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ON RESIDUAL-BASED A POSTERIORI ERROR ESTIMATORS FOR LOWEST-ORDER RAVIART-THOMAS ELEMENT APPROXIMATION TO CONVECTION-DIFFUSION-REACTION EQUATIONS*

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Abstract

A new technique of residual-type a posteriori error analysis is developed for the lowest-order Raviart-Thomas mixed finite element discretizations of convection-diffusion-reaction equations in two- or three-dimension. Both centered mixed scheme and upwind-weighted mixed scheme are considered. The a posteriori error estimators, derived for the stress variable error plus scalar displacement error in L^2-norm, can be directly computed with the solutions of the mixed schemes without any additional cost, and are proven to be reliable. Local efficiency dependent on local variations in coefficients is obtained without any saturation assumption, and holds from the cases where convection or reaction is not present to convection- or reaction-dominated problems. The main tools of the analysis are the postprocessed approximation of scalar displacement, abstract error estimates, and the property of modified Oswald interpolation. Numerical experiments are carried out to support our theoretical results and to show the competitive behavior of the proposed posteriori error estimates.

Mathematics subject classification: 65N15, 65N30, 76S05.
Key words: Convection-diffusion-reaction equation, Centered mixed scheme, Upwind-weighted mixed scheme, Postprocessed approximation, A posteriori error estimators.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded polygonal or polyhedral domain in $\mathbb{R}^d, d = 2$ or 3. We consider the following homogeneous Dirichlet boundary value problem for the convection-diffusion-reaction equations:

$$\begin{cases} -\nabla \cdot (S\nabla p) + \nabla \cdot (p\mathbf{w}) + rp = f & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $S \in L^\infty(\Omega; \mathbb{R}^{d\times d})$ denotes an inhomogeneous and anisotropic diffusion-dispersion tensor, $\mathbf{w}$ is a (dominating) velocity field, r a reaction function, f a source term. The choice of boundary conditions is made for ease of presentation, since similar results are valid for other boundary conditions. This type of equations arise in many chemical and biological settings. For instance,

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in hydrology these equations govern the transport and degradation of adsorbing contaminants and microbe-nutrient systems in groundwater.

Reliable and efficient a posteriori error estimators are an indispensable tool for adaptive algorithms. For second-order elliptic problems without convection term, the theory of a posteriori error estimation has reached a degree of maturity for finite elements of conforming, nonconforming and mixed types; see [1-9, 11-14, 18, 20, 22-23, 27, 31-34] and the references therein. For convection-diffusion(-reaction) problems, on the contrary, the theory is still under development.

The mathematical analysis of robustness of a-posteriori estimators for convection-diffusion-reaction equations was first addressed by *Verfürth* [35] in the singular perturbation case, namely $S = \varepsilon I$ with I the identical matrix and $0 < \varepsilon \ll 1$. The proposed estimators for the standard Galerkin approximation and the SUPG disctetization give global upper and local lower bounds on the error measured in the energy norm, and are robust when the Péclet number is not large. In [36] *Verfürth* improved the results of [35] in the sense that the derived estimates are fully robust with respect to convection dominance and uniform with respect to the size of the zero-order reaction term. *Sangalli* [31] developed an a posteriori estimator for the residual-free bubbles methods applied to convection-diffusion problems. Later he presented a residual-based a posteriori estimator for the one-dimensional convection-diffusion-reaction model problem [32]. In [23,24] *Kunert* carried out a posteriori error estimation for the SUPG approach to a singularly perturbed convection- or reaction-diffusion problem on anisotropic meshes. One may also refer to [26, 27] for a posteriori error estimation in the framework of finite volume approximations.

For the convection-diffusion-reaction model (1.1), following an idea of postprocessing in [25] *Vohralík* [37] established residual a posteriori error estimates for lowest-order Raviart-Thomas mixed finite element discretizations on simplicial meshes. Global upper bounds and local lower bounds for the postprocessed approximation error, $p - \tilde{p}_h$, in the energy norm were derived with $\tilde{p}_h$ the postprocessed approximation to the finite element solution p_h, and the local efficiency of the estimators was shown to depend only on local variations in the coefficients and on the local Péclet number. Moreover, the developed general framework allows for asymptotic exactness and full robustness with respect to inhomogeneities and anisotropies.

In this paper, we develop a new technique for residual-based a posteriori estimation of the lowest-order Raviart-Thomas mixed finite element schemes (centered mixed scheme and upwind-mixed scheme) over both the stress error, $\mathbf{u} - \mathbf{u}_h$, and the displacement error, $p - p_h$, of the mixed finite element solutions $(\mathbf{u}_h, p_h)$ for the problem (1.1) with $\mathbf{u} := -S\nabla p$. Local efficiency dependent only on local variations of the coefficients is obtained without any saturation assumption, holds for the convection or reaction dominated equations. Especially for the centered mixed scheme in the singular perturbation case, the proposed estimator yields global upper and local lower bounds which differ by multiplicative constants depending only on the shape regularity parameter and the local mesh-Péclet number. Compared with the standard analysis to the diffusion equations, our analysis avoids, by using the postprocessed approximation $\tilde{p}_h$ as a transition, Helmholtz decomposition of stress variables and dual arguments of displacement error in L^2-norm, and then does not need any weak regularity assumption on the diffusion-dispersion tensor. We note that although being employed in our analysis, the post-processed displacement approximation and its modified Oswald interpolation are not involved in our estimators.

The rest of this paper is organized as follows. In Section 2 we give notations, assumptions of data, and the weak problem. We introduce in Section 3 the mixed finite element schemes (include the centered and upwind-weighted mixed scheme) and the post-processed techniques. Section 4 includes the main results. Section 5 collects some preliminary results and remarks. Sections 6 and 7 analyze respectively the a posteriori error estimates and the local efficiency. Finally, we present several numerical examples in Section 8 to test our estimators.

2. Notations, Assumptions and Weak Problem

For a domain $A \subset \mathbb{R}^d$, we denote by $L^2(A)$ and $\mathbf{L}^2(A) =: (L^2(A))^d$ the spaces of square-integrable functions, by $(\cdot,\cdot)_A$ the $L^2(A)$ or $\mathbf{L}^2(A)$ inner product, by $||\cdot||_A$ the associated norm, and by $|A|$ the Lebesgue measure of A. Let $H^k(A)$ be the usual Sobolev space consisting of functions defined on A with all derivatives of order up to k square-integrable; $H_0^1(A) := \{v \in H^1(A) : \ v|_{\partial A} = 0\}$, $\mathbf{H}(\mathrm{div}, A) := \{\mathbf{v} \in \mathbf{L}^2(A) : \mathrm{div}\ \mathbf{v} \in L^2(A)\}$. $< \cdot,\cdot >_{\partial A}$ denotes $d-1$-dimensional inner product on ∂A for the duality paring between $H^{-1/2}(\partial A)$ and $H^{1/2}(\partial A)$.

Let $\mathcal{T}_h$ be a shape regular triangulation in the sense of [16] which satisfies the angle condition, namely there exists a constant c_0 such that for all $K \in \mathcal{T}_h$ with $h_K := \mathrm{diam}(K)$,

$$c_0^{-1} h_K^d \leq |K| \leq c_0 h_K^d.$$

We denote by ε_h the set of element sides in $\mathcal{T}_h$, by $\varepsilon_h^{\mathrm{int}}$ and $\varepsilon_h^{\mathrm{ext}}$ the sets of all interior and exterior sides of $\mathcal{T}_h$, respectively. For $K \in \mathcal{T}_h$, denote by ε_K the set of sides of K, especially by $\varepsilon_K^{\mathrm{ext}}$ the set of the boundary sides of K. Furthermore, we denote by ω_σ and $\tilde{\omega}_\sigma$ the union of all elements in $\mathcal{T}_h$ sharing a side σ and the union of all elements sharing at least one point of σ, respectively. For an element $K \in \mathcal{T}_h$ the set $\tilde{\omega}_K$ is defined analogously. We also use the "broken Sobolev space"

$$H^1(\bigcup \mathcal{T}_h) := \{\varphi \in L^2(\Omega) : \varphi|_K \in H^1(K), \forall K \in \mathcal{T}_h\},$$

and denote by $[v]|_\sigma := (v|_K)|_\sigma - (v|_L)|_\sigma$ the jump of $v \in H^1(\bigcup \mathcal{T}_h)$ over an interior side $\sigma := \bar{K} \cap \bar{L}$ of diameter $h_\sigma := \mathrm{diam}(\sigma)$, shared by the two neighboring (closed) elements $K, L \in \mathcal{T}_h$. Especially, $[v]|_\sigma := (v|_K)|_\sigma$ if $\sigma \in \varepsilon_K^{\mathrm{ext}}$.

We consider $d = 2, 3$ simultaneously and let $m := 1$ if $d = 2$ and $m := 3$ if $d = 3$. The Curl of a function $\psi \in H^1(\Omega)^m$ is defined by

$$\mathrm{Curl}\psi := (-\partial\psi/\partial x_2, \partial\psi/\partial x_1) \text{ if } d = 2 \ \text{ and } \ \mathrm{Curl}\psi := \nabla \times \psi \ \text{ if } d = 3,$$

where $\times$ denotes the usual vector product of two vectors in $\mathbb{R}^3$. Given a unit normal vector $\mathbf{n} = (n_1, n_2)$ along the side σ, we define the tangential component of a vector $\mathbf{v} \in \mathbb{R}^d$ by

$$\gamma_{\mathbf{t}_\sigma}(\mathbf{v}) := \begin{cases} \mathbf{v} \cdot (-n_2, n_1) & \text{if } \ d = 2, \\ \mathbf{v} \times \mathbf{n} & \text{if } \ d = 3. \end{cases}$$

We need in our analysis the following inequalities, Poincaré inequality and Friedrichs inequalities [10, 28]: for $K \in \mathcal{T}_h$ and $\varphi \in H^1(K)$,

$$||\varphi - \varphi_K||_K^2 \lesssim h_K^2 ||\nabla\varphi||_K^2, \tag{2.1}$$

$$(\varphi_K - \varphi_\sigma)^2 \leq \frac{3dh_K^2}{|K|} ||\nabla\varphi||_K^2, \quad ||\varphi - \varphi_\sigma||_K^2 \leq 3dh_K^2 ||\nabla\varphi||_K^2. \tag{2.2}$$

Here $\varphi_K := (1, \varphi)_K/|K|$ and $\varphi_\sigma :=< 1, \varphi >_\sigma /|\sigma|$ denote the integrable means of φ over K and over $\sigma \in \varepsilon_K$, respectively.

For convenience, throughout the paper we use the notation $a \lesssim b$ to represent that there exists a generic positive constant C depending only on the shape regularity parameter, c_0, of $\mathcal{T}_h$ such that $a \leq Cb$.

Following [37], we suppose that there exists an original triangulation $\mathcal{T}_0$ of Ω such that data of the problem (1.1) are given in the following way.

