nonconforming rectangular elements were introduced, with the convergence order of

* Received August 9, 2011 / Revised version received December 4, 2013 / Accepted January 15, 2014 / Published online March 31, 2014 /

$\mathcal{O}(h)$ in L^2-norm for both the stress and the displacement, and the simplest element employed 12 degrees of freedom for the stress and 4 for the displacement. In [6], a family of conforming rectangular MFEs were proposed. It is closely related to one of the elements in [5]. Actually, the same finite element space is used for the displacement, while the space used to approximate the stress space is an extension of [5]. The lowest order pair in this family, with 17 degrees of freedom for the stress and 4 for the displacement, results in a convergence rate of $\mathcal{O}(h^2)$ for the stress and $\mathcal{O}(h)$ for the displacement in L^2-norm, respectively. In [7], a new family of minimal, any space-dimensional, symmetric, nonconforming mixed finite elements were presented. In 1D, it is nothing else but the 1D Raviart-Thomas element, which is the only conforming element in this family. In 2D and higher dimensions, they are new elements but of the minimal degrees of freedom. The total degrees of freedom for per element are 2 plus 1 in 1D, 7 plus 2 in 2D, and 15 plus 3 in 3D, respectively. In [8], the elements used in [7] were extended to conforming elements by enriching the spaces for both the stress and displacement, and the number of total degrees of freedom for per element are 10 plus 4 in 2D, and 21 plus 6 in 3D respectively, which are the simplest conforming rectangular elements so far.

On the other hand, the superconvergence study of the finite element methods is one of the most active topics for a long time in theoretical analysis and practical computations, and many valuable results about conforming and nonconforming finite elements have been obtained for different problems [9–16], but no consideration on this aspect is known about the finite elements of [6]. In this paper, at the first attempt, we will have a try to fill this gap. We obtain the supercloseness property of $\mathcal{O}(h^2)$ order for the stress and displacement and the superconvergence result of $\mathcal{O}(h^2)$ order for the displacement in L^2-norm through a Clément interpolation, an integral identity and interpolation postprocessing techniques.

The rest of this paper is organized as follows. In next section, some notations and preliminaries are introduced and the weak coercivity is established by the V-elliptic property and the B-B condition. Then we present the construction of finite element spaces in section 3. The last section is devoted to derive the supercloseness and global superconvergence of the displacement field.

2. Notations and Preliminaries

In this part, firstly we introduce some special functional spaces and operators. Let $\Omega \subset \mathcal{R}^2$ be a bounded convex domain, and $p, v = (v^{[1]}, v^{[2]})$ and $\tau = (\tau_{ij})_{2\times 2}$ be a function, vector-valued field and symmetric tensor, respectively. We define the following notions:

$$\operatorname{grad} p = \begin{pmatrix} \partial p/\partial x \\ \partial p/\partial y \end{pmatrix}, \qquad \operatorname{div} \tau = \begin{pmatrix} \partial \tau_{11}/\partial x + \partial \tau_{12}/\partial y \\ \partial \tau_{21}/\partial x + \partial \tau_{22}/\partial y \end{pmatrix},$$

$$\operatorname{grad} v = \begin{pmatrix} \partial v^{[1]}/\partial x & \partial v^{[1]}/\partial y \\ \partial v^{[2]}/\partial x & \partial v^{[2]}/\partial y \end{pmatrix}, \qquad \epsilon(v) = \frac{1}{2}(\operatorname{grad} v + (\operatorname{grad})^T v).$$

Let $\mathbb{S}$ denote the space of symmetric tensors, equipped with the inner product

$$(\sigma, \tau) = \int_\Omega \sigma : \tau, \quad \text{where} \quad \sigma : \tau = \sum_{i,j=1}^{2} \sigma_{ij} \tau_{ij}.$$

The space $H^k(\Omega, X)$ is defined as

$$H^k(\Omega, X) = \Big\{ v \in L^2(\Omega, X) \,|\, D^\alpha v \in L^2(\Omega, X), \forall\, |\alpha| \le k \Big\},$$

where X ranges $\mathcal{R}, \mathcal{R}^2$ or $\mathbb{S}$. If $X = \mathcal{R}$, we will simply use $H^k(\Omega)$ instead. The space $H(\mathrm{div}, \Omega, \mathbb{S})$ is defined by

$$H(\mathrm{div}, \Omega, \mathbb{S}) = \Big\{ \tau \in L^2(\Omega, \mathbb{S}) \,|\, \mathrm{div}\tau \in L^2(\Omega, \mathcal{R}^2) \Big\}.$$

