Journal of Computational Mathematics
Vol.32, No.4, 2014, 442–455.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1403-m4390

PRECONDITIONED HSS-LIKE ITERATIVE METHOD FOR SADDLE POINT PROBLEMS*

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Abstract

A new HSS-like iterative method is first proposed based on HSS-like splitting of non-Hermitian (1,1) block for solving saddle point problems. The convergence analysis for the new method is given. Meanwhile, we consider the solution of saddle point systems by preconditioned Krylov subspace method and discuss some spectral properties of the preconditioned saddle point matrices. Numerical experiments are given to validate the performances of the preconditioners.

Mathematics subject classification: 65F10, 65F50.
Key words: Saddle point problem, Non-Hermitian positive definite matrix, HSS-like splitting, Preconditioning.

1. Introduction

We consider the solution of the following saddle point linear system

$$\mathcal{A}\mathbf{x} = \begin{bmatrix} A & B^* \\ -B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} = \mathbf{b}, \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}$ is a non-Hermitian positive definite matrix, that is, the matrix $H = (A+A^*)/2$, the Hermitian part of A, is positive definite, $B \in \mathbb{C}^{m\times n}$, with $m \le n$, has full row rank. Such linear systems arise in a large number of scientific computing and engineering applications (see for instance [11-12, 18-21, 25-26, 31-32, 34]). As such systems are typically large and sparse, iterative methods become more attractive than direct methods for solving the saddle point problem (1.1). Solution by iterative methods can be found in the literature, such as Uzawa-type schemes [16, 18, 36], SOR-like and GSOR iterative methods [14, 16, 31, 37], matrix splitting methods [1-4, 6-14, 17, 23-24, 27-30, 33], iterative projection methods [35], restrictively preconditioned conjugate gradient (RPCG) methods [5, 15] and iterative null space methods [18], and so on.

In [6], Bai, Golub and Ng presented an Hermitian and skew-Hermitian splitting (HSS) method for solving non-Hermitian positive definite linear systems. The use of HSS as a stationary iteration for solving saddle point systems has been proposed in [2-3, 9, 13], where it

* Received May 14, 2013 / Revised version received December 23, 2013 / Accepted March 25, 2014 / Published online July 3, 2014 /

was shown that the iteration converges for a large class of problems. Bai, Golub and Ng [8, 13] further generalized HSS to positive-definite and skew-Hermitian splitting (PSS), normal and skew-Hermitian splitting (NSS) and considered preconditioners based on these splittings. Pan, Ng and Bai [28 proposed two preconditioners for the saddle point problem with a non-Hermitian positive definite (1,1) block A, using the HSS and PSS of A, not based on using of the coefficient matrix $\mathcal{A}$ as a preconditioner for Krylov subspace methods. Recently, Jiang and Cao [23] presented a local Hermitian and skew-Hermitian iterative method and analyzed the convergence of the LHSS method. Zhang, Ren and Zhou [33] also presented an HSS-based constraint preconditioner, in which the (1,1) block of the preconditioner is constructed by the HSS method for solving the non-Hermitian positive definite linear systems.

In this paper, we propose a new HSS-like iterative method for the saddle point problem (1.1) based on the HSS of the (1,1) block A. We mainly focus on the case that A is a non-Hermitian positive definite matrix with the Hermitian part. We first establish a new HSS-like iterative method for the saddle point problem (1.1) and then give the convergence analysis of the new method in Section 2. In Section 3, we will show that the HSS-like iteration can provide an effective preconditioner for Krylov subspace methods applied to (1.1). Meanwhile, we present a modified HSS-like preconditioner and give spectral analysis of the preconditioned matrix. Numerical experiments are presented in Section 4. Meanwhile, we draw some conclusions.

2. The New HSS-like Iteration Method

From now on, we will adopt the general notation

$$\mathcal{A} = \begin{bmatrix} A & B^* \\ -B & 0 \end{bmatrix} \tag{2.1}$$

to represent the non-Hermitian saddle point matrix of Eq. (1). We assume that A is non-Hermitian positive definite, and that B is of size $m\times n$ and has full row rank. Let $H = (A+A^*)/2$ and $S = (A - A^*)/2$ be its Hermitian and skew-Hermitian parts.

Let $\alpha > 0$ be a parameter, and consider the following splitting of A,

$$A = M_\alpha - N_\alpha = \frac{1}{2\alpha}(\alpha I + H)(\alpha I + S) - \frac{1}{2\alpha}(\alpha I - H)(\alpha I - S).$$

Note that A is non-Hermitian positive definite. Then $M_\alpha = \frac{1}{2\alpha}(\alpha I+H)(\alpha I+S)$ is nonsingular. Thus we make the following special splitting:

$$\begin{bmatrix} A & B^* \\ -B & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2\alpha}(\alpha I + H)(\alpha I + S) & 0 \\ -B & Q \end{bmatrix} - \begin{bmatrix} \frac{1}{2\alpha}(\alpha I - H)(\alpha I - S) & -B^* \\ 0 & Q \end{bmatrix},$$

where Q is a Hermitian positive definite matrix. We propose a new iterative method based on this special splitting.