Assumptions of data :

- (D1) $S_K := S|_K$ is a constant, symmetric, and uniformly positive definite tensor such that $c_{S,K}\mathbf{v}\cdot\mathbf{v} \leq S_K\mathbf{v}\cdot\mathbf{v} \leq C_{S,K}\mathbf{v}\cdot\mathbf{v}$ holds for all $\mathbf{v} \in \mathbb{R}^d$ and all $K \in \mathcal{T}_0$ with constants $c_{S,K}, C_{S,K} > 0$;
- (D2) $\mathbf{w} \in RT_0(\mathcal{T}_0)$ (cf, Section 3 below) such that $|\mathbf{w}|_K| \leq C_{\mathbf{w},K}$ holds for all $K \in \mathcal{T}_0$ with constant $C_{\mathbf{w},K} \geq 0$;
- (D3) $r_K := r|_K$ is a constant for all $K \in \mathcal{T}_0$;
- (D4) $c_{\mathbf{w},r,K} := \frac{1}{2}\nabla\cdot\mathbf{w}|_K + r_K \geq 0$ and $C_{\mathbf{w},r,K} := |\nabla\cdot\mathbf{w}|_K + r_K|$ for all $K \in \mathcal{T}_0$;
- (D5) $f|_K$ is a polynomial for each $K \in \mathcal{T}_0$;
- (D6) if $c_{\mathbf{w},r,K} = 0$, then $C_{\mathbf{w},r,K} = 0$.

As pointed out in [37], all the assumptions are made for the sake of simplicity and are usually satisfied in practice. If data do not satisfy these assumptions, we may employ the interpolation or projection of data with additional occurrence of data oscillation.

Finally we show the weak problem of the model (1.1): Find $p \in H_0^1(\Omega)$ such that

$$\mathcal{B}(p,\varphi) = (f,\varphi) \quad \text{for all } \ \varphi \in H_0^1(\Omega), \tag{2.3}$$

where the bilinear form is given by

$$\mathcal{B}(p,\varphi) := \sum_{K\in\mathcal{T}_h} \{(S\nabla p,\nabla\varphi)_K + (\nabla\cdot(p\mathbf{w}),\varphi)_K + (rp,\varphi)_K\}, \quad p,\varphi \in H^1(\bigcup\mathcal{T}_h),$$

and $\mathcal{T}_h$ is a refinement of $\mathcal{T}_0$. We define as following an energy (semi) norm corresponding to the bilinear form $\mathcal{B}$:

$$|||\varphi|||_\Omega^2 := \sum_{K\in\mathcal{T}_h} |||\varphi|||_K^2,\ |||\varphi|||_K^2 := (S\nabla\varphi,\nabla\varphi)_K + c_{\mathbf{w},r,K}||\varphi||_K^2, \quad \varphi \in H^1(\bigcup\mathcal{T}_h).$$

We note that the weak problem (2.3) admits a unique solution under Assumptions (D1)-(D6), see, e.g., [37].

3. Mixed Finite Element Schemes and Postprocessing

Since it is of interest in many applications, the stress variable $\mathbf{u} := -S\nabla p$ are usually approximated by using the mixed finite elements for the problem (1.1). We introduce in this section the centered and upwind-weighted mixed finite element schemes, and show the postprocessed techniques presented by Vohralík in [37].

We define the lowest order Raviart-Thomas finite element and piecewise constant space respectively as following:

$$
\begin{aligned}
RT_0(\mathcal{T}_h) &:= \left\{ \begin{array}{c} \mathbf{q}_h \in \mathbf{H}(\mathrm{div},\Omega):\ \forall K \in \mathcal{T}_h,\ \exists \mathbf{a} \in \mathbb{R}^d,\ \exists b \in \mathbb{R}, \\ \text{such that } \mathbf{q}_h(\mathbf{x}) = \mathbf{a} + b\mathbf{x}, \text{for all } \mathbf{x} \in K. \end{array} \right\}, \\
P_0(\mathcal{T}_h) &:= \{v_h \in L^\infty(\Omega):\ \forall K \in \mathcal{T}_h,\ v_h|_K \in P_0(K)\}.
\end{aligned}
$$

Here $\mathbf{n}$ is the unit outer normal vector along $\sigma \in \varepsilon_h$, and $P_0(K)$ denotes the set of constant functions on each $K \in \mathcal{T}_h$. We note that $\nabla \cdot (RT_0(\mathcal{T}_h)) \subset P_0(\mathcal{T}_h)$.

The centered mixed finite element scheme [18,37] reads as: Find $(\mathbf{u}_h, p_h) \in RT_0(\mathcal{T}_h) \times P_0(\mathcal{T}_h)$ such that

$$
(S^{-1}\mathbf{u}_h, \mathbf{v}_h)_\Omega - (p_h, \nabla \cdot \mathbf{v}_h)_\Omega = 0 \quad \text{for all} \quad \mathbf{v}_h \in RT_0(\mathcal{T}_h), \tag{3.1}
$$

$$
\begin{aligned}
&(\nabla \cdot \mathbf{u}_h, \varphi_h)_\Omega - (S^{-1}\mathbf{u}_h \cdot \mathbf{w}, \varphi_h)_\Omega + ((r + \nabla \cdot \mathbf{w})p_h, \varphi_h)_\Omega \\
&\qquad = (f, \varphi_h)_\Omega \quad \text{for all} \quad \varphi_h \in P_0(\mathcal{T}_h).
\end{aligned} \tag{3.2}
$$

The upwind-weighted mixed finite element scheme [17,37] reads as: Find $(\mathbf{u}_h, p_h) \in RT_0(\mathcal{T}_h) \times P_0(\mathcal{T}_h)$ such that

$$
(S^{-1}\mathbf{u}_h, \mathbf{v}_h)_\Omega - (p_h, \nabla \cdot \mathbf{v}_h)_\Omega = 0 \quad \text{for all } \mathbf{v}_h \in RT_0(\mathcal{T}_h), \tag{3.3}
$$

$$
\begin{aligned}
&(\nabla \cdot \mathbf{u}_h, \varphi_h)_\Omega + \sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \varepsilon_K} \hat{p}_\sigma w_{K,\sigma} \varphi_K + (r p_h, \varphi_h)_\Omega \\
&\qquad = (f, \varphi_h)_\Omega \quad \text{for all} \quad \varphi_h \in P_0(\mathcal{T}_h),
\end{aligned} \tag{3.4}
$$

where $w_{K,\sigma} :=< 1, \mathbf{w} \cdot \mathbf{n} >_\sigma$ for $\sigma \in \varepsilon_K$, with $\mathbf{n}$ the unit normal vector of σ, outward to K, $\varphi_K = (1, \varphi_h)_K/|K| = \varphi_h|_K$ for all $K \in \mathcal{T}_h$, and $\hat{p}_\sigma$ is the weighted upwind value given by

$$
\hat{p}_\sigma := \left\{ \begin{array}{ll} (1-\nu_\sigma)p_K + \nu_\sigma p_L & \text{if} \quad w_{K,\sigma} \geq 0, \\ (1-\nu_\sigma)p_L + \nu_\sigma p_K & \text{if} \quad w_{K,\sigma} < 0 \end{array} \right. \tag{3.5}
$$

when σ is an interior side sharing by elements K and L, and by

$$
\hat{p}_\sigma := \left\{ \begin{array}{ll} (1-\nu_\sigma)p_K & \text{if} \quad w_{K,\sigma} \geq 0, \\ \nu_\sigma p_K & \text{if} \quad w_{K,\sigma} < 0 \end{array} \right. \tag{3.6}
$$

when σ is a boundary side included in ε_K. Here p_K and p_L denote respectively the restrictions of p_h over K and L, $\nu_\sigma \in [0, 1/2]$ denotes the coefficient of the amount of upstream weighting which may be chosen as [37]

$$
\nu_\sigma := \left\{ \begin{array}{ll} \min\left\{ c_{S,\sigma} \frac{|\sigma|}{h_\sigma |w_{K,\sigma}|}, \frac{1}{2} \right\} & \text{if} \quad w_{K,\sigma} \neq 0 \ \text{ and } \ \sigma \in \varepsilon_h^{\mathrm{int}}, \\ & \text{or if} \quad \sigma \in \varepsilon_h^{\mathrm{ext}} \ \text{ and } \ w_{K,\sigma} > 0; \\ 0 & \text{if} \quad w_{K,\sigma} = 0 \text{ or if } \sigma \in \varepsilon_h^{\mathrm{ext}} \ \text{ and } \ w_{K,\sigma} < 0, \end{array} \right. \tag{3.7}
$$

where $c_{S,\sigma}$ is the harmonic average of $c_{S,K}$ and $c_{S,L}$ if $\sigma \in \partial K \cap \partial L$ and $c_{S,K}$ otherwise.

We now introduce the postprocessed technique in [37], where a postprocessed approximation $\tilde{p}_h$ to the displacement p is constructed which links p_h and $\mathbf{u}_h$ on each simplex in the following way:

$$
-S_K \nabla \tilde{p}_h|_K = \mathbf{u}_h \quad \text{for all} \quad K \in \mathcal{T}_h, \tag{3.8}
$$

$$\frac{1}{|K|}\int_K \tilde{p}_h d\mathbf{x} = p_K \quad \text{for all} \quad K \in \mathcal{T}_h. \tag{3.9}$$

We refer to [37] for the existence of $\tilde{p}_h$. It has been shown that the new quantity $\tilde{p}_h \in W_0(\mathcal{T}_h)$ but $\notin H_0^1(\Omega)$ (see Lemma 6.1 in [37]), where

$$W_0(\mathcal{T}_h) := \Big\{\varphi \in L^2(\Omega) : \varphi|_K \in H^1(K) \text{ for all } K \in \mathcal{T}_h,\ <1, \varphi|_K - \varphi|_L>_{\sigma_{K,L}} = 0$$
$$\text{for all} \ \ \sigma_{K,L} \in \varepsilon_h^{\text{int}},\ \ <1,\varphi>_\sigma = 0 \ \text{ for all } \sigma \in \varepsilon_h^{\text{ext}}\Big\}.$$

We note that the postprocessing (3.8)-(3.9) is only valid for the lowest order Raviart-Thomas element.