We can check that $H(\mathrm{div}, \Omega, \mathbb{S})$ is a Hilbert space equipped with the norm

$$\|\tau\|_{H(\mathrm{div})} = \sqrt{\|\tau\|_0^2 + \|\mathrm{div}\tau\|_0^2}$$

and inner product

$$(\sigma, \tau)_{H(\mathrm{div})} = \int_\Omega \sigma : \tau + \mathrm{div}\sigma \cdot \mathrm{div}\tau.$$

Next we consider the linear elasticity problem in $\mathcal{R}^2$: given a body force f, find a symmetric stress tensor σ and a displacement u such that

$$\begin{cases} A\sigma = \epsilon(u), & \text{in } \ \Omega, \\ -\mathrm{div}\sigma = f, & \text{in } \ \Omega, \\ u = 0, & \text{on } \ \partial\Omega, \end{cases} \tag{2.1}$$

where the compliance tensor $A = A(x, y) : \mathbb{S} \to \mathbb{S}$ is bounded and symmetric positive definite uniformly for $(x, y) \in \Omega$.

The weak formulation of (2.1), based on the Hellinger-Reissner variational principle, is to find $(\sigma, u) \in \Sigma \times V$ such that

$$\begin{cases} a(\sigma, \tau) + b(\tau, u) = 0, & \forall \tau \in \Sigma, \\ b(\sigma, v) = G(v), & \forall v \in V, \end{cases} \tag{2.2}$$

where $\Sigma = H(\mathrm{div}, \ \Omega, \mathbb{S}), \ V = L^2(\Omega, \mathcal{R}^2)$,

$$a(\sigma, \tau) = \int_\Omega A\sigma : \tau, \quad b(\tau, v) = \int_\Omega \mathrm{div}\tau \cdot v, \quad G(v) = -\int_\Omega f \cdot v.$$

The bilinear form $a(\cdot, \cdot)$ is V-elliptic in the collection $Z = \{\tau \in \Sigma \,|\, b(\tau, v) = 0, \forall v \in V\}$ and the B-B condition is satisfied in the sense that there exist positive constants α and β such that

$$a(\tau, \tau) \geq \alpha \|\tau\|^2_{H(\mathrm{div})}, \qquad \forall \tau \in Z, \tag{2.3}$$

$$\sup_{\tau \in \Sigma} \frac{\int_\Omega \mathrm{div}\tau \cdot v}{\|\tau\|_{H(\mathrm{div})}} \geq \beta \|v\|_0, \quad \forall v \in V, \tag{2.4}$$

therefore, the problem (2.2) has a unique solution.

We rewrite this problem as: find $(\sigma, u) \in \Sigma \times V$ such that

$$Q((\sigma, u), (\tau, v)) = -G(v), \quad \forall (\tau, v) \in \Sigma \times V, \tag{2.5}$$

where

$$Q((\sigma, u), (\tau, v)) = \int_\Omega A\sigma : \tau + \int_\Omega \mathrm{div}\tau \cdot u - \int_\Omega \mathrm{div}\sigma \cdot v.$$

Then we define the norm on $\Sigma \times V$ by its square:

$$\|(\tau, v)\|^2 = \|\tau\|^2_{H(\mathrm{div})} + \|v\|_0^2.$$

The following lemma will play an important role in our supercloseness analysis, of which the similar proof can be found in [17]. For the sake of completeness, we give the proof sketchily.