Given an initial guess $x_0 \in \mathbb{R}^n, y_0 \in \mathbb{R}^m$, the new HSS-like iteration is given as follows:

$$\begin{bmatrix} M_\alpha & 0 \\ -B & Q \end{bmatrix} \begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} N_\alpha & -B^* \\ 0 & Q \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix} + \begin{bmatrix} f \\ g \end{bmatrix},$$

or equivalently, it can be written as

$$\begin{cases} x_{k+1} = x_k + M_\alpha^{-1}(f - Ax_k - B^*y_k), \\ y_{k+1} = y_k + Q^{-1}(Bx_{k+1} + g). \end{cases} \tag{2.2}$$

Remark 2.1. When the (1,1) block A of the saddle point matrix is Hermitian positive definite, the HSS-like iterative method (2.2) is a special case of the PIU method in [16].

In the following, we consider the convergence property of the local HSS-like iteration. Note that the iteration matrix of this iteration scheme is

$$\mathcal{T}_\alpha = \begin{bmatrix} M_\alpha & 0 \\ -B & Q \end{bmatrix}^{-1} \begin{bmatrix} N_\alpha & -B^* \\ 0 & Q \end{bmatrix}. \tag{2.3}$$

Let $\rho(\mathcal{T}_\alpha)$ denote the spectral radius of the iteration matrix $\mathcal{T}_\alpha$. Then the local HSS-like iteration converges if and only if $\rho(\mathcal{T}_\alpha) < 1$. Let λ be an eigenvalue of $\mathcal{T}_\alpha$ and $(u^*, v^*)^*$ be a corresponding eigenvector, where $u \in \mathbb{C}^n$ and $v \in \mathbb{C}^m$. Then we have

$$\mathcal{T}_\alpha \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} u \\ v \end{bmatrix},$$

or equivalently,

$$\begin{cases} (\alpha^2 - \alpha A + HS)u - 2\alpha B^* v = \lambda(\alpha^2 + \alpha A + HS)u, \\ \lambda B u = (\lambda - 1) Q v. \end{cases} \tag{2.4}$$

To prove the convergence of the iterative scheme (2.2), we first assume $\lambda \neq 0$ and give some useful lemmas.

Lemma 2.1. *Let A be a non-Hermitian matrix, with the Hermitian part $H = (A + A^*)/2$ being positive definite, and the matrix B has full row rank. If λ is an eigenvalue of iteration matrix $\mathcal{T}_\alpha$ defined by* (2.3)*, then $\lambda \neq 1$.*

Proof. If $\lambda = 1$ and $(u^*, v^*)^*$ be the corresponding eigenvector, then from (2.4) we have

$$\begin{cases} Au + B^* v = 0, \\ -Bu = 0. \end{cases} \tag{2.5}$$

Note that the above equation can be rewritten as

$$\begin{bmatrix} A & B^* \\ -B & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0. \tag{2.6}$$

It is easy to know that the coefficient matrix of (2.6) is nonsingular. Hence $u = 0$ and $v = 0$, which contradicts the assumption that $(u^*, v^*)^*$ is an eigenvector of the iteration matrix $\mathcal{T}_\alpha$. So $\lambda \neq 1$. □

Lemma 2.2. *Let A be a non-Hermitian matrix with the positive definite Hermitian part $H = (A + A^*)/2$, and the skew-Hermitian part $S = (A - A^*)/2$. Let the matrix B have full row rank. If $(u^*, v^*)^*$ is an eigenvector of the iteration matrix $\mathcal{T}_\alpha$ corresponding to the eigenvalue λ, then $u \neq 0$. Moreover, if $v = 0$, then $|\lambda| < 1$.*

Proof. If $u = 0$, then from (2.4) we have $B^* v = 0$ and $Qv = 0$. Since B has full row-rank and Q is a Hermitian positive definite matrix, we have $v = 0$, which contradicts the assumption that $(u^*, v^*)^*$ is an eigenvector. Therefore, $u \neq 0$.

If $v = 0$, then from (2.4), we have

$$(\alpha^2 - \alpha A + HS)u = \lambda(\alpha^2 + \alpha A + HS)u.$$

That is to say,

$$(\alpha I - H)(\alpha I - S)u = \lambda(\alpha I + H)(\alpha I + S)u,$$

which is equivalent to

$$\frac{1}{2\alpha}(\alpha I - H)(\alpha I - S)u = \lambda\frac{1}{2\alpha}(\alpha I + H)(\alpha I + S)u. \tag{2.7}$$

If we define $M_\alpha = \frac{1}{2\alpha}(\alpha I + H)(\alpha I + S)$ and $N_\alpha = \frac{1}{2\alpha}(\alpha I - H)(\alpha I - S)$, then (2.7) can be rewritten as

$$M_\alpha^{-1}N_\alpha u = \lambda u.$$

Note that $u \neq 0$, we know that λ is an eigenvalue of $M_\alpha^{-1}N_\alpha$. From [6], we get that $|\lambda| < 1$, for $\forall\alpha > 0$. $\square$