4. Main Results

With the stress variable $\mathbf{u} = -S\nabla p$, we define the global and local errors, $\mathcal{E}$ and $\mathcal{E}_K$, of the stress and displacement variables as

$$\mathcal{E} := \Big\{\sum_{K\in\mathcal{T}_h} \mathcal{E}_K^2\Big\}^{1/2}, \quad \mathcal{E}_K^2 := ||S^{-1/2}(\mathbf{u}-\mathbf{u}_h)||_K^2 + c_{\mathbf{w},r,K}||p - p_h||_K^2. \tag{4.1}$$

Denote respectively by $\eta_{D,K}$ and $\eta_{R,K}$ the elementwise displacement and residual estimator with

$$\eta_{D,K}^2 := c_{\mathbf{w},r,K} h_K^2 ||S^{-1}\mathbf{u}_h||_K^2, \tag{4.2}$$
$$\eta_{R,K}^2 := \alpha_K^2 ||f - \nabla\cdot\mathbf{u}_h + (S^{-1}\mathbf{u}_h)\cdot\mathbf{w} - (r + \nabla\cdot\mathbf{w})p_h||_K^2 + \beta_K^2||S^{-1}\mathbf{u}_h||_K^2. \tag{4.3}$$

Here the residual weight factors

$$\alpha_K := \min\Big\{\frac{h_K}{\sqrt{c_{S,K}}}, \frac{1}{\sqrt{c_{\mathbf{w},r,K}}}\Big\}, \quad \beta_K := C_{\mathbf{w},r,K} h_K \alpha_K. \tag{4.4}$$

Note that in (4.4), if $c_{\mathbf{w},r,K} = 0$, α_K should be understood as $h_K/\sqrt{c_{S,K}}$. Let ν_σ be given in (3.7) for each side $\sigma \in \varepsilon_h$. We denote

$$\hat{\hat{p}}_\sigma := \begin{cases} (1/2 - \nu_\sigma)(p_K - p_L) & \text{if} \quad w_{K,\sigma} \ge 0, \\ (1/2-\nu_\sigma)(p_L - p_K) & \text{if} \quad w_{K,\sigma} < 0 \end{cases} \tag{4.5}$$

when σ is an interior side sharing by elements K and L, and

$$\hat{\hat{p}}_\sigma := \begin{cases} -\nu_\sigma p_K & \text{if} \quad w_{K,\sigma} \ge 0, \\ -(1-\nu_\sigma)p_K & \text{if} \quad w_{K,\sigma} < 0, \end{cases} \tag{4.6}$$

when σ is a boundary side included in ε_K. We thus define an elementwise upwind estimator $\eta_{U,K}$ by

$$\eta_{U,K}^2 := \frac{h_K}{c_{S,K}} \sum_{\sigma\in\varepsilon_K} \Big((\mathbf{w}\cdot\mathbf{n})|_\sigma\Big)^2 \Big(||\hat{\hat{p}}_\sigma||_\sigma^2 + h_\sigma ||S^{-1}\mathbf{u}_h||_{\omega_\sigma}^2\Big). \tag{4.7}$$

In order to reflect the change of the maximum eigenvalue of the coefficients matrix S over the patch $\tilde{\omega}_\sigma$ of a side $\sigma \in \varepsilon_h$, we introduce a quantity

$$\Lambda_\sigma := \max_{K, \bar{K}\cap\bar{\sigma}\neq\emptyset} \{C_{S,K}\}.$$

Similarly, the change of one variation $c_{\mathbf{w},r,K}$ of the coefficients over the patch $\tilde{\omega}_K$ of an element $K \in \mathcal{T}_h$ is described by the quantity

$$\Lambda_{\mathbf{w},r,K} := \max_{K', \bar{K}' \cap \bar{K} \neq \emptyset} \{c_{\mathbf{w},r,K}\}. \tag{4.8}$$

Thus we define $\eta_{NC,K}$ as the elementwise nonconforming estimator by

$$\eta_{NC,K}^2 := \Lambda_{\mathbf{w},r,K} h_K^2 ||S^{-1}\mathbf{u}_h||_K^2 + \sum_{\sigma \in \varepsilon_K} \delta_\sigma \Lambda_\sigma h_\sigma ||[\gamma_{\mathbf{t}_\sigma}(S^{-1}\mathbf{u}_h)]||_\sigma^2, \tag{4.9}$$

where $\delta_\sigma = 1/2$ if $\sigma \in \varepsilon_h^{\text{int}}$, $\delta_\sigma = 1$ if $\sigma \in \varepsilon_h^{\text{ext}}$.

Since the convection occurs in the equations, we need to define two numbers $\Lambda_{\nabla\cdot\mathbf{w},K}$ and $\Lambda_{\mathbf{w},\sigma}$ similar to Péclet numbers describing the convection-dominated. To this end, for each $K \in \mathcal{T}_h$ we denote

$$C_{\nabla\cdot\mathbf{w},K} := |\nabla \cdot \mathbf{w}|_K|, \quad \Lambda_{\nabla\cdot\mathbf{w},K} := \max_{K' : \bar{K}' \cap \bar{K} \neq \emptyset} \left\{ \frac{C_{\nabla\cdot\mathbf{w},K'}}{\sqrt{c_{\mathbf{w},r,K'}}} \right\}, \tag{4.10}$$

and for each $\sigma \in \varepsilon_h$ we set $\Lambda_{\mathbf{w},\sigma} := \min\{\lambda_{\mathbf{w},\sigma}, p_{\mathbf{w},\sigma}\}$ with

$$\lambda_{\mathbf{w},\sigma} := \max_{K : \bar{K} \cap \bar{\sigma} \neq \emptyset} \left\{ \frac{C_{\mathbf{w},K}\sqrt{C_{S,K}}}{\sqrt{c_{\mathbf{w},r,K} c_{S,K}}} \right\}, \quad p_{\mathbf{w},\sigma} := \max_{K : \bar{K} \cap \bar{\sigma} \neq \emptyset} \left\{ \frac{h_K C_{\mathbf{w},K}\sqrt{C_{S,K}}}{\sqrt{c_{S,K}}} \right\}. \tag{4.11}$$

We then define $\eta_{C,K}$ as an elementwise convection estimator by

$$\eta_{C,K}^2 := \Lambda_{\nabla\cdot\mathbf{w},K}^2 h_K^2 ||S^{-1}\mathbf{u}_h||_K^2 + \sum_{\sigma \in \varepsilon_K} \delta_\sigma \Lambda_{\mathbf{w},\sigma}^2 h_\sigma ||[\gamma_{\mathbf{t}_\sigma}(S^{-1}\mathbf{u}_h)]||_\sigma^2. \tag{4.12}$$

We now state a posteriori error estimates for the global error of stress and displacement.

Theorem 4.1. (Global error estimate for the centered mixed scheme) *Let $p \in H_0^1(\Omega)$ be the weak solution of the problem* (2.3), *$\mathbf{u} = -S\nabla p$ be the continuous stress vector, $(\mathbf{u}_h, p_h)$ be the solution of the centered mixed scheme* (3.1)-(3.2). *Let $\mathcal{E}$ be the error of the stress and displacement in the weighted norm defined in* (4.1), *$\eta_{D,K}, \eta_{R,K}, \eta_{NC,K}$, and $\eta_{C,K}$ are the corresponding elementwise displacement estimator, residual estimator, convection estimator, and nonconforming estimator, defined in* (4.2)-(4.3) *and* (4.9)-(4.12), *respectively. Then it holds*

$$\mathcal{E}^2 \lesssim \sum_{K \in \mathcal{T}_h} \left(\eta_{D,K}^2 + \eta_{R,K}^2 + \eta_{C,K}^2 + \eta_{NC,K}^2 \right). \tag{4.13}$$

Theorem 4.2. (Global error estimate for the upwind-weighted scheme) *Let $p \in H_0^1(\Omega)$ be the weak solution of the problem* (2.3), *$\mathbf{u} = -S\nabla p$ be the continuous stress vector, $(\mathbf{u}_h, p_h)$ be the solution of the upwind-weighted mixed scheme* (3.3)-(3.4). *Let $\mathcal{E}$ be the error of the stress and displacement in the weighted norm defined in* (4.1), *$\eta_{D,K}, \eta_{R,K}, \eta_{U,K}$, $\eta_{NC,K}$, and $\eta_{C,K}$ are the corresponding elementwise displacement estimator, residual estimator, upwind estimator, convection estimator, and nonconforming estimator, defined in* (4.2)-(4.3) *and* (4.7)-(4.12), *respectively. Then it holds*

$$\mathcal{E}^2 \lesssim \sum_{K \in \mathcal{T}_h} \left(\eta_{D,K}^2 + \eta_{R,K}^2 + \eta_{C,K}^2 + \eta_{NC,K}^2 + \eta_{U,K}^2 \right). \tag{4.14}$$

Remark 4.1. In [12] Carstensen presented a posteriori error estimates of the Raviart-Thomas, Brezzi-Douglas-Morini, Brezzi-Douglas-Fortin-Marini elements (M_h, L_h) for the diffusion equations (the case $\mathbf{w} = r = 0$ in the model (1.1)). In his estimators, the term $\min_{v_h \in L_h} ||h(S^{-1}\mathbf{u}_h - \nabla_h v_h)||_\Omega$ is included, where ∇_h denotes the $\mathcal{T}_h-$piecewise gradient operator. In practice one may substitute it with the term $||h(S^{-1}\mathbf{u}_h - \nabla_h p_h)||_\Omega$, where $(\mathbf{u}_h, p_h) \in M_h \times L_h$ is a pair of finite element solutions. For the lowest order Raviart-Thomas element, it holds $\nabla_h p_h = 0$, then the term $||h(S^{-1}\mathbf{u}_h - \nabla_h p_h)||_\Omega$ is reduced to $||hS^{-1}\mathbf{u}_h||_\Omega$, which shows that occurrence of $||hS^{-1}\mathbf{u}_h||_\Omega$ is reasonable in the a posteriori error estimators $\eta_{D,K}$ defined in (4.2). In addition, we note that the postprocessing (3.8) can remove the term $||h\mathrm{curl}_h(S^{-1}\mathbf{u}_h)||_\Omega$, which is also contained in Carstensen's estimators. Here curl_h denotes the $\mathcal{T}_h-$piecewise curl operator with $\mathrm{curl}\psi := \partial_x\psi_2 - \partial_y\psi_1$ for a vector-valued function $\psi = (\psi_1, \psi_2)$.

The global error estimates above show that the a posteriori indicator over each element consists of a series of estimators. Thus, the local efficiency of each component ensures the local efficiency of the a posteriori indicator over an element. Here, we point out the local efficiency is in the sense that its converse estimate holds up to a different multiplicative constant.

Theorem 4.3. (Local efficiency for the displacement and residual estimators) *For $K \in \mathcal{T}_h$, let $\eta_{D,K}$ and $\eta_{R,K}$ denote the elementwise displacement and residual estimators defined in* (4.2) *and* (4.3)*, respectively. Then it holds*

$$\eta_{D,K}^2 + \eta_{R,K}^2 \lesssim \alpha_{*,K}^2 \mathcal{E}_K^2 \tag{4.15}$$

with

$$\begin{aligned}\alpha_{*,K} :&= \max\left\{\sqrt{\frac{C_{S,K}}{c_{S,K}}} + \frac{h_K C_{\mathbf{w},K}}{c_{S,K}}, \frac{h_K C_{\mathbf{w},r,K}}{\sqrt{c_{\mathbf{w},r,K} c_{S,K}}}\right\} \\ &\quad + \max\left\{\frac{h_K^2 C_{\mathbf{w},r,K}}{c_{S,K}}, \frac{h_K C_{\mathbf{w},r,K}}{\sqrt{c_{S,K} c_{\mathbf{w},r,K}}}\right\} + \max\left\{\frac{h_K\sqrt{c_{\mathbf{w},r,K}}}{\sqrt{c_{S,K}}}, 1\right\}.\end{aligned}$$

Theorem 4.4. (Local efficiency for the nonconforming and convection estimators) *Let $\eta_{NC,K}$ and $\eta_{C,K}$ be the elementwise nonconforming and convection estimators defined in* (4.9) *and* (4.12)*, respectively. Then it holds*

$$\eta_{NC,K}^2 + \eta_{C,K}^2 \lesssim \beta_{*,K}^2 \mathcal{E}_K^2 + \sum_{\sigma \in \varepsilon_K} c_{\omega_\sigma}^2 (\Lambda_\sigma + \Lambda_{\mathbf{w},\sigma}^2) ||S^{-1/2}(\mathbf{u} - \mathbf{u}_h)||_{\omega_\sigma}^2, \tag{4.16}$$

where

$$\beta_{*,K}^2 := (\Lambda_{\mathbf{w},r,K} + \Lambda_{\nabla\cdot\mathbf{w},K}^2) \max\{h_K^2/c_{S,K}, 1/c_{\mathbf{w},r,K}\},$$

$$c_{\omega_\sigma} := \begin{cases} \max(c_{S,K}^{-1/2}, c_{S,L}^{-1/2}) & \text{if} \quad \sigma = \bar{K} \cap \bar{L}, \\ c_{S,K}^{-1/2} & \text{if} \quad \sigma \in \varepsilon_K \cap \varepsilon_h^{\mathrm{ext}}, \end{cases}$$

and $\Lambda_{\mathbf{w},r,K}, \Lambda_{\nabla\cdot\mathbf{w},K}$ are given by (4.8) and (4.10), respectively.