Lemma 2.1. *If* (2.3) *and* (2.4) *hold, the following weak coercivity condition is true*

$$\sup_{(\tau,v)\in\Sigma\times V}\frac{Q((\sigma,u),(\tau,v))}{\|(\tau,v)\|}\geq C\,\|(\sigma,u)\|\,,\quad\forall(\sigma,u)\in\Sigma\times V. \tag{2.6}$$

Proof. Given $(\sigma,u)\in\Sigma\times V$, we consider the following two auxiliary problems:
Problem 1: find $(\sigma_1,u_1)\in\Sigma\times V$, such that

$$\begin{cases} a(\tau,\sigma_1)-b(\tau,u_1)=(\sigma,\tau)_{H(\mathrm{div})}, & \forall\tau\in\Sigma,\\ b(\sigma_1,v)=0, & \forall v\in V.\end{cases} \tag{2.7}$$

Problem 2: find $(\sigma_2,u_2)\in\Sigma\times V$, such that

$$\begin{cases} a(\tau,\sigma_2)-b(\tau,u_2)=0, & \forall\tau\in\Sigma,\\ b(\sigma_2,v)=(u,v), & \forall v\in V.\end{cases} \tag{2.8}$$

Obviously, each of (2.7) and (2.8) has a unique solution if (2.3) and (2.4) hold. We now set $(\tau,v)=((\sigma_1+\sigma_2),(u_1+u_2))$, then

$$\begin{aligned} Q((\sigma,u),(\tau,v))&=a(\sigma,\sigma_1+\sigma_2)+b(\sigma_1+\sigma_2,u)-b(\sigma,u_1+u_2)\\ &=\|\sigma\|^2_{H(\mathrm{div})}+\|u\|_0^2=\|(\sigma,u)\|^2,\end{aligned}$$

where (2.7) and (2.8) are used. Since $\|(\sigma_1,u_1)\|\leq C\|\sigma\|_{H(\mathrm{div})}$ and $\|(\sigma_2,u_2)\|\leq C\|u\|_0$, we get $\|(\tau,v)\|\leq C\,\|(\sigma,u)\|$, and finally

$$Q((\sigma,u),(\tau,v))\geq C\,\|(\sigma,u)\|\,\|(\tau,v)\|\,,$$

which implies the desired result. □

3. Construction of the Finite Element Spaces

Let $\Omega\subset\mathcal{R}^2$ be a rectangular domain with the boundary $\partial\Omega$ parallel to x-axis or y-axis in the plane, $\mathcal{T}_h$ be a family of axiparallel rectangular meshes of Ω, and h be the mesh size. For a given $K\in\mathcal{T}_h$, we denote the central point of the element K by (x_K,y_K), the length of edges parallel x-axis and y-axis by $2h_{x_K}$ and $2h_{y_K}$, respectively. Denote the four vertices by $d_i(i=1,2,3,4)$, and the four sides by $l_i=\overline{d_id_{i+1}}(i=1,2,3,4 \bmod 4)$, where $d_1=(x_K-h_{x_K},y_K-h_{y_K}),\ d_2=(x_K+h_{x_K},y_K-h_{y_K}),\ d_3=(x_K+h_{x_K},y_K+h_{y_K})$ and $d_4=(x_K-h_{x_K},y_K+h_{y_K})$. $n=(n_1,n_2)$ and $t=(-n_2,n_1)$ denote the unit normal vector and the unit tangential vector, respectively. The space $Q_{i,j}(K)$ consists of polynomials on K of degree at most i for x and j for y. The space $P_i(K)$ consists of polynomials on K of total degree at most i for x and y.

Let Σ_h and V_h be the finite element spaces for the stress field Σ and the displacement field V, respectively. To get a unique solution and to ensure a good approximation of the true solution, the pair Σ_h and V_h must satisfy the B-B condition. In [2], Arnold and Winther established a set of stability conditions instead of the B-B condition for the elasticity problem, that is

- (A1) $\mathrm{div}\Sigma_h\subset V_h$,
- (A2) There exists a linear operator $\Pi_h:H^1(\Omega,\ \mathbb{S})\to\Sigma_h$, bounded in $\mathcal{L}(H^1,L^2)$ uniformly with respect to h, such that $\mathrm{div}\Pi_h\sigma=P_h\mathrm{div}\sigma$ for all $\sigma\in H^1(\Omega,\mathbb{S})$, where $P_h:L^2(\Omega,\mathbb{R}^2)\to V_h$ denotes the L^2-projection operator.

In this paper, we define the finite element space V_h similar to the zero-degree Raviart-Thomas rectangular element to get the supercloseness and superconvergence result, and adapt the finite element space Σ_h and the linear operator Π_h as [6], which are constructed to satisfy conditions (A1) and (A2). Now we introduce the definitions of V_h, Σ_h, and Π_h.