Lemma 2.3. ([16, 38]) *Both roots of the complex quadratic equation $\lambda^2 + \phi\lambda + \psi = 0$ have modulus less than one if and only if $|\psi| < 1$ and $|\phi - \bar{\phi}\psi| < 1 - |\psi|^2$, where $\bar{\phi}$ denotes the conjugate complex of ϕ.*

Theorem 2.1. *Let A be a non-Hermitian matrix with the positive definite Hermitian part $H = (A + A^*)/2$ and the skew-Hermitian part $S = (A - A^*)/2$. Let the matrix B have full row rank and Q be a Hermitian positive definite matrix. Assume that $(u^*, v^*)^*$ is an eigenvector of the iteration matrix $\mathcal{T}_\alpha$ corresponding to the eigenvalue λ. Denote*

$$u^*Au = a + bi, \qquad u^*HSu = c + di, \qquad u^*B^*Q^{-1}Bu = e.$$

Then the local HSS-like iteration is convergent if a, b, c, d, e satisfy the following condition:

$$a\alpha^2 + ac + bd > 0 \quad and \quad 0 \le e < \frac{2a\alpha(a\alpha^2 + ac + bd)}{a^2\alpha^2 + d^2}.$$

Proof. Let λ be an eigenvalue of $\mathcal{T}_\alpha$ and $(u^*, v^*)^*$ be a corresponding eigenvector. From Lemmas 2.1 and 2.2, we have $\lambda \neq 1$ and $u \neq 0$, without loss of generality, we further assume $u^*u = 1$. From the second equality of the Eq. (2.4), we have

$$v = \frac{\lambda}{\lambda - 1}Q^{-1}Bu. \tag{2.8}$$

If $Bu = 0$, it follows from (2.8) that $v = 0$. From Lemma 2.2, we have $|\lambda| < 1$.

If $Bu \neq 0$, which means that $e > 0$ according to the definition of e. By substituting (2.8) into the first equality of (2.4), we get

$$(\lambda - 1)(\alpha^2 I - \alpha A + HS)u - 2\alpha\lambda B^*Q^{-1}Bu = \lambda(\lambda - 1)(\alpha^2 I + \alpha A + HS)u.$$

Multiplying both sides of this equality from left with u^*, we have

$$(\alpha^2 + \alpha u^*Au + u^*HSu)\lambda^2 - 2(\alpha^2 + u^*HSu - \alpha u^*B^*Q^{-1}Bu)\lambda + (\alpha^2 - \alpha u^*Au + u^*HSu) = 0. \tag{2.9}$$

If $(\alpha^2 + \alpha u^*Au + u^*HSu) = 0$, then $\alpha^2 + u^*HSu = -\alpha u^*Au$. From (2.9), we have

$$\lambda = \frac{\alpha^2 - \alpha u^*Au + u^*HSu}{2(\alpha^2 + u^*HSu - \alpha u^*B^*Q^{-1}Bu)} = \frac{u^*Au}{u^*Au + u^*B^*Q^{-1}Bu}.$$

Since $u^*Au = a + bi$, $a > 0$, $u^*B^*Q^{-1}Bu = e > 0$, we get $|\lambda| < 1$.

If $(\alpha^2+\alpha u^*Au+u^*HSu)\neq 0$, from (2.9), we have

$$\lambda^2-2\frac{\alpha^2+u^*HSu-\alpha u^*B^*Q^{-1}Bu}{\alpha^2+\alpha u^*Au+u^*HSu}\lambda+\frac{\alpha^2-\alpha u^*Au+u^*HSu}{\alpha^2+\alpha u^*Au+u^*HSu}=0.$$

Note that $u^*Au=a+bi, u^*HSu=c+di$ and $u^*B^*Q^{-1}Bu=e$, we have

$$\lambda^2-2\frac{(\alpha^2-e\alpha+c)+di}{(\alpha^2+a\alpha+c)+(d+b\alpha)i}\lambda+\frac{(\alpha^2-a\alpha+c)+(d-b\alpha)i}{(\alpha^2+a\alpha+c)+(d+b\alpha)i}=0. \tag{2.10}$$

Now, according to Lemma 2.3 we know that both roots of the complex quadratic Eq. (2.10) satisfy $|\lambda|<1$ if and only if

$$\left|\frac{(\alpha^2-a\alpha+c)+(d-b\alpha)i}{(\alpha^2+a\alpha+c)+(d+b\alpha)i}\right|<1, \tag{2.11}$$

and

$$\left|\frac{-4a\alpha(\alpha^2-e\alpha+c)-4bd\alpha-4de\alpha i}{(\alpha^2+a\alpha+c)^2+(d+b\alpha)^2}\right|<\frac{4a\alpha(\alpha^2+c)+4bd\alpha}{(\alpha^2+a\alpha+c)^2+(d+b\alpha)^2}. \tag{2.12}$$

By simplifying the inequality (2.11) and (2.12) we immediately obtain the condition that we demonstrate. □

Remark 2.2. For the iteration matrix $\mathcal{T}_\alpha$, it needs compute the inverse of M_α. We remark that exact inverses of the matrices $\alpha I+H$ and $\alpha I+S$ are quite expensive, and therefore, some further approximations, e.g., the incomplete Cholesky factorization and the incomplete orthogonal-triangular factorization to these two matrices may be respectively adopted in actual applications [6].