We finally need the following quantities for the local efficiency of the upwind estimator over an element, where ν_σ is given in (3.7) for each side $\sigma \in \varepsilon_h$:

$$\lambda_\sigma := \begin{cases} \frac{|(\mathbf{w}\cdot\mathbf{n})|_\sigma|}{\sqrt{c_{S,K}}} \left((\frac{1}{2} - \nu_\sigma)\max(\frac{1}{\sqrt{c_{S,K}}}, \frac{1}{\sqrt{c_{S,L}}}) + \max(\frac{h_K}{\sqrt{c_{S,K}}}, \frac{h_L}{\sqrt{c_{S,L}}})\right) & \text{if} \;\; \sigma = \bar{K} \cap \bar{L}, \\ \frac{|(\mathbf{w}\cdot\mathbf{n})|_\sigma|}{\sqrt{c_{S,K}}} \left((1 - \nu_\sigma)\frac{1}{\sqrt{c_{S,K}}} + \frac{h_K}{\sqrt{c_{S,K}}}\right) & \text{if} \;\; \sigma \in \varepsilon_K^{\mathrm{ext}}, \end{cases}$$

$$\rho_\sigma := \begin{cases} \frac{|(\mathbf{w}\cdot\mathbf{n})|_\sigma|}{\sqrt{c_{S,K}}}\left((\frac{1}{2}-\nu_\sigma)|\sigma|^{-\frac{1}{2}}+1\right)\max(\frac{1}{\sqrt{c_{\mathbf{w},r,K}}},\frac{1}{\sqrt{c_{\mathbf{w},r,L}}}) & \text{if } \quad \sigma=\bar{K}\cap\bar{L}, \\ \frac{|(\mathbf{w}\cdot\mathbf{n})|_\sigma|}{\sqrt{c_{S,K}}}\left((1-\nu_\sigma)|\sigma|^{-\frac{1}{2}}+1\right)\frac{1}{\sqrt{c_{\mathbf{w},r,K}}} & \text{if } \quad \sigma\in\varepsilon_K^{\text{ext}}, \end{cases}$$

and

$$\mathcal{E}_{D,\omega_\sigma} := \begin{cases} \left(c_{\mathbf{w},r,K}||p-p_h||_K^2+c_{\mathbf{w},r,L}||p-p_h||_L^2\right)^{1/2} & \text{if } \quad \sigma=\bar{K}\cap\bar{L}, \\ \sqrt{c_{\mathbf{w},r,K}}||p-p_h||_K & \text{if } \quad \sigma\in\varepsilon_K^{\text{ext}}. \end{cases}$$

Theorem 4.5. (Local efficiency for the upwind estimator) *Let $\eta_{U,K}$ be the elementwise upwind estimator defined in* (4.7). *Then, it holds*

$$\eta_{U,K} \lesssim \sum_{\sigma\in\varepsilon_K}\left(\lambda_\sigma||S^{-1/2}(\mathbf{u}-\mathbf{u}_h)||_{\omega_\sigma}+\rho_\sigma\mathcal{E}_{D,\omega_\sigma}\right). \tag{4.17}$$

Remark 4.2. We note that in the reliability estimates of Theorems 4.1-4.2 the multiplicative constants depend only on the shape regularity parameter, while in the efficiency estimates of Theorems 4.3-4.5 the multiplicative constants depend only on the shape regularity parameter and local variations of the coefficients. In particular, for the centered mixed scheme in the singular perturbation case with $S=\varepsilon I$, Theorems 4.3-4.4 show that the proposed estimator yields local lower bounds depending on the local mesh-Péclet number, and the upper and lower bounds of the estimator differ by a factor $c(1+h_K D_{w,r}/\varepsilon)$, where h_K is the local mesh size, $D_{w,r}$ denotes one local variation of w and r, and c is a positive constant depending only on the shape regularity parameter.

5. Preliminary Results

This section shows the abstract error estimates developed by *Vohralík* in [37]. For any $\varphi\in H_0^1(\Omega)$, $s\in H_0^1(\Omega)$, we define

$$T_R(\varphi) := \sum_{K\in\mathcal{T}_h}(f+\nabla\cdot(S\nabla\tilde{p}_h)-\nabla\cdot(\tilde{p}_h\mathbf{w})-r\tilde{p}_h,\varphi-\varphi_K), \tag{5.1}$$

$$T_C(\varphi,s) := \sum_{K\in\mathcal{T}_h}(\nabla\cdot((\tilde{p}_h-s)\mathbf{w})-1/2(\tilde{p}_h-s)\nabla\cdot\mathbf{w},\varphi)_K, \tag{5.2}$$

$$T_U(\varphi) := \sum_{K\in\mathcal{T}_h}\sum_{\sigma\in\varepsilon_K} < (\hat{p}_\sigma-\tilde{p}_h)\mathbf{w}\cdot\mathbf{n},\varphi_K >_\sigma, \tag{5.3}$$

where φ_K is the mean of φ over K, $\tilde{p}_h$ the postprocessed approximation solution given by (3.8)-(3.9), and $\hat{p}_\sigma$ the weighted upwind value defined in (3.5)-(3.6).

Lemma 5.1. (Abstract error estimates, [37]) *Let $p\in H_0^1(\Omega)$ be the weak solution of the problem* (2.3), *and let $s\in H_0^1(\Omega)$ be arbitrary. Then it holds*

$$|||p-\tilde{p}_h|||_\Omega \le |||\tilde{p}_h-s|||_\Omega + \sup_{\varphi\in H_0^1(\Omega),|||\varphi|||_\Omega=1}\{T_R(\varphi)+T_C(\varphi,s)\} \tag{5.4}$$

if $\tilde{p}_h$ is the postprocessed solution, given by (3.8)-(3.9), *of the centered mixed finite element scheme* (3.1)-(3.2), *and holds*

$$|||p-\tilde{p}_h|||_\Omega \le |||\tilde{p}_h-s|||_\Omega + \sup_{\varphi\in H_0^1(\Omega),|||\varphi|||_\Omega=1}\{T_R(\varphi)+T_C(\varphi,s)+T_U(\varphi)\} \tag{5.5}$$

if $\tilde{p}_h$ is the postprocessed solution, given by (3.8)-(3.9)*, of the upwind-weighted mixed finite element scheme* (3.3)-(3.4).

Remark 5.1. In Vohralík's work [37], the modified Oswald interpolation, $\mathcal{I}_{\mathrm{MO}}(\tilde{p}_h) \in H_0^1(\Omega)$, of $\tilde{p}_h$ is introduced to replace s in the abstract error estimates (5.4)-(5.5) so as to obtain computable estimates of the terms.

We now state our abstract error estimates for the global error of stress and displacement in the weighted norm.

Lemma 5.2. *(*Abstract error estimates for the global error*) Let $p \in H_0^1(\Omega)$ denote the weak solution of the problem* (2.3)*, and $s \in H_0^1(\Omega)$ be arbitrary. Let $\mathcal{E}$ be the global error defined in* (4.1) *and $\eta_{D,K}$ be the elementwise displacement estimator defined in* (4.2)*. Then it holds*

$$\mathcal{E} \lesssim \left\{ |||\tilde{p}_h - s|||_\Omega + \sup_{\varphi \in H_0^1(\Omega), |||\varphi|||_\Omega = 1} (T_R(\varphi) + T_C(\varphi, s)) + \left(\sum_{K \in \mathcal{T}_h} \eta_{D,K}^2 \right)^{1/2} \right\} \tag{5.6}$$

if $\tilde{p}_h$ is the postprocessed solution, given by (3.8)-(3.9)*, of the centered mixed finite* (3.1)-(3.2)*, and holds*

$$\begin{aligned} \mathcal{E} \lesssim \Bigg\{ |||\tilde{p}_h - s|||_\Omega + & \sup_{\varphi \in H_0^1(\Omega), |||\varphi|||_\Omega = 1} (T_R(\varphi) + T_C(\varphi, s) + T_U(\varphi)) \\ & + \left(\sum_{K \in \mathcal{T}_h} \eta_{D,K}^2 \right)^{1/2} \Bigg\} \end{aligned} \tag{5.7}$$

if $\tilde{p}_h$ is the postprocessed solution, given by (3.8)-(3.9)*, of the upwind-weighted mixed finite element scheme* (3.3)-(3.4).

Proof. By the postprocessed formulations (3.8)-(3.9) and the generalized Friedrichs inequality (2.2), we have

$$\begin{aligned} ||p - p_h||_K &\leq ||p - \tilde{p}_h||_K + ||\tilde{p}_h - p_h||_K \leq ||p - \tilde{p}_h||_K + h_K ||\nabla \tilde{p}_h||_K \\ &= ||p - \tilde{p}_h||_K + h_K ||S^{-1} \mathbf{u}_h||_K \quad \text{for all} \quad K \in \mathcal{T}_h. \end{aligned} \tag{5.8}$$

On the other hand, it holds

$$||S^{-1/2}(\mathbf{u} - \mathbf{u}_h)||_K^2 = ||S^{1/2} \nabla (p - \tilde{p}_h)||_K^2 \quad \text{for all} \quad K \in \mathcal{T}_h. \tag{5.9}$$

Summing (5.9) and (5.8) with a multiplier $c_{\mathbf{w},r,K}^{1/2}$ over all $K \in \mathcal{T}_h$ yields

$$\mathcal{E} \lesssim \left\{ |||p - \tilde{p}_h|||_\Omega + \left(\sum_{K \in \mathcal{T}_h} c_{\mathbf{w},r,K} h_K^2 ||S^{-1} \mathbf{u}_h||_K^2 \right)^{1/2} \right\}. \tag{5.10}$$

The desired results (5.6)-(5.7) then follows from Lemma 5.1. □

Lemma 5.3. *For any $K \in \mathcal{T}_h$ and $\varphi \in H^1(K)$, it holds*

$$||\varphi - \varphi_K||_K \lesssim \alpha_K |||\varphi|||_K, \tag{5.11}$$

where φ_K denotes the mean of φ over K, and α_K is given by (4.4).