- Firstly, let $V_K = (V_K^{[1]}, V_K^{[2]})$, where $V_K^{[1]} = Q_{0,1}(K)$, $V_K^{[2]} = Q_{1,0}(K)$. We define the degrees of freedom for V_K as follows:
  (e) $\int_{l_1} v^{[1]}ds$, $\int_{l_3} v^{[1]}ds$; $\int_{l_2} v^{[2]}ds$, $\int_{l_4} v^{[2]}ds$,
  that is, the two horizontal and vertical edge integral values are employed as the degrees of freedom for the finite element spaces defined by $Q_{0,1}(K)$ and $Q_{1,0}(K)$ respectively, which is similar to that for the zero-degree Raviart-Thomas rectangular element. The only difference lies that the order of the vector component is reversed.

- Secondly, we define

$$\Sigma_K = \left\{ \tau \in \begin{pmatrix} Q_{3,1}(K) & Q_{2,2}(K) \\ Q_{2,2}(K) & Q_{1,3}(K) \end{pmatrix} \Big| \, \mathrm{div}\tau \in V_K \right\}.$$

It can be checked that the explicit representation of Σ_K is as follows:

$$\Sigma_K = \begin{pmatrix} Q_{1,1}(K) & P_1(K) \oplus \mathrm{span}\{x^2, y^2\} \\ P_1(K) \oplus \mathrm{span}\{x^2, y^2\} & Q_{1,1}(K) \end{pmatrix}$$
$$\oplus \,\mathrm{span} \left\{ \begin{pmatrix} -\frac{1}{2}x^2 & xy \\ xy & -\frac{1}{2}y^2 \end{pmatrix}, \begin{pmatrix} -\frac{1}{3}x^3 & x^2y \\ x^2y & -xy^2 \end{pmatrix}, \begin{pmatrix} -x^2y & xy^2 \\ xy^2 & -\frac{1}{3}y^3 \end{pmatrix}, \begin{pmatrix} -\frac{2}{3}x^3y & x^2y^2 \\ x^2y^2 & -\frac{2}{3}xy^3 \end{pmatrix} \right\}.$$

The degrees of freedom for Σ_K are defined as

- (a) $\int_{l_i} \tau n \cdot nvds, \forall v \in P_1(l_i)$, (b) $\int_{l_i} \tau n \cdot tds$,
- (c) $\tau_{12}(d_i)$, (d) $\int_K \tau_{12}$,

where $P_1(l_i)$ denotes the space of linear polynomials on l_i (i=1,2,3,4). Then a stress field $\tau \in \Sigma_K$ can be determined by the degrees of freedom of (a), (b), (c) and (d) uniquely, which is the result of Lemma 1 in [6].

Thirdly, we introduce how to construct the operator Π_h. Because the vertex degrees of freedom (c) is employed for Σ_K, the canonical interpolation operator is not bounded with respect to the norm in $H^1(\Omega, \mathbb{S})$. To construct an effective interpolation operator with respect to the norm $H^1(\Omega)$, we first let $R_h : L^2(\Omega, \mathbb{S}) \to \Sigma_h^0 \subset \Sigma_h \cap H^1(\Omega, \mathbb{S})$ be a Clément interpolation [18] satisfying

$$\|R_h\tau - \tau\|_0 \le Ch^2 \|\tau\|_2, \tag{3.1}$$

where $\Sigma_h^0 = \{\tau \in C^0(\Omega, \mathbb{S}) \big| \, \tau_{ij}|_K \in Q_{1,1}(K), \forall K \in \mathcal{T}_h\}$. Then the interpolation operator $\Pi_h : H^1(\Omega, \mathbb{S}) \to \Sigma_h$, $\Pi_h|_K = \Pi_K$ is defined by

$$(\Pi_K\tau)_{12}(d_i) = (R_h\tau)_{12}(d_i), \tag{3.2}$$

$$\int_{l_i} (\Pi_K\tau - \tau)n \cdot nvds = 0, \forall v \in P_1(l_i), \tag{3.3}$$

$$\int_{l_i} (\Pi_K\tau - \tau)n \cdot tds = 0, \tag{3.4}$$

$$\int_K (\Pi_K\tau - \tau) : \phi = 0, \forall \phi \in \epsilon(V_K), \tag{3.5}$$

and the following properties of Π_h hold

$$\mathrm{div}\Pi_h\tau = P_h\mathrm{div}\tau, \tag{3.6}$$
$$\|\Pi_h\tau\|_0 \le C\|\tau\|_1, \tag{3.7}$$
$$\|\Pi_h\tau - \tau\|_0 \le Ch^2\|\tau\|_2. \tag{3.8}$$

In fact, property (3.6) can be proved by use of (3.3)-(3.5) and Green's formula, which is the result of lemma 2 in [6]. Properties (3.7) and (3.8) can be proved by use of (3.1),(3.2) and the standard scaling arguments, and a direct proof was given in [6]. Then (3.6) and (3.7) imply that condition (A2) is satisfied.