3. Krylov Subspace Acceleration

In this Section, we will show that the HSS-like iteration method can provide effective preconditioners for Krylov subspace methods applied to (1.1).

Let

$$\mathcal{M}_\alpha=\begin{bmatrix}\frac{1}{2\alpha}(\alpha I+H)(\alpha I+S) & 0\\ -B & Q\end{bmatrix},\quad \mathcal{N}_\alpha=\begin{bmatrix}\frac{1}{2\alpha}(\alpha I-H)(\alpha I-S) & -B^*\\ 0 & Q\end{bmatrix}.$$

It is easy to see that there is a unique splitting $\mathcal{A}=\mathcal{M}_\alpha-\mathcal{N}_\alpha$ with $\mathcal{M}_\alpha$ nonsingular such that the iteration matrix $\mathcal{T}_\alpha$ is the matrix induced by that splitting, i.e.,

$$\mathcal{T}_\alpha=\mathcal{M}_\alpha^{-1}\mathcal{N}_\alpha=\mathcal{I}-\mathcal{M}_\alpha^{-1}\mathcal{A},$$

where $\mathcal{I}$ denotes the identity matrix. It is therefore possible to rewrite the iteration (2.2) in correction form:

$$\mathbf{x}_{n+1}=\mathbf{x}_n+\mathcal{M}_\alpha^{-1}\mathbf{r}_n,\quad \mathbf{r}_n=\mathbf{b}-\mathcal{A}\mathbf{x}_n,$$

where $\mathbf{x}=(x^*,y^*)^*$, $\mathbf{b}=(f^*,g^*)^*$. This will be useful when we consider Krykov subspace acceleration.

Obviously, the linear system $\mathcal{A}\mathbf{x}=\mathbf{b}$ is equivalent to the linear system

$$(\mathcal{I}-T_\alpha)\mathbf{x}=\mathcal{M}_\alpha^{-1}\mathcal{A}\mathbf{x}=\mathcal{M}_\alpha^{-1}\mathbf{b}.$$

This equivalent system can be solved with GMRES. Hence, the matrix $\mathcal{M}_\alpha$ can be seen as a preconditioner for GMRES. That is, the preconditioner $\mathcal{M}_\alpha$ is used to accelerate the convergence rate GMRES applied to $\mathcal{A}\mathbf{x} = \mathbf{b}$.

We can use

$$\mathcal{M}_\alpha = \begin{bmatrix} \frac{1}{2\alpha}(\alpha I + H)(\alpha I + S) & 0 \\ -B & Q \end{bmatrix}$$

as an HSS-like preconditioner. Application of the preconditioner within GMRES requires solving a linear system of the form

$$\mathcal{M}_\alpha \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2\alpha}(\alpha I + H)(\alpha I + S) & 0 \\ -B & Q \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

at each iteration. This is done by first solving

$$(\alpha I + H)(\alpha I + S) z_1 = 2\alpha r_1 \tag{3.1}$$

for z_1, and followed by

$$-Bz_1 + Qz_2 = r_2.$$

For Eq. (3.1), we can solve it by first solving $(\alpha I + H)v = 2\alpha r_1$ and followed by $(\alpha I + S)z_1 = v$.

Under the assumptions of Theorem 2.1, since $\mathcal{M}_\alpha^{-1}\mathcal{A} = \mathcal{I} - T_\alpha$, it is easy to see that for all $\alpha > 0$ the eigenvalues of the preconditioned matrix $\mathcal{M}_\alpha^{-1}\mathcal{A}$ are entirely contained in the open disk of radius 1 centered at $(1, 0)$. In particular, the smaller the spectral radius of T_α is, the more clustered the eigenvalues of the preconditioned matrix (around 1) will be. A clustered spectrum often translates in rapid convergence of GMRES, see [18], but careful attention must be paid to the conditioning and eigenvalue distribution of the matrix $\mathcal{A}$ itself, which determines convergence rate of the inner iteration, see [22] for a comprehensive survey.

We next present a modified preconditioner $\widehat{\mathcal{M}}_\alpha$ and give the spectral property of the preconditioned saddle point matrix $\widehat{\mathcal{M}}_\alpha^{-1}\mathcal{A}$, where

$$\widehat{\mathcal{M}}_\alpha = \begin{bmatrix} \alpha I + A & 0 \\ -B & Q \end{bmatrix}.$$

It is easy to see that the eigenvalues of the preconditioned matrix $\widehat{\mathcal{M}}_\alpha^{-1}\mathcal{A}$ satisfy the generalized eigenvalue problem

$$\begin{bmatrix} A & B^* \\ -B & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} \alpha I + A & 0 \\ -B & Q \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \tag{3.2}$$

Note that the above equation can be equivalently written as

$$\begin{cases} Au + B^*v = \lambda(\alpha I + A)u, \\ (\lambda - 1)Bu = \lambda Qv. \end{cases} \tag{3.3}$$

If $u = 0$, then from the second equality of (3.3) we have $Qv = 0$. Since Q is positive definite, we have $v = 0$, which contradicts the assumption that $(u^*, v^*)^*$ is an eigenvector. Therefore, $u \neq 0$.