Proof. From (4.4), it holds $\alpha_K = h_K c_{S,K}^{-1/2}$ when $h_K c_{S,K}^{-1/2} \le c_{\mathbf{w},r,K}^{-1/2}$. By Bramble-Hilbert lemma we have

$$\begin{aligned}||\varphi - \varphi_K||_K &\lesssim h_K||\nabla\varphi||_K \lesssim h_K c_{S,K}^{-1/2}||S^{1/2}\nabla\varphi||_K \\ &= \alpha_K||S^{1/2}\nabla\varphi||_K \le \alpha_K|||\varphi|||_K. \end{aligned} \tag{5.12}$$

On the other hand, it holds $\alpha_K = c_{\mathbf{w},r,K}^{-1/2}$ when $h_K c_{S,K}^{-1/2} > c_{\mathbf{w},r,K}^{-1/2}$. By the property of L^2-projection we get

$$\begin{aligned}||\varphi - \varphi_K||_K &\le ||\varphi||_K = c_{\mathbf{w},r,K}^{-1/2} c_{\mathbf{w},r,K}^{1/2}||\varphi||_K \\ &= \alpha_K c_{\mathbf{w},r,K}^{1/2}||\varphi||_K \le \alpha_K|||\varphi|||_K. \end{aligned} \tag{5.13}$$

The assertion (5.11) follows from (5.12)-(5.13). □

6. A Posteriori Error Analysis

We devote this section to the a posteriori error estimation for the global error of stress and displacement for both the centered mixed scheme (3.1)-(3.2) and the upwind-weighted mixed scheme (3.3)-(3.4). We respectively derive computable estimates for the right-side terms of the abstract error estimates (5.6)-(5.7), $T_R(\varphi), T_U(\varphi)$, $T_C(\varphi, s)$ and $|||\tilde{p}_h - s|||_\Omega$ with $s := I_{MO}(\tilde{p}_h)$, the modified Oswald interpolation of the postprocessing solution $\tilde{p}_h$, and finally show the proof of Theorems 4.1-4.2.

Lemma 6.1. (Residual estimator) *Let $T_R(\varphi)$ be defined as in* (5.1) *with* $|||\varphi|||_\Omega = 1$, *and $\eta_{R,K}$ be defined as in* (4.3). *Then it holds*

$$T_R(\varphi) \lesssim \{\sum_{K\in\mathcal{T}_h} \eta_{R,K}^2\}^{1/2}. \tag{6.1}$$

Proof. A combination of Assumption (D4), Lemma 5.3, Friedrichs inequality (2.2), and the postprocessing (3.8)-(3.9), yields

$$\begin{aligned} T_R(\varphi) &= \sum_{K\in\mathcal{T}_h} (f + \nabla\cdot(S\nabla\tilde{p}_h) - \nabla\cdot(\tilde{p}_h\mathbf{w}) - r\tilde{p}_h, \varphi - \varphi_K)_K \\ &= \sum_{K\in\mathcal{T}_h} (f - \nabla\cdot\mathbf{u}_h + (S^{-1}\mathbf{u}_h)\cdot\mathbf{w} - (r + \nabla\cdot\mathbf{w})p_h, \varphi - \varphi_K)_K \\ &\quad + \sum_{K\in\mathcal{T}_h} ((r+\nabla\cdot\mathbf{w})(p_h - \tilde{p}_h), \varphi - \varphi_K)_K \\ &\lesssim \sum_{K\in\mathcal{T}_h} \alpha_K||f - \nabla\cdot\mathbf{u}_h + (S^{-1}\mathbf{u}_h)\cdot\mathbf{w} - (r+\nabla\cdot\mathbf{w})p_h||_K|||\varphi|||_K \\ &\quad + \sum_{K\in\mathcal{T}_h} C_{\mathbf{w},r,K} h_K||\nabla\tilde{p}_h||_K\alpha_K|||\varphi|||_K \\ &= \sum_{K\in\mathcal{T}_h} \alpha_K||f - \nabla\cdot\mathbf{u}_h + (S^{-1}\mathbf{u}_h)\cdot\mathbf{w} - (r+\nabla\cdot\mathbf{w})p_h||_K|||\varphi|||_K \\ &\quad + \sum_{K\in\mathcal{T}_h} \beta_K||S^{-1}\mathbf{u}_h||_K|||\varphi|||_K, \end{aligned} \tag{6.2}$$

where α_K, β_K are given by (4.4). Then the desired result (6.1) follows with $|||\varphi|||_\Omega = 1$.

Lemma 6.2. (Upwind estimator) *Let $T_U(\varphi)$ be defined as in* (5.3) *with* $|||\varphi|||_\Omega = 1$, *and* $\eta_{U,K}$ *be defined as in* (4.7). *Then it holds*

$$T_U(\varphi) \lesssim \left\{ \sum_{K\in\mathcal{T}_h} \eta_{U,K}^2 \right\}^{1/2}. \tag{6.3}$$

Proof. We denote by $\tilde{p}_\sigma$ the mean of $\tilde{p}_h$ over $\sigma \in \varepsilon_h$, i.e., $\tilde{p}_\sigma :=< 1, \tilde{p}_h >_\sigma /|\sigma|$. By recalling that $w_{K,\sigma} :=< 1, \mathbf{w}\cdot\mathbf{n} >_\sigma$ is a constant for $\sigma \in \varepsilon_K$, the definition of $T_U(\varphi)$, together with Assumption $(D2)$ of the velocity field $\mathbf{w}$, imply

$$T_U(\varphi) = \sum_{K\in\mathcal{T}_h} \sum_{\sigma\in\varepsilon_K} (\hat{p}_\sigma - \tilde{p}_\sigma) w_{K,\sigma}\varphi_K. \tag{6.4}$$

For an element $K \in \mathcal{T}_h$, it holds $\sigma \in \varepsilon_K \cap \varepsilon_L$ or $\sigma \in \varepsilon_K^{\rm ext}$. For the former case, recalling $p_K = p_h|_K, p_L = p_h|_L$, from the postprocessing (3.9) we obtain

$$\begin{aligned}
\hat{p}_\sigma - \tilde{p}_\sigma &= \hat{p}_\sigma - \frac{1}{2}(p_K + p_L) + \frac{1}{2}(p_K - \tilde{p}_\sigma) + \frac{1}{2}(p_L - \tilde{p}_\sigma) \\
&= \hat{p}_\sigma - \frac{1}{2}(p_K + p_L) + \frac{1}{2}\left(\frac{1}{|K|}\int_K \tilde{p}_h dx - \frac{1}{|\sigma|}\int_\sigma \tilde{p}_h ds\right) \\
&\quad + \frac{1}{2}\left(\frac{1}{|L|}\int_L \tilde{p}_h dx - \frac{1}{|\sigma|}\int_\sigma \tilde{p}_h ds\right).
\end{aligned} \tag{6.5}$$

For the latter case, we similarly have

$$\hat{p}_\sigma - \tilde{p}_\sigma = \hat{p}_\sigma - p_K + \left(\frac{1}{|K|}\int_K \tilde{p}_h dx - \frac{1}{|\sigma|}\int_\sigma \tilde{p}_h ds\right). \tag{6.6}$$

For convenience, we denote in what follows

$$\hat{p}_{\omega_\sigma} := \frac{1}{2}\left(\frac{1}{|K|}\int_K \tilde{p}_h dx - \frac{1}{|\sigma|}\int_\sigma \tilde{p}_h ds\right) + \frac{1}{2}\left(\frac{1}{|L|}\int_L \tilde{p}_h dx - \frac{1}{|\sigma|}\int_\sigma \tilde{p}_h ds\right)$$

when $\sigma \in \varepsilon_K \cap \varepsilon_L$, and

$$\hat{p}_{\omega_\sigma} := \frac{1}{|K|}\int_K \tilde{p}_h dx - \frac{1}{|\sigma|}\int_\sigma \tilde{p}_h ds$$

when $\sigma \in \varepsilon_K^{\rm ext}$.

In light of the definitions of $\hat{p}_\sigma$ and $\hat{\hat{p}}_\sigma$ in (3.5)-(3.6) and (4.5)-(4.6), and from (6.4)-(6.6) we have

$$T_U(\varphi) = \sum_{K\in\mathcal{T}_h} \sum_{\sigma\in\varepsilon_K} (\hat{\hat{p}}_\sigma + \hat{p}_{\omega_\sigma}) w_{K,\sigma}\varphi_K. \tag{6.7}$$

Since $\varphi \in H_0^1(\Omega)$, and $\hat{\hat{p}}_\sigma$, $\hat{p}_{\omega_\sigma}$ and $\mathbf{w}\cdot\mathbf{n}$ are constants over a side $\sigma \in \varepsilon_h$, it holds

$$\sum_{K\in\mathcal{T}_h} \sum_{\sigma\in\varepsilon_K} \hat{\hat{p}}_\sigma w_{K,\sigma}\varphi_K = \sum_{K\in\mathcal{T}_h} \sum_{\sigma\in\varepsilon_K} \int_\sigma \hat{\hat{p}}_\sigma \mathbf{w}\cdot\mathbf{n}(\varphi_K - \varphi), \tag{6.8}$$

$$\sum_{K\in\mathcal{T}_h} \sum_{\sigma\in\varepsilon_K} \hat{p}_{\omega_\sigma} w_{K,\sigma}\varphi_K = \sum_{K\in\mathcal{T}_h} \sum_{\sigma\in\varepsilon_K} \int_\sigma \hat{p}_{\omega_\sigma} \mathbf{w}\cdot\mathbf{n}(\varphi_K - \varphi). \tag{6.9}$$

From Friedrichs inequality (2.2) and the postprocessing (3.8) we have

$$|\hat{p}_{\omega_\sigma}| \lesssim h_\sigma^{1-d/2}||S^{-1}\mathbf{u}_h||_{\omega_\sigma}. \tag{6.10}$$

The trace inequality (see Lemma 3.1 in [35]) and local shape regularity of elements indicate

$$\begin{aligned} ||\varphi_K-\varphi||_\sigma &\lesssim h_\sigma^{-1/2}||\varphi-\varphi_K||_K+||\varphi-\varphi_K||_K^{1/2}||\nabla(\varphi-\varphi_K)||_K^{1/2} \\ &\lesssim h_K^{1/2}||\nabla\varphi||_K\leq h_K^{1/2}c_{S,K}^{-1/2}||S^{1/2}\nabla\varphi||_K. \end{aligned} \tag{6.11}$$

A combination of (6.10)- (6.11) then yields

$$\begin{aligned} &\sum_{K\in\mathcal{T}_h}\sum_{\sigma\in\varepsilon_K}\int_\sigma \hat{p}_{\omega_\sigma}\mathbf{w}\cdot\mathbf{n}(\varphi_K-\varphi) \\ \lesssim &\sum_{K\in\mathcal{T}_h}\left\{\sum_{\sigma\in\varepsilon_K}|(\mathbf{w}\cdot\mathbf{n})|_\sigma|h_\sigma^{1/2}||S^{-1}\mathbf{u}_h||_{\omega_\sigma}\right\}h_K^{1/2}c_{S,K}^{-1/2}|||\varphi|||_K. \end{aligned} \tag{6.12}$$

Similarly we can obtain

$$\sum_{K\in\mathcal{T}_h}\sum_{\sigma\in\varepsilon_K}\int_\sigma \hat{p}_\sigma\mathbf{w}\cdot\mathbf{n}(\varphi_K-\varphi)\lesssim\sum_{K\in\mathcal{T}_h}\left\{\sum_{\sigma\in\varepsilon_K}|(\mathbf{w}\cdot\mathbf{n})|_\sigma|||\hat{p}_\sigma||_\sigma\right\}h_K^{1/2}c_{S,K}^{-1/2}|||\varphi|||_K. \tag{6.13}$$

The desired result (6.3) follows from (6.7)-(6.9) and (6.12)-(6.13) with $|||\varphi|||_\Omega=1$. □

For the first term, $|||\tilde{p}_h-s|||_\Omega$, of the right side of the abstract error estimate (5.6) or (5.7), we follow [37] to take $s:=I_{MO}(\tilde{p}_h)$ in the sequel, where $I_{MO}(\tilde{p}_h)$ is the modified Oswald interpolation of $\tilde{p}_h$. Recall an estimate on the modified Oswald interpolation [21],

$$||\nabla(\varphi_h-\mathcal{I}_{\rm MO}(\varphi_h))||_K^2\lesssim\sum_{\sigma:\sigma\cap K\neq\Phi}h_\sigma^{-1}||[\varphi_h]||_\sigma^2,\quad \varphi_h\in\mathbb{P}_d(\mathcal{T}_h)\cap W_0(\mathcal{T}_h), \tag{6.14}$$

where $\mathcal{I}_{\rm MO}(\varphi_h)\in\mathbb{P}_d(\mathcal{T}_h)\cap H_0^1(\Omega)$ is the modified Oswald interpolation of φ_h, $\mathbb{P}_d(\mathcal{T}_h)$ (d =2 or 3) denotes the set of polynomials of degree at most d on each simplex, $\sigma\cap K\neq\emptyset$ when σ contains a vertex of K.