Finally, we give the finite element spaces as follows

$$V_h = \Big\{ v \in L^2(\Omega, \mathcal{R}^2) \,|\, v|_K \in V_K, \forall K \in \mathcal{T}_h \Big\},$$
$$\Sigma_h = \Big\{ \tau \in L^2(\Omega, \mathbb{S}) \,|\, \tau|_K \in \Sigma_K, \forall K \in \mathcal{T}_h \Big\}.$$

From the definitions of Σ_K and V_K, we can check that condition (A1) is also satisfied. The conforming finite element approximation of (2.2) reads as: find $(\sigma_h, u_h) \in \Sigma_h \times V_h$ such that

$$\begin{cases} a(\sigma_h, \tau_h) + b(\tau_h, u_h) = 0, & \forall \tau_h \in \Sigma_h, \\ b(\sigma_h, v_h) = G(v_h), & \forall v_h \in V_h. \end{cases} \tag{3.9}$$

Because (A1) and (A2) are satisfied, the approximation problem (3.9) has a unique solution.

The error equation can be obtained instantly from (2.2) and (3.9) as follows

$$Q((\sigma - \sigma_h, u - u_h), (\tau_h, v_h)) = 0, \quad \forall (\tau_h, v_h) \in \Sigma_h \times V_h. \tag{3.10}$$

4. Supercloseness and Superconvergence Analysis

To get the supercloseness and superconvergence result, we first define the interpolation operator $I_K : L^2(\Omega, \mathcal{R}^2) \to V_K, I_K = (I_K^{[1]}, I_K^{[2]})$ as follows

$$\int_{l_i} (v^{[1]} - I_K^{[1]} v^{[1]}) ds = 0, \;\; i = 1, 3; \quad \int_{l_i} (v^{[2]} - I_K^{[2]} v^{[2]}) ds = 0, \;\; i = 2, 4, \tag{4.1}$$

then $I_h = (I_h^{[1]}, I_h^{[2]})$ is defined by $I_h|_K = I_K$.

Let $w^{[1]} = u^{[1]} - I_K^{[1]} u^{[1]}, w^{[2]} = u^{[2]} - I_K^{[2]} u^{[2]}$, and the error functions $F(y) = \frac{1}{2}((y - y_K)^2 - h_{y_K}^2)$, $F^2(y) = (F(y))^2$, then we will introduce the following important lemma. It can be derived similarly from Lemma 1.1 in [15], but for the sake of completeness, we give the proof.

Lemma 4.1. *If $u \in H^2(\Omega, \mathcal{R}^2)$, then for all $v \in V_h$, there holds*

$$\int_\Omega (u - I_h u) \cdot v = O(h^2)|u|_2\|v\|_0. \tag{4.2}$$

Proof. For $v^{[1]} \in V_h^{[1]}$ and $K \in \mathcal{T}_h$, there holds

$$v^{[1]}(x, y) = v^{[1]}(x_K, y_K) + (y - y_K)v_y^{[1]}.$$

Then

$$\int_K w^{[1]} v^{[1]} = \int_K w^{[1]} v^{[1]}(x_K, y_K) + \int_K w^{[1]} (y - y_K) v_y^{[1]}.$$