If $Bu = 0$, from the second equality of the Eq. (3.3), we have $v = 0$ and

$$Au = \lambda(\alpha I + A)u.$$

Multiplying both sides of the above equality from left with u^*, we have

$$u^*Au = \lambda(\alpha u^*u + u^*Au).$$

Thus, we have $\lambda = \frac{u^*Au}{\alpha u^*u+u^*Au}$. It is easy to see that $\lambda \to 1$ when $\alpha \to 0$.

If $Bu \neq 0$, it is easy to know that $\lambda \neq 0$. The second equality of Eq. (3.3) gives $v = \frac{\lambda-1}{\lambda}Q^{-1}Bu$. Substituting it into the first equality of Eq. (3.3) gives

$$\lambda Au + (\lambda - 1)B^*Q^{-1}Bu = \lambda^2(\alpha I + A)u.$$

Let $u^*u = 1$. Multiplying both sides of the above equality from left with u^* we have

$$\lambda u^*Au + (\lambda - 1)u^*B^*Q^{-1}Bu = \lambda^2(\alpha + u^*Au).$$

That is to say,

$$(\alpha + u^*Au)\lambda^2 - (u^*B^*Q^{-1}Bu + u^*Au)\lambda + u^*B^*Q^{-1}Bu = 0. \tag{3.4}$$

Let $u^*Au = a + bi$, $u^*B^*Q^{-1}Bu = e > 0$. We have

$$(\alpha + a + bi)\lambda^2 - (e + a + bi)\lambda + e = 0. \tag{3.5}$$

For the Eq. (3.5) with complex coefficients, the quadratic formula for the roots of this quadratic equation is

$$\lambda = \frac{(e + a + bi) \pm \sqrt{d}}{2(\alpha + a + bi)}, \tag{3.6}$$

where the discriminant $d = (e + a + bi)^2 - 4e(\alpha + a + bi)$ is the complex number. We can write it in the form

$$d = d_1 + d_2 i,$$

where $d_1 = (a - e)^2 - b^2 - 4\alpha e$ and $d_2 = 2b(a - e)$ are real numbers. It was shown in the lesson on taking a square root of a complex number of this module that the square root of the complex number $d = d_1 + d_2 i$ has two values.

The first value is the complex number

$$w_1 = s_1 + t_1 i,$$

where

$$s_1 = \sqrt{\frac{\sqrt{d_1^2 + d_2^2} + d_1}{2}}, \quad t_1 = \sqrt{\frac{\sqrt{d_1^2 + d_2^2} - d_1}{2}},$$

and the second value is $w_2 = -w_1$.

By computation, we have

$$s_1 = \sqrt{\frac{\sqrt{[(a-e)^2 + b^2]^2 + 8\alpha e[2\alpha e + b^2 - (a-e)^2} + (a-e)^2 - b^2 - 4\alpha e}{2}}, \tag{3.7}$$

and

$$t_1 = \sqrt{\frac{\sqrt{[(a-e)^2 + b^2]^2 + 8\alpha e[2\alpha e + b^2 - (a-e)^2} - (a-e)^2 + b^2 + 4\alpha e}{2}}. \tag{3.8}$$

From (3.7) and (3.8), we have $s_1 \to \sqrt{(a-e)^2}$ and $t_1 \to b$ when $\alpha \to 0$. Thus we have

$$\lambda \to \frac{(e+a+bi) \pm (\sqrt{(a-e)^2}+bi)}{2(a+bi)}. \tag{3.9}$$

From (3.9), we know that $\lambda \to 2$ and $\lambda \to \frac{e(a-bi)}{a^2+b^2}$ when $a > e$, and if $a < e$, we have $\lambda \to 1$ and $\lambda \to \frac{a(a-bi)}{a^2+b^2}$.

4. Numerical Experiments

In this section, we present numerical experiments for the saddle point linear system (1.1) in order to verify the effectiveness of the local HSS-like iterative method. All the numerical experiments were performed with MATLAB 7.0. The machine we have used is a PC-AMD, CPU T7400 2.2GHz process. The GMRES method is used to solve the above test problem. The initial guess is taken to be $x^{(0)} = 0$ and the stopping criterion is chosen as $\frac{\|\mathbf{b}-\mathcal{A}\mathbf{x}^{(k)}\|_2}{\|\mathbf{b}\|_2} \le 10^{-6}$. IT and CPU represent the number of iteration steps and elapsed CPU time in seconds, respectively.