By the definition of the norm $|||\cdot|||_\Omega$ we have

$$|||\tilde{p}_h-s|||_\Omega=\left\{\sum_{K\in\mathcal{T}_h}(S\nabla(\tilde{p}_h-s),\nabla(\tilde{p}_h-s))_K+\sum_{K\in\mathcal{T}_h}c_{\mathbf{w},r,K}||\tilde{p}_h-s||_K^2\right\}^{1/2}.$$

Lemmas 6.3-6.4 show respectively computable estimates of the two right-side terms of the above identity in terms of of $\mathbf{u}_h$ and p_h.

Lemma 6.3. *Let $\gamma_{\mathbf{t}_\sigma}(\cdot)$ be defined as in Section 2.1, and $s:=I_{MO}(\tilde{p}_h)$. Then it holds*

$$\sum_{K\in\mathcal{T}_h}||S^{1/2}\nabla(\tilde{p}_h-s)||_K^2\lesssim\sum_{\sigma\in\varepsilon_h}\Lambda_\sigma h_\sigma||[\gamma_{\mathbf{t}_\sigma}(S^{-1}\mathbf{u}_h)]||_\sigma^2, \tag{6.15}$$

where Λ_σ is given in Section 4, and $\mathbf{t}_\sigma$ denotes the unit tangent vector along σ.

Proof. From the estimate (6.14) we have

$$||\nabla(\tilde{p}_h-s)||_K^2\lesssim\sum_{\sigma,\sigma\cap K\neq\emptyset}h_\sigma^{-1}||[\tilde{p}_h]||_\sigma^2,\quad\text{for all }\ K\in\mathcal{T}_h. \tag{6.16}$$

Since the mean of $\tilde{p}_h$ over interior side is continuous and its mean on exterior side vanishes, i.e., $\int_\sigma [\tilde{p}_h] ds = 0$ for all $\sigma \in \varepsilon_h$, by Poincaré inequality it holds

$$||[\tilde{p}_h]||_\sigma = ||[\tilde{p}_h] - \int_\sigma [\tilde{p}_h]||_\sigma \lesssim h_\sigma ||\gamma_{\mathbf{t}_\sigma}(\nabla([\tilde{p}_h]))||_\sigma. \tag{6.17}$$

The postprocessing (3.8) indicates

$$\gamma_{\mathbf{t}_\sigma}(\nabla([\tilde{p}_h])) = -[\gamma_{\mathbf{t}_\sigma}(S^{-1}\mathbf{u}_h)], \quad \text{for all } \ \sigma \in \varepsilon_h. \tag{6.18}$$

A combination of (6.16)-(6.18) yields

$$||S^{1/2}\nabla(\tilde{p}_h - s)||_K^2 \lesssim C_{S,K} \sum_{\sigma, \sigma\cap K \neq \emptyset} h_\sigma ||[\gamma_{\mathbf{t}_\sigma}(S^{-1}\mathbf{u}_h)]||_\sigma^2. \tag{6.19}$$

Summing (6.19) over each element K, noticing that the number of summation over a side $\sigma \in \varepsilon_h$ is bounded by a positive constant (depends only on the shape regularity of the triangulation), and combining the definition of Λ_σ, we obtain

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h} ||S^{1/2}\nabla(\tilde{p}_h - s)||_K^2 \\ \lesssim &\sum_{K\in\mathcal{T}_h} C_{S,K} \sum_{\sigma,\sigma\cap K\neq\emptyset} h_\sigma ||[\gamma_{\mathbf{t}_\sigma}(S^{-1}\mathbf{u}_h)]||_\sigma^2 \lesssim \sum_{\sigma\in\varepsilon_h} \Lambda_\sigma h_\sigma ||[\gamma_{\mathbf{t}_\sigma}(S^{-1}\mathbf{u}_h)]||_\sigma^2.\end{aligned} \tag{6.20}$$

The desired result (6.15) follows from (6.20). □

Remark 6.1. The node with respect to which the quasi-monotone condition is violated is called singular node (cf. [15, 29]). We can derive an alternative form of (6.20) as following:

$$\sum_{K\in\mathcal{T}_h} ||S^{1/2}\nabla(\tilde{p}_h - s)||_K^2 \lesssim \sum_{K\in\mathcal{T}_h} \xi_K^2,$$

where

$$\xi_K^2 := \begin{cases} \sum\limits_{\sigma\in\varepsilon_K} h_\sigma ||[\gamma_{\mathbf{t}_\sigma}(S^{-1/2}\mathbf{u}_h)]||_\sigma^2, & \text{if} \quad K \text{ has no singular nodes}, \\ \sum\limits_{\sigma\in\varepsilon_K} C_{S,\omega_K} h_\sigma ||[\gamma_{\mathbf{t}_\sigma}(S^{-1}\mathbf{u}_h)]||_\sigma^2, & \text{if} \quad K \text{ includes a singular node} \end{cases}$$

with $C_{S,\omega_K} := \max_{K'\in\tilde{\omega}_K} C_{S,K'}$.

Lemma 6.4. *Let $\Lambda_{\mathbf{w},r,K}$ be the same as in* (4.9) *and $s := I_{MO}(\tilde{p}_h)$. Then it holds*

$$\sum_{K\in\mathcal{T}_h} c_{\mathbf{w},r,K} ||\tilde{p}_h - s||_K^2 \lesssim \sum_{K\in\mathcal{T}_h} \Lambda_{\mathbf{w},r,K} h_K^2 ||S^{-1}\mathbf{u}_h||_K^2. \tag{6.21}$$

Proof. Following the line of the proof of Theorem 2.2 in [21], we obtain

$$||\tilde{p}_h - s||_K^2 \lesssim \sum_{\sigma:\sigma\cap K\neq\emptyset} h_\sigma ||[\tilde{p}_h]||_\sigma^2. \tag{6.22}$$

Let $\tilde{p}_\sigma :=< 1, \tilde{p}_h >_\sigma /|\sigma|$ denote the mean of the postprocessed scalar variable $\tilde{p}_h$ over a side $\sigma \in \varepsilon_h$. From the trace theory and generalized Friedrichs inequality (2.2), we obtain

$$||[\tilde{p}_h]||_\sigma \lesssim h_\sigma^{1/2} ||\nabla_h \tilde{p}_h||_{\omega_\sigma}. \tag{6.23}$$

A combination of (6.22), (6.23) and the postprocessing (3.8) yields that

$$||\tilde{p}_h - s||_K^2 \lesssim \sum_{\sigma,\sigma\cap K\neq\emptyset} h_\sigma^2 ||S^{-1}\mathbf{u}_h||_{\omega_\sigma}^2. \tag{6.24}$$

Summing (6.24) over each element K, noticing that the mesh is local quasi-uniform, and combining the definition of $\Lambda_{\mathbf{w},r,K}$, we finally get

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h} c_{\mathbf{w},r,K}||\tilde{p}_h - s||_K^2\\ \lesssim &\sum_{K\in\mathcal{T}_h} c_{\mathbf{w},r,K} \sum_{\sigma,\sigma\cap K\neq\emptyset} h_\sigma^2||S^{-1}\mathbf{u}_h||_{\omega_\sigma}^2 \lesssim \sum_{K\in\mathcal{T}_h} \Lambda_{\mathbf{w},r,K} h_K^2 ||S^{-1}\mathbf{u}_h||_K^2.\end{aligned}$$

This completes the proof of the lemma. $\square$

The following corollary follows from Lemmas 6.3-6.4.

Corollary 6.1. *Let $\eta_{NC,K}$ be defined as in* (4.9) *and $s := I_{MO}(\tilde{p}_h)$. Then it holds*

$$|||\tilde{p}_h - s|||_\Omega \lesssim \{\sum_{K\in\mathcal{T}_h} \eta_{NC,K}^2\}^{1/2}. \tag{6.25}$$

Lemma 6.5. (Convection estimator.) *Let $T_C(\varphi, s)$ be defined as in* (5.2) *with $|||\varphi|||_\Omega = 1$ and $s := I_{MO}(\tilde{p}_h)$, and $\eta_{C,K}$ be defined as in* (4.12)*. Then it holds*

$$T_C(\varphi, s) \lesssim \left\{ \sum_{K\in\mathcal{T}_h} \eta_{C,K}^2 \right\}^{1/2}. \tag{6.26}$$

Proof. By triangle inequality and Hölder inequality we obtain

$$\begin{aligned}T_C(\varphi, s) &\leq \sum_{K\in\mathcal{T}_h} \left\{ C_{\mathbf{w},K}||\nabla(\tilde{p}_h - s)||_K ||\varphi||_K + \frac{1}{2} C_{\nabla\cdot\mathbf{w},K}||\tilde{p}_h - s||_K ||\varphi||_K \right\}\\ &\leq \left\{ \sum_{K\in\mathcal{T}_h} \left(\frac{C_{\mathbf{w},K}^2}{c_{S,K}c_{\mathbf{w},r,K}} ||S^{1/2}\nabla(\tilde{p}_h - s)||_K^2 + \frac{C_{\nabla\cdot\mathbf{w},K}^2}{4c_{\mathbf{w},r,K}} ||\tilde{p}_h - s||_K^2\right) \right\}^{1/2}. \end{aligned}\tag{6.27}$$