Note that $(F(y))', (F^2(y))''$ are constants and $F(y), (F^2(y))'$ vanish when restricted to $l_1,\ l_3$, where superscripts $'$ and $''$ denote derivative, and $\int_{l_1} w^{[1]}ds = 0,\ \int_{l_3} w^{[1]}ds = 0$, we have

$$\begin{aligned}\int_K w^{[1]} &= \int_K (F(y))'' w^{[1]} = \int_{l_3} (F(y))' w^{[1]} ds - \int_{l_1} (F(y))' w^{[1]} ds) - \int_K (F(y))' w^{[1]}_y \\ &= -\Big(\int_{l_3} F(y) w^{[1]}_y ds - \int_{l_1} F(y) w^{[1]}_y ds\Big) + \int_K F(y) w^{[1]}_{yy} = \int_K F(y) u^{[1]}_{yy},\end{aligned}$$

and

$$\begin{aligned}\int_K w^{[1]}(y-y_K) &= \frac{1}{6}\int_K (F^2(y))''' w^{[1]} \\ &= \frac{1}{6}\Big(\int_{l_3} (F^2(y))'' w^{[1]} ds - \int_{l_1} (F^2(y))'' w^{[1]} ds\Big) - \frac{1}{6}\int_K (F^2(y))'' w^{[1]}_y \\ &= -\frac{1}{6}\Big(\int_{l_3} (F^2(y))' w^{[1]}_y ds - \int_{l_1} (F^2(y))' w^{[1]}_y ds\Big) + \frac{1}{6}\int_K (F^2(y))' w^{[1]}_{yy} \\ &= \frac{1}{3}\int_K F(y)(y-y_K) u^{[1]}_{yy}.\end{aligned}$$

By Hölder inequality and inverse inequality, we get

$$\begin{aligned}\int_K w^{[1]} v^{[1]} &= \int_K F(y) u^{[1]}_{yy} (v^{[1]} - (y-y_K) v^{[1]}_y) + \frac{1}{3}\int_K F(y)(y-y_K) u^{[1]}_{yy} v^{[1]}_y \\ &= O(h^2_{y_K}) \left|u^{[1]}\right|_{2,K} \left\|v^{[1]}\right\|_{0,K}.\end{aligned} \tag{4.3}$$

Similarly, we can get

$$\int_K w^{[2]} v^{[2]} = O(h^2_{x_K}) \left|u^{[2]}\right|_{2,K} \left\|v^{[2]}\right\|_{0,K}. \tag{4.4}$$

Then the desired result follows from (4.3) and (4.4) by summing all K over $\mathcal{T}_h$. $\square$

Based on Lemma 4.1, we have

Theorem 4.1. *Assume that (σ, u) and (σ_h, u_h) are the solutions of (2.2) and (3.9), respectively. If $(\sigma, u) \in H^2(\Omega, \mathbb{S}) \times H^2(\Omega, \mathcal{R}^2)$, then there holds the following superclose property*

$$\|(\sigma_h - \Pi_h\sigma, u_h - I_h u)\| = O(h^2)(\|\sigma\|_2 + \|u\|_2). \tag{4.5}$$

Proof. By (2.6) and (3.10), we have

$$\begin{aligned}\|(\sigma_h - \Pi_h\sigma, u_h - I_h u)\| &\le C \sup_{(\tau_h, v_h)\in\Sigma_h\times V_h} \frac{Q((\sigma_h - \Pi_h\sigma, u_h - I_h u), (\tau_h, v_h))}{\|(\tau_h, v_h)\|} \\ &\le C \sup_{(\tau_h, v_h)\in\Sigma_h\times V_h} \frac{Q((\sigma - \Pi_h\sigma, u - I_h u), (\tau_h, v_h))}{\|(\tau_h, v_h)\|} \\ &= C \sup_{(\tau_h, v_h)\in\Sigma_h\times V_h} \frac{1}{\|(\tau_h, v_h)\|}\Big\{ \int_\Omega A(\sigma - \Pi_h\sigma) : \tau_h \\ &\qquad + \int_\Omega \mathrm{div}\tau_h \cdot (u - I_h u) - \int_\Omega \mathrm{div}(\sigma - \Pi_h\sigma)\cdot v_h \Big\}.\end{aligned}$$

Then, (3.6), (3.8) and (4.2) imply that

$$\int_\Omega \mathrm{div}(\sigma-\Pi_h\sigma)\cdot v_h=\int_\Omega(\mathrm{div}\sigma-P_h\mathrm{div}\sigma)\cdot v_h=0, \tag{4.6}$$

$$\int_\Omega A(\sigma-\Pi_h\sigma):\tau_h\le C\|\sigma-\Pi_h\sigma\|_0\|\tau_h\|_0\le Ch^2\|\sigma\|_2\|\tau_h\|_0, \tag{4.7}$$

$$\int_\Omega \mathrm{div}\tau_h\cdot(u-I_hu)\le Ch^2\,\|u\|_2\,\|\mathrm{div}\tau_h\|_0. \tag{4.8}$$

Combining the above three estimates yields the desired result.