We consider the saddle point matrix $\mathcal{A}$ of the following form:

$$\mathcal{A} = \begin{bmatrix} A & B^T \\ -B & 0 \end{bmatrix}, \tag{4.1}$$

where the sub-matrices $A = \upsilon A_1 + N$, υ can be regarded as the viscosity, and N has only two diagonal lines of nonzero, which start from the 2nd and the nth colomns, i.e.,

$$N = \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 & -1 & \ddots & 0 \\ 0 & 0 & 0 & -1 & 0 & \cdots & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \ddots & -1 \\ 0 & \cdots & 0 & 0 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & -1 & 0 & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 & \ddots & 0 & \ddots & -1 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}, \tag{4.2}$$

and A_1, B are taken from [7], i.e.,

$$A_1 = \begin{bmatrix} I \otimes T + T \otimes I & 0 \\ 0 & I \otimes T + T \otimes I \end{bmatrix}, \quad B = \begin{bmatrix} I \otimes F \\ F \otimes I \end{bmatrix},$$

and

$$T = \frac{1}{h^2} tridiag(-1, 2, -1) \in R^{p \times p}, \quad F = \frac{1}{h} tridiag(-1, 1, 0) \in R^{p \times p},$$

with $\otimes$ being the Kronecker product symbol and $h = \frac{1}{p+1}$ the discretization meshsize. The right vectors are defined as

$$f = (1, 1, \cdots, 1) \in R^n, g = (0, 0, \cdots, 0) \in R^m, n = 2p^2, m = p^2.$$

For this example, the matrix A is nonsymmetric and positive real.

Table 4.1: Spectral radius of the iteration matrix $\mathcal{T}_\alpha$ with $v = 0.01$ for different α.

γ	0.01	0.02	0.03	0.04	0.05	0.07	0.1
$\rho(\mathcal{T}_\alpha)$	0.9920	0.9840	0.9773	0.9831	0.9878	0.9918	0.9945

Table 4.2: Spectral radius of the iteration matrix $\mathcal{T}_\alpha$ with $\alpha = 0.03$ for different v.

δ	0.001	0.002	0.005	0.008	0.01	0.015	0.02
$\rho(\mathcal{T}_\alpha)$	0.9981	0.9961	0.9890	0.9813	0.9773	0.9755	0.9736

In the following experiments, we take $Q = \frac{1}{\gamma} I_m$ with $\gamma = \frac{\|A\|}{\|B\|_2}$. In Tables 4.1-4.2, we first present some result on the spectral radius of the iteration matrix $\mathcal{T}_\alpha$ with different values of v and α. The purpose of these experiments is just to investigate the convergence behavior of HSS-like iterative method. Clearly, all results show that the HSS-like iterative method is convergent. Meanwhile, we see that the spectral radius is very close to 1. This also shows the the convergence of HSS-like iterative algorithm. So we discuss the preconditioned Krylov subspace method with two preconditioners.

It is well known that the spectral properties of the preconditioned matrix give important insight in the convergence behavior of the preconditioned Krylov subspace methods. For symmetric problems, the rate of convergence of Krylov subspace methods like CG or MINRES depends on the distribution of the eigenvalues of $\mathcal{A}$. A key for the rapid convergence of an iterative method for a linear system of the form $\mathcal{A}x = b$ is the availability of an effective preconditioner. Thus, in this subsection, based on the above-mentioned ideas in order to illustrate the above results in Section 3, there is a need to test the eigenvalue distributions of the preconditioned matrix $\mathcal{M}_\alpha^{-1}\mathcal{A}$ and $\widehat{\mathcal{M}}_\alpha^{-1}\mathcal{A}$. The eigenvalue distributions of the preconditioned matrix $\mathcal{M}_\alpha^{-1}\mathcal{A}$ with $v = 1$ and $v = 0.1$ are plotted in Fig. 4.1 and Fig. 4.3, respectively. Thus, from

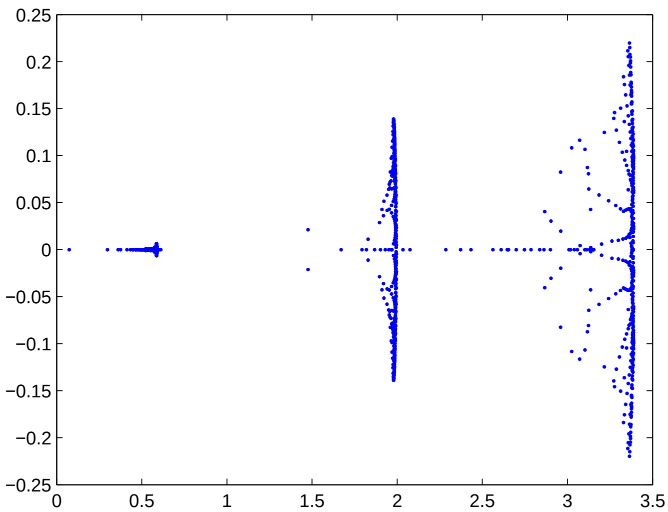

Fig. 4.1. Eigenvalue distribution of the preconditioned matrix $\mathcal{M}_1^{-1}\mathcal{A}$ with $v = 0.1$.