Apply (6.19) and (6.24) to the inequality (6.27), and combine the definitions of $\Lambda_{\nabla\cdot\mathbf{w},K}$ and $\lambda_{\mathbf{w},\sigma}$ in (4.10) and (4.11), we then arrive at

$$\begin{aligned}T_C(\varphi, s) \lesssim &\Bigg\{ \sum_{K\in\mathcal{T}_h} \Big(\frac{C_{\mathbf{w},K}^2 C_{S,K}}{c_{S,K}c_{\mathbf{w},r,K}} \sum_{\sigma:\sigma\cap K\neq\emptyset} h_\sigma ||[\gamma_{\mathbf{t}_\sigma}(S^{-1}\mathbf{u}_h)]||_\sigma^2\\ &\qquad + \frac{C_{\nabla\cdot\mathbf{w},K}^2}{4c_{\mathbf{w},r,K}} \sum_{\sigma:\sigma\cap K\neq\emptyset} h_\sigma^2 ||S^{-1}\mathbf{u}_h||_{\omega_\sigma}^2\Big)\Bigg\}^{1/2}\\ \lesssim &\left\{ \sum_{\sigma\in\varepsilon_h} \lambda_{\mathbf{w},\sigma}^2 h_\sigma ||[\gamma_{\mathbf{t}_\sigma}(S^{-1}\mathbf{u}_h)]||_\sigma^2 + \sum_{K\in\mathcal{T}_h} \Lambda_{\nabla\cdot\mathbf{w},K}^2 h_K^2 ||S^{-1}\mathbf{u}_h||_K^2 \right\}^{1/2}\end{aligned} \tag{6.28}$$

Since the modified Oswald interpolation $s = \mathcal{I}_{MO}(\tilde{p}_h)$ preserves the mean of $\tilde{p}_h$ on the side, and $\mathbf{w}\cdot\mathbf{n}$ is constant over a side, it holds

$$(\nabla\cdot((\tilde{p}_h - s)\mathbf{w}), \varphi_K)_K = < (\tilde{p}_h - s)\mathbf{w}\cdot\mathbf{n}, \varphi_K >_{\partial K} = 0,$$

where φ_K is the mean of φ over K. Write $v := \tilde{p}_h - s$, then we have

$$\begin{aligned}&(\nabla\cdot(v\mathbf{w}) - 1/2v\nabla\cdot\mathbf{w}, \varphi)_K\\ =&(\nabla v\cdot\mathbf{w}, \varphi-\varphi_K)_K + (1/2v\nabla\cdot\mathbf{w}, \varphi)_K - (v\nabla\cdot\mathbf{w}, \varphi_K)_K.\end{aligned} \tag{6.29}$$

A combination of (6.29), (2.1), (6.19), (6.24) and Hölder inequality yields

$$\begin{aligned} T_C(\varphi, s) \quad &\le \sum_{K\in\mathcal{T}_h}\Big(\frac{h_K C_{\mathbf{w},K}}{\sqrt{c_{S,K}}}||S^{1/2}\nabla(\tilde{p}_h - s)||_K + \frac{3C_{\nabla\cdot\mathbf{w},K}}{2\sqrt{c_{\mathbf{w},r,K}}}||\tilde{p}_h - s||_K\Big)|||\varphi|||_K\\ &\lesssim \Big\{\sum_{K\in\mathcal{T}_h}\frac{h_K^2 C_{\mathbf{w},K}^2 C_{S,K}}{c_{S,K}}\sum_{\sigma,\sigma\cap K\neq\emptyset} h_\sigma||[\gamma_{\mathbf{t}_\sigma}(S^{-1}\mathbf{u}_h)]||_\sigma^2\\ &\quad + \sum_{K\in\mathcal{T}_h}\frac{C_{\nabla\cdot\mathbf{w},K}^2}{c_{\mathbf{w},r,K}}\sum_{\sigma,\sigma\cap K\neq\emptyset} h_\sigma^2||S^{-1}\mathbf{u}_h||_{\omega_\sigma}^2\Big\}^{1/2}. \end{aligned} \tag{6.30}$$

This estimate, together with the definitions of $p_{\mathbf{w},\sigma}$ and $\Lambda_{\nabla\cdot\mathbf{w},K}$, indicates $T_C(\varphi, s)$ from (6.30)

$$T_C(\varphi, s) \lesssim \{\sum_{\sigma\in\varepsilon_h} p_{\mathbf{w},\sigma}^2 h_\sigma||[\gamma_{\mathbf{t}_\sigma}(S^{-1}\mathbf{u}_h)]||_\sigma^2 + \sum_{K\in\mathcal{T}_h}\Lambda_{\nabla\cdot\mathbf{w},K}^2 h_K^2||S^{-1}\mathbf{u}_h||_K^2\}^{1/2}. \tag{6.31}$$

The desired result (6.26) follows from (6.28) and (6.31).

Proof of Theorem 4.1: For the centered mixed scheme (3.1)-(3.2), the desired result (4.13) follows from Lemmas 5.2, 6.1, 6.5, and Corollary 6.1.

Proof of Theorem 4.2: For the upwind-weighted mixed scheme (3.3)-(3.4), the assertion (4.14) follows from Lemmas 5.2, 6.1, 6.2, 6.5, and Corollary 6.1.

Remark 6.2. (Two approaches in a posteriori error analysis) In literature there are usually two analysis approaches of the a posteriori error analysis. One is directly based on the solution of the discretization scheme, the other one is based on the postprocessed approximation. Seemingly, these two approaches are fully different. Our analysis establishes a link between them, i.e. a posteriori error estimates based on the discretization solution can be derived with the help of the postprocessing technique. In doing so, one can avoid the use of Helmholtz decomposition of the stress variable which is required in traditional a posteriori error analysis for mixed finite elements.

Remark 6.3. (Pure diffusion problem) When $\mathbf{w} = r = 0$, the model (1.1) is reduced to a pure diffusion problem. In this case, the fact that $-\nabla\cdot(S_K\nabla\tilde{p}_h|_K) = \nabla\cdot\mathbf{u}_h|_K = f_K$ for all $K\in\mathcal{T}_h$ with f_K the mean value of f over K indicates

$$\begin{aligned}&\eta_{D,K} = 0, \quad \eta_{R,K}^2 = \tfrac{h_K^2}{c_{S,K}}||f - f_K||_K^2,\\ &\eta_{C,K} = 0, \quad \eta_{NC,K}^2 = \textstyle\sum_{\sigma\in\varepsilon_K}\delta_\sigma\Lambda_\sigma h_\sigma||[\gamma_{\mathbf{t}_\sigma}(S^{-1}\mathbf{u}_h)]||_\sigma^2.\end{aligned}$$

Thus the a posteriori error estimate (4.13) is reduced to

$$\mathcal{E}^2 \lesssim \sum_{K\in\mathcal{T}_h}\left(\frac{h_K^2}{c_{S,K}}||f - f_K||_K^2 + \sum_{\sigma\in\varepsilon_K}\delta_\sigma\Lambda_\sigma h_\sigma||[\gamma_{\mathbf{t}_\sigma}(S^{-1}\mathbf{u}_h)]||_\sigma^2\right) \tag{6.32}$$

with $\mathcal{E} = \{\sum\limits_{K\in\mathcal{T}_h} ||S^{-1/2}(\mathbf{u}-\mathbf{u}_h)||_K^2\}^{1/2}$. In addition, Remark 6.1 implies an alternative estimate

$$\mathcal{E} \lesssim \left\{ \sum_{K\in\mathcal{T}_h} (\frac{h_K^2}{c_{S,K}}||f - f_K||_K^2 + \xi_K^2 \right\}^{1/2}. \tag{6.33}$$

Note that being an oscillation term, the first term in the right side of (6.32) or (6.33) may not be computed in practice.

7. Analysis of Local Efficiency

7.1. Some lemmas

Using standard arguments we can easily derive the following two lemmas.

Lemma 7.1. *Denote* $v := f - \nabla\cdot\mathbf{u}_h + (S^{-1}\mathbf{u}_h)\cdot\mathbf{w} - (r + \nabla\cdot\mathbf{w})p_h$, *and let* $\mathcal{E}_K$ *be the local error for the stress and displacement defined in* (4.1). *Under Assumption* (*D*5) *for* f, *for all* $K \in \mathcal{T}_h$ *it holds*

$$h_K||v||_K \lesssim \max\left\{ \sqrt{C_{S,K}} + \frac{C_{\mathbf{w},K}}{\sqrt{c_{S,K}}}h_K, \frac{C_{\mathbf{w},r,K}}{\sqrt{c_{\mathbf{w},r,K}}}h_K \right\}\mathcal{E}_K. \tag{7.1}$$

Lemma 7.2. *For all* $\sigma \in \varepsilon_h$ *it holds*

$$h_\sigma^{1/2}||[\gamma_{\mathbf{t}_\sigma}(S^{-1}\mathbf{u}_h)]||_\sigma \lesssim c_{\omega_\sigma}||S^{-1/2}(\mathbf{u}-\mathbf{u}_h)||_{\omega_\sigma}. \tag{7.2}$$

Lemma 7.3. *For all* $K \in \mathcal{T}_h$ *it holds*

$$h_K||S^{-1}\mathbf{u}_h||_K \lesssim \max\{\frac{h_K}{\sqrt{c_{S,K}}}, \frac{1}{\sqrt{c_{\mathbf{w},r,K}}}\}\mathcal{E}_K. \tag{7.3}$$

Proof. For all $K \in \mathcal{T}_h$, let ψ_K denote the bubble function on K with zero boundary values on K and $0 \le \psi_K \le 1$. The relation $S^{-1}\mathbf{u}_h|_K = (S^{-1}\mathbf{u}_h + \nabla p_h)|_K$ for all $K \in \mathcal{T}_h$ shows

$$\begin{aligned}||S^{-1}\mathbf{u}_h||_K^2 &= ||S^{-1}\mathbf{u}_h + \nabla p_h||_K^2 \lesssim ||\psi_K^{1/2}(S^{-1}\mathbf{u}_h + \nabla p_h)||_K^2\\ &= (\psi_K S^{-1}\mathbf{u}_h, S^{-1}\mathbf{u}_h + \nabla p_h)_K\\ &= (\psi_K S^{-1}\mathbf{u}_h, S^{-1}(\mathbf{u}_h - \mathbf{u}))_K + (\psi_K S^{-1}\mathbf{u}_h, S^{-1}\mathbf{u} + \nabla p_h)_K.\end{aligned} \tag{7.4}$$

Integration by parts implies

$$\begin{aligned}&(\psi_K S^{-1}\mathbf{u}_h, S^{-1}\mathbf{u} + \nabla p_h)_K\\ =&(\psi_K S^{-1}\mathbf{u}_h, \nabla(p_h - p))_K = -(\nabla\cdot(\psi_K S^{-1}\mathbf{u}_h), p_h - p)_K.\end{aligned} \tag{7.5}$$

A combination of (7.4), (7.5) and inverse inequality imply

$$||S^{-1}\mathbf{u}_h||_K^2 \lesssim \left(\frac{1}{\sqrt{c_{S,K}}}||S^{-1/2}(\mathbf{u}-\mathbf{u}_h)||_K + h_K^{-1}||p - p_h||_K\right)||S^{-1}\mathbf{u}_h||_K. \tag{7.6}$$

The desired result (7.3) then follows from (7.6). $\square$

Lemma 7.4. *For all $\sigma \in \varepsilon_h$ it holds*

$$h_K^{\frac{1}{2}}||\hat{p}||_\sigma \lesssim \begin{cases} |\sigma|^{-\frac{1}{2}}(\frac{1}{2} - \nu_\sigma)\Big(h_K||S^{-1}\mathbf{u}_h||_K + h_L||S^{-1}\mathbf{u}_h||_L\Big) & \text{if} \quad \sigma = \bar{K} \cap \bar{L}, \\ |\sigma|^{-\frac{1}{2}}(1 - \nu_\sigma)h_K||S^{-1}\mathbf{u}_h||_K & \text{if} \quad \sigma \in \varepsilon_K \cap \varepsilon_h^{\text{ext}}. \end{cases}$$

Proof. If $\sigma = \bar{K} \cap \bar{L}$, from $\displaystyle\int_\sigma \tilde{p}_h|_K ds = \int_\sigma \tilde{p}_h|_L ds$ we have

$$\int_\sigma |\hat{p}_\sigma| = \begin{cases} \displaystyle\int_\sigma (1/2 - \nu_\sigma)\left((p_h - \tilde{p}_h)|_K - (p_h - \tilde{p}_h)|_L\right) & \text{if} \quad p_K \geq p_L, \\ \displaystyle\int_\sigma (1/2 - \nu_\sigma)\left((p_h - \tilde{p}_h)|_L - (p_h - \tilde{p}_h)|_K\right) & \text{if} \quad p_K < p_L. \end{cases}$$

This relation, together with trace theorem and the postprocessing (3.9), indicates

$$\begin{aligned} \int_\sigma |\hat{p}_\sigma| &\leq (1/2 - \nu_\sigma)\Big(||p_h - \tilde{p}_h||_{\partial K} + ||p_h - \tilde{p}_h||_{\partial L}\Big) \\ &\lesssim (1/2 - \nu_\sigma)\Big(h_K^{1/2}||\nabla \tilde{p}_h||_K + h_L^{1/2}||\nabla \tilde{p}_h||_L\Big), \end{aligned}$$

which, together with the local shape regularity of elements and the postprocessing (3.8), implies

$$\begin{aligned} h_K^{1/2}||\hat{p}||_\sigma &= h_K^{1/2}|\hat{p}||\sigma|^{1/2} = h_K^{1/2}|\sigma|^{-1/2}\int_\sigma |\hat{p}_\sigma| ds \\ &\lesssim |\sigma|^{-1/2}(1/2 - \nu_\sigma)\Big(h_K||S^{-1}\mathbf{u}_h||_K + h_L||S^{-1}\mathbf{u}_h||_L\Big). \end{aligned}$$

If $\sigma \in \varepsilon_K \cap \varepsilon_h^{\text{ext}}$, the second assertion of the lemma follows from the fact $\displaystyle\int_\sigma \tilde{p}_h|_K ds = 0$ and $\nu_\sigma \leq 1/2$. □

7.2. Proof of Theorem 4.3

From the definition of $\alpha_{*,K}$ in this theorem, Lemma 7.3 shows

$$\eta_{D,K} \lesssim \alpha_{*,K}\mathcal{E}_K. \tag{7.7}$$

Denote $v := f - \nabla \cdot \mathbf{u}_h + (S^{-1}\mathbf{u}_h) \cdot \mathbf{w} - (r + \nabla \cdot \mathbf{w})p_h$, and then it holds

$$\eta_{R,K} \leq \alpha_K||v||_K + \beta_K||S^{-1}\mathbf{u}_h||_K \leq \frac{h_K}{\sqrt{c_{S,K}}}||v||_K + C_{\mathbf{w},r,K}h_K\frac{h_K}{\sqrt{c_{S,K}}}||S^{-1}\mathbf{u}_h||_K. \tag{7.8}$$

A combination of (7.8), Lemma 7.1 and Lemma 7.3, leads to

$$\eta_{R,K} \lesssim \alpha_{*,K}\mathcal{E}_K. \tag{7.9}$$

The desired result (4.15) follows from (7.7) and (7.9). □

7.3. Proof of Theorem 4.4

Notice that

$$\eta_{NC,K} \leq \sqrt{\Lambda_{\mathbf{w},r,K}}h_K||S^{-1}\mathbf{u}_h||_K + \sum_{\sigma \in \varepsilon_K} \Lambda_\sigma^{1/2}h_\sigma^{1/2}||[\gamma_{\mathbf{t}_\sigma}(S^{-1}\mathbf{u}_h)]||_\sigma, \tag{7.10}$$

$$\eta_{C,K} \leq \Lambda_{\nabla\cdot\mathbf{w},K} h_K ||S^{-1}\mathbf{u}_h||_K + \sum_{\sigma\in\varepsilon_K} \Lambda_{\mathbf{w},\sigma} h_\sigma^{1/2} ||[\gamma_{\mathbf{t}_\sigma}(S^{-1}\mathbf{u}_h)]||_\sigma. \tag{7.11}$$

From the definitions of $\beta_{*,K}$ and c_{ω_σ} in this theorem, we respectively apply Lemmas 7.2-7.3 to the above two inequalities so as to obtain

$$\eta_{NC,K} \lesssim \beta_{*,K}\mathcal{E}_K + \sum_{\sigma\in\varepsilon_K} c_{\omega_\sigma}\Lambda_\sigma^{1/2} ||S^{-1/2}(\mathbf{u}-\mathbf{u}_h)||_{\omega_\sigma}, \tag{7.12}$$

$$\eta_{C,K} \lesssim \beta_{*,K}\mathcal{E}_K + \sum_{\sigma\in\varepsilon_K} \Lambda_{\mathbf{w},\sigma} c_{\omega_\sigma} ||S^{-1/2}(\mathbf{u}-\mathbf{u}_h)||_{\omega_\sigma}. \tag{7.13}$$

The assertion (4.16) follows from (7.12) and (7.13). □

7.4. Proof of Theorem 4.5

The local shape regularity of elements implies

$$\eta_{U,K} \lesssim \frac{1}{\sqrt{c_{S,K}}} \sum_{\sigma\in\varepsilon_K} |(\mathbf{w}\cdot\mathbf{n})|_\sigma| (h_K^{1/2} ||\hat{\hat{p}}_\sigma||_\sigma + h_\sigma ||S^{-1}\mathbf{u}_h||_{\omega_\sigma}). \tag{7.14}$$

Then the desired estimate (4.17) follows from (7.14), Lemma 7.4 and the definitions of the constants λ_σ, ρ_σ, and $\mathcal{E}_{D,\omega_\sigma}$.

8. Numerical Experiments

In this section, we test our proposed posteriori error estimators on three model problems.

8.1. Model problem with singularity at the origin

We consider the problem (1.1) in an L-shape domain $\Omega = \{(-1,1)\times(0,1)\} \cup \{(-1,0)\times(-1,0)\}$ with $\mathbf{w} = r = 0$ and $f = 0$. The exact solution is given by

$$p(\rho,\theta) = \rho^{2/3}\sin(2\theta/3),$$

where ρ, θ are the polar coordinates.

It is well known that this model possesses singularity at the origin. The original mesh consists of 6 right-angled triangles. We employ the centered mixed scheme described in Section

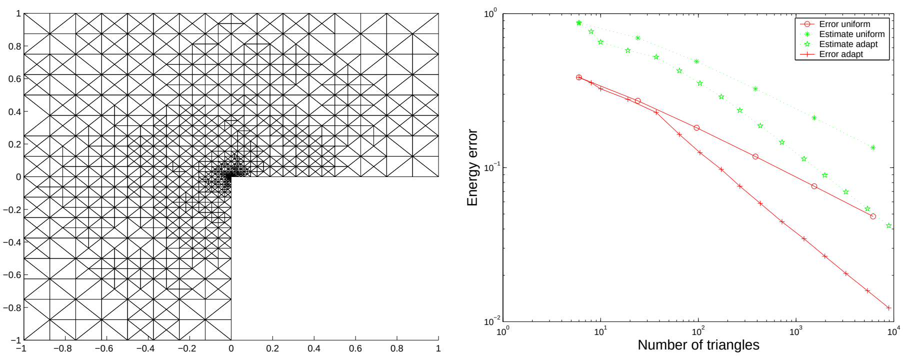

Fig. 8.1. A mesh with 1635 triangles (left) and the estimated and actual errors in uniformly / adaptively refined meshes (right).

3.1 to compute the approximaton solution, mark elements in terms of Dörfler marking with the marking parameter $\theta = 0.5$, and then use the "longest edge" refinement to recover an admissible mesh. Specially, the uniform refinement means that all elements should be marked. We note that in the given case, the residual estimators $\eta_{R,K}$ vanish over all $K \in \mathcal{T}_h$.

We see in the first figure of Fig. 8.1 with 1635 elements that the refinement concentrates around the origin, which means the predicted error estimator captures well the singularity of the solution. The second graph of Fig. 8.1 reports the estimated and actual errors of the numerical solutions on uniformly and adaptively refined meshes. It can be seen that one can substantially reduce the number of unknowns necessary to obtain the prescribed accuracy by using the a posteriori error estimates and adaptively refined meshes, and that the error of the flux in L^2 norm uniformly reduces with a fixed factor on two successive meshes, and that the adaptive mixed finite element method is a contraction with respect to the energy error.

8.2. Model problem with inhomogeneous diffusion tensor [19, 30, 37]

We consider the problem (1.1) in a square domain $\Omega = (-1, 1) \times (-1, 1)$ with $\mathbf{w} = r = 0$ and $f = 0$, where Ω is divided into four subdomains Ω_i $(i = 1, 2, 3, 4)$ corresponding to the axis quadrants (in the counterclockwise direction), and the diffusion-dispersion tensor S is piecewise constant matrix with $S = s_i I$ in Ω_i. We suppose the exact solution of this model has the form

$$p(r, \theta) = r^{\alpha}(a_i \sin(\alpha\theta) + b_i \cos(\alpha\theta))$$

in each Ω_i with Dirichlet boundary conditions. Here r, θ are the polar coordinates in Ω, a_i and b_i are constants depending on Ω_i, and α is a parameter. We note that the stress solution $\mathbf{u} = -S\nabla p$ is not continuous across the interfaces, and only its normal component is continuous. It finally exhibits a strong singularity at the origin. We consider two sets of coefficients in the following table:

Case 1	Case 2
$s_1 = s_3 = 5$, $s_2 = s_4 = 1$	$s_1 = s_3 = 100$, $s_2 = s_4 = 1$
$\alpha = 0.53544095$	$\alpha = 0.12690207$
$a_1 = 0.44721360$, $b_1 = 1.00000000$	$a_1 = 0.10000000$, $b_1 = 1.00000000$
$a_2 = -0.74535599$, $b_2 = 2.33333333$	$a_2 = -9.60396040$, $b_2 = 2.96039604$
$a_3 = -0.94411759$, $b_3 = 0.55555555$	$a_3 = -0.48035487$, $b_3 = -0.88275659$
$a_4 = -2.40170264$, $b_4 = -0.48148148$	$a_4 = 7.70156488$, $b_4 = -6.45646175$

The origin mesh consists of 8 right-angled triangles. We use the centered scheme compute the approximation solution, and mark elements in terms of Dörfler marking with the marking parameter $\theta = 0.7$ in the first case and $\theta = 0.94$ in the second case. We note that the elementwise estimators ξ_K are used as the a posteriori error indicators, since the residual estimators $\eta_{R,K}$ vanish over $K \in \mathcal{T}_h$.

In Table 8.1 we show for Case 1 some results of the actual error E_k, the a posteriori indicator η_k, the experimental convergence rate, EOC_E, of E_k, and the experimental convergence rate, EOC_η, of η_k, where

$$\mathrm{EOC}_E := \frac{\log(E_{k-1}/E_k)}{\log(\mathrm{DOF}_k/\mathrm{