Remark 4.1. We point out that the result of Theorem 4.1 can be extended to arbitrary degree polynomial finite elements of [3] and [6] easily.

Next we derive the superconvergence by employing interpolation postprocessing techniques. In order to do this, we merge the adjacent four elements into one big element $\tilde{K}$, $\tilde{K}=\bigcup_{i=1}^{4}K_i$ (see Fig. 4.1), and denote the partition by $\mathcal{T}_{2h}$.

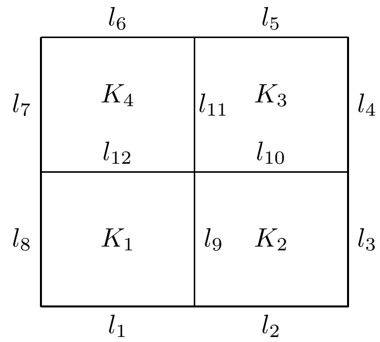

Fig. 4.1 Big element $\tilde{K}$.

Similar to the zero-degree Raviart-Thomas rectangular element (see [15] or [16]), we construct the following postprocessing interpolation operator $I_{2h}=(I_{2h}^{[1]},I_{2h}^{[2]})$ on $\tilde{K}\in\mathcal{T}_{2h}$ as

$$\begin{cases} I_{2h}u|_{\tilde{K}}\in Q_{1,1}(\tilde{K})\times Q_{1,1}(\tilde{K}),\\ \int_{l_i}(u^{[1]}-I_{2h}^{[1]}u^{[1]})\,ds=0, \quad i=1,2,5,6,\\ \int_{l_i}(u^{[2]}-I_{2h}^{[2]}u^{[2]})\,ds=0, \quad i=3,4,7,8. \end{cases} \tag{4.9}$$

Then, there holds the following lemma.

Lemma 4.2. *For all $u\in H^2(\Omega,\mathcal{R}^2)$, the interpolation operator I_{2h} satisfies*

$$I_{2h}I_hu=I_{2h}u, \tag{4.10}$$

$$\|I_{2h}u-u\|_0\le Ch^2\|u\|_2, \tag{4.11}$$

$$\|I_{2h}v\|_0\le C\|v\|_0,\forall v\in V_h. \tag{4.12}$$

Proof. Firstly, (4.11) can be obtained directly from the interpolation theory. Secondly, by the definitions of I_{2h} and I_h, we can see

$$\int_{l_i}I_{2h}^{[1]}I_h^1u^{[1]}ds=\int_{l_i}I_h^{[1]}u^{[1]}ds=\int_{l_i}u^{[1]}ds=\int_{l_i}I_{2h}^{[1]}u^{[1]}ds, \quad i=1,2,5,6.$$

Then, from the well-poseness of I_{2h}, we have $I_{2h}^{[1]}I_h^{[1]}u^{[1]} = I_{2h}^{[1]}u^{[1]}$. Similarly, $I_{2h}^{[2]}I_h^{[2]}u^{[2]} = I_{2h}^{[2]}u^{[2]}$. So (4.10) is true.

Lastly, we will prove (4.12). Let $2h_{x_{\tilde{K}}}$ and $2h_{y_{\tilde{K}}}$ be the length of edges of $\tilde{K}$ parallel x-axis and y-axis, respectively, $\hat{\tilde{K}}$ be the reference element of of $\tilde{K}$, F be an affine mapping from $\hat{\tilde{K}}$ to $\tilde{K}$. $\forall v \in V_h, \hat{v} = v \circ F$, let

$$\hat{v}_i^{[1]} \triangleq \int_{\hat{l}_i} \hat{v}^{[1]} d\hat{s}, \quad i = 1, 2, 5, 6.$$