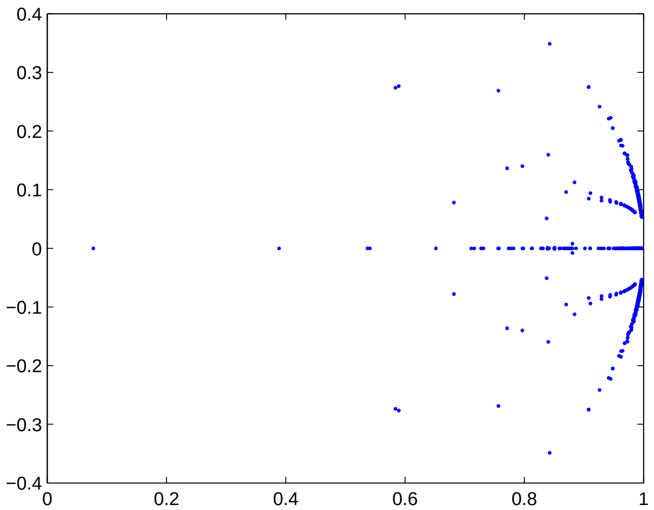

Fig. 4.2. Eigenvalue distribution of the preconditioned matrix $\widehat{\mathcal{M}}_1^{-1}\mathcal{A}$ with $v = 0.1$.

Fig. 4.1 and Fig. 4.3, we see that the eigenvalue distribution of the preconditioned matrix $\mathcal{M}_\alpha^{-1}\mathcal{A}$ is regular and gathering. In the end, in Fig. 4.2 and Fig. 4.4 we display the eigenvalue distribution of the preconditioned matrix $\widehat{\mathcal{M}}_\alpha^{-1}\mathcal{A}$ with $v = 1$ and $v = 0.1$, respectively. Clearly, the eigenvalues of the preconditioned matrix $\widehat{\mathcal{M}}_\alpha^{-1}\mathcal{A}$ have $\lambda \to 1$ when $\alpha \to 0$, the rest of the eigenvalues is close to $\frac{b}{a}$ when $\alpha \to 0$, which is in accordance with the spectral analysis of the preconditioned matrix $\widehat{\mathcal{M}}_\alpha^{-1}\mathcal{A}$ in Section 3.

To illustrate the validity of our preconditioners, we next to test the performance of four preconditioners, one is the alternating LHSS preconditioner $\mathcal{P}_\alpha$ [27], and another is the preconditioner $\widehat{\mathcal{P}}_\alpha$ in [30] which are defined as follows, respectively.

$$\mathcal{P}_\alpha = \frac{1}{2\alpha}(\alpha I + \mathcal{H})(\alpha I + \mathcal{S}) \quad \text{and} \quad \widehat{\mathcal{P}}_\alpha = \frac{1}{2\alpha}(\alpha I + \widehat{\mathcal{H}})(\alpha I + \widehat{\mathcal{S}})$$

with

$$\mathcal{H} = \begin{bmatrix} H & B^T \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{S} = \begin{bmatrix} S & 0 \\ -B & 0 \end{bmatrix},$$

and

$$\widehat{\mathcal{H}} = \begin{bmatrix} A & B^T \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \widehat{\mathcal{S}} = \begin{bmatrix} 0 & 0 \\ -B & 0 \end{bmatrix}.$$

In Tables 4.3-4.4, we give some results to illustrate the convergence behavior of GMRES(10) preconditioned by $\mathcal{P}_\alpha$, $\widehat{\mathcal{P}}_\alpha$, $\mathcal{M}_\alpha$ and $\widehat{\mathcal{M}}_\alpha$ with the different values of v and α. "IT" denotes the number of iterations. "CPU(s)" denotes the CPU time (in seconds) required to solve a problem. The purpose of these experiments is to investigate the influence of the eigenvalue distribution on the convergence behavior of GMRES(10).

Tables 4.3-4.4 contain experimental results for alternating LHSS and block HSS-like preconditioned GMRES(10) on different orders of matrix. From Table 4.3, shows that the pre-

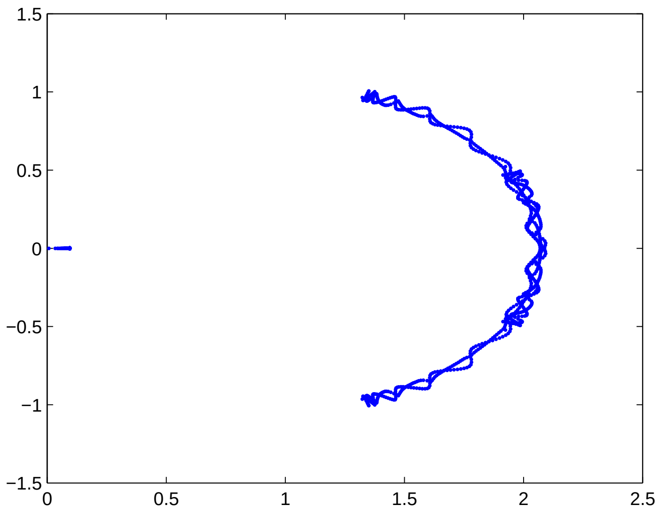

Fig. 4.3. Eigenvalue distribution of the preconditioned matrix $\mathcal{M}_{0.1}^{-1}\mathcal{A}$ with $v = 1$.

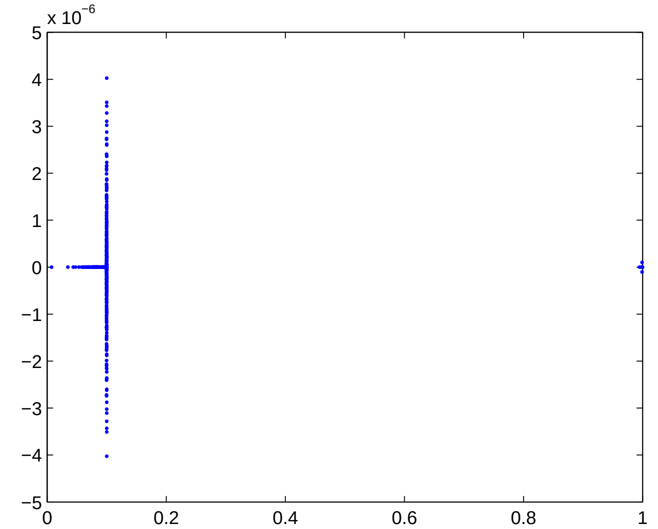

Fig. 4.4. Eigenvalue distribution of the preconditioned matrix $\widehat{\mathcal{M}}_{0.1}^{-1}\mathcal{A}$ with $v = 1$.

conditioners $\widehat{\mathcal{P}}_\alpha$ and $\widehat{\mathcal{M}}_\alpha$ are more effective than the preconditioners $\mathcal{P}_\alpha$ and $\mathcal{M}_\alpha$ for outer iterations of the preconditioned matrices, and inner iteration of four exact preconditioners $\mathcal{P}_\alpha$, $\widehat{\mathcal{P}}_\alpha$, $\mathcal{M}_\alpha$ and $\widehat{\mathcal{M}}_\alpha$ are hardly sensitive to change on the order of the coefficient matrix. From Table 4.4 we see that the preconditioners $\mathcal{M}_\alpha$ and $\widehat{\mathcal{M}}_\alpha$ are more effective than the preconditioners $\mathcal{P}_\alpha$ and $\widehat{\mathcal{P}}_\alpha$ for outer iterations of the preconditioned matrices, and inner iterations of four exact preconditioners are relatively stable. From Table 4.3-4.4 we can see that changes of outer iterations of the preconditioner $\mathcal{M}_\alpha$ and $\widehat{\mathcal{M}}_\alpha$ are very obvious on different values of v and

Table 4.3: Outer(inner) iterations and CPU(s) of GMRES(10) with $v = 1$ and $\alpha = 0.1$.

	$m+n$	300	675	1200	1875	2700
$\mathcal{P}_\alpha$	IT	12(9)	10(6)	7(6)	8(6)	8(7)
	CPU(s)	0.3277	1.9266	4.8028	17.1657	44.1211
$\widehat{\mathcal{P}}_\alpha$	IT	3(9)	4(5)	4(8)	4(7)	4(8)
	CPU(s)	0.0975	1.6216	2.7936	8.0511	20.9726
$\mathcal{M}_\alpha$	IT	26(7)	24(6)	16(6)	13(10)	13(7)
	CPU(s)	0.5412	3.3398	9.5012	27.3087	71.4579
$\widehat{\mathcal{M}}_\alpha$	IT	2(4)	2(8)	2(8)	2(8)	2(8)
	CPU(s)	0.0380	0.2042	1.0390	3.6227	10.0373

Table 4.4: Outer(inner) iterations and CPU(s) of GMRES(10) with $v = 0.01$ and $\alpha = 1$.

	$m+n$	300	675	1200	1875	2700
$\mathcal{P}_\alpha$	IT	13(10)	15(6)	14(7)	19(1)	21(1)
	CPU(s)	0.4005	2.4773	9.6170	39.9245	113.3451
$\widehat{\mathcal{P}}_\alpha$	IT	14(9)	10(7)	15(1)	14(8)	13(10)
	CPU(s)	0.3435	1.8039	9.4101	29.1805	68.5944
$\mathcal{M}_\alpha$	IT	6(10)	6(10)	6(5)	7(2)	7(10)
	CPU(s)	0.1515	1.0157	4.3173	13.6992	42.5647
$\widehat{\mathcal{M}}_\alpha$	IT	8(6)	8(1)	7(9)	7(10)	7(9)
	CPU(s)	0.1748	1.1974	5.0780	15.1742	37.9604

α. All results show that four preconditioners indeed improve the convergence of GMRES(10) efficiently, and compared with preconditioners $\mathcal{P}_\alpha$, $\widehat{\mathcal{P}}_\alpha$ and $\mathcal{M}_\alpha$, the preconditioner $\widehat{\mathcal{M}}_\alpha$ may be competitive under certain conditions.

Acknowledgments. The authors are very much indebted to the referees for providing very valuable suggestions and comments, which greatly improved the original manuscript of this paper. The research was supported by the National Natural Science Foundation of China (11071079, 11201362), the National Natural Science Foundation of Zhejiang Province (Y14A0100 23) and Ningbo Natural Science Foundation (2012A610037, 2013A610096).

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