Then

$$\begin{aligned}\hat{I}_{2h}^{[1]}\hat{v}^{[1]} =& \frac{1}{4}\left(\hat{v}_1^{[1]} + \hat{v}_2^{[1]} + \hat{v}_5^{[1]} + \hat{v}_6^{[1]}\right) + \frac{1}{2}\left(-\hat{v}_1^{[1]} + \hat{v}_2^{[1]} + \hat{v}_5^{[1]} - \hat{v}_6^{[1]}\right)\xi \\ &+ \frac{1}{4}\left(-\hat{v}_1^{[1]} - \hat{v}_2^{[1]} + \hat{v}_5^{[1]} + \hat{v}_6^{[1]}\right)\eta + \frac{1}{2}\left(\hat{v}_1^{[1]} - \hat{v}_2^{[1]} + \hat{v}_5^{[1]} - \hat{v}_6^{[1]}\right)\xi\eta.\end{aligned}$$

By trace theory and norm equivalence lemma, we have

$$\begin{aligned}\left\|\hat{I}_{2h}^{[1]}\hat{v}^{[1]}\right\|_{0,\hat{\tilde{K}}} &\le C\left(\left|\hat{v}_1^{[1]}\right| + \left|\hat{v}_2^{[1]}\right| + \left|\hat{v}_5^{[1]}\right| + \left|\hat{v}_6^{[1]}\right|\right)\left(\left|\hat{\tilde{K}}\right| + \|\xi\|_{0,\hat{\tilde{K}}} + \|\eta\|_{0,\hat{\tilde{K}}} + \|\xi\eta\|_{0,\hat{\tilde{K}}}\right) \\ &\le C\sum_{i=1,2,5,6}\left\|\hat{v}^{[1]}\right\|_{0,\hat{l}_i} \le C\sum_{i=1,2,3,4}\left\|\hat{v}^{[1]}\right\|_{1,\hat{K}_i} \le C\left\|\hat{v}^{[1]}\right\|_{0,\hat{\tilde{K}}}, \\ \left\|I_{2h}^{[1]}v^{[1]}\right\|_{0,\tilde{K}} &= h_{x_{\tilde{K}}}h_{y_{\tilde{K}}}\left\|\hat{I}_{2h}^{[1]}\hat{v}^{[1]}\right\|_{0,\hat{\tilde{K}}} \le Ch_{x_{\tilde{K}}}h_{y_{\tilde{K}}}\left\|\hat{v}^{[1]}\right\|_{0,\hat{\tilde{K}}} \le C\left\|v^{[1]}\right\|_{0,\tilde{K}}.\end{aligned}$$

Similarly,

$$\left\|I_{2h}^{[2]}v^{[2]}\right\|_{0,\tilde{K}} \le C\left\|v^{[2]}\right\|_{0,\tilde{K}}.$$

So (4.12) holds. The proof is complete. □

Theorem 4.2. *Under the assumptions of Theorem* 4.1*, we have the following superconvengence result*

$$\|I_{2h}u_h - u\|_0 = O(h^2)(\|\sigma\|_2 + \|u\|_2). \tag{4.13}$$

Proof. By Theorem 4.1 and Lemma 4.2, we have

$$\begin{aligned}\|I_{2h}u_h - u\|_0 &\le \|I_{2h}u_h - I_{2h}I_hu\|_0 + \|I_{2h}I_hu - u\|_0 \\ &\le C\|u_h - I_hu\|_0 + \|I_{2h}u - u\|_0 \\ &= O(h^2)(\|\sigma\|_2 + \|u\|_2).\end{aligned}$$

The proof is complete. □

Remark 4.2. The superconvergence result in L^2-norm of Theorem 4.2 cannot be derived by the elements of [5] for their consistency errors can only be estimated with order $O(h)$ instead of order $O(h^2)$ in the sense of the broken $H(\text{div})$ norm.

5. Conclusion

In this paper, we propose a method to analyze the superconvergence phenomenon of the linear elasticity problem for some rectangular conforming MFEMs in [3] and [6]. Indeed, these rectangular MFEMs involve many degrees of freedom, and the finite element space of stress is not easy to be constructed, so the numerical implementation is not convenient. Recently, some more simple and excellent elements have been developed in [7] and [8], moreover, the superconvergence phenomenon has been observed in numerical tests of [8]. In our next work we will apply our method to analyze the element of [8].

Acknowledgments. This research is supported by NSF of China (10971203, 11271340, 11101381), Specialized Research Fund for the Doctoral Program of Higher Education (20094101 110006).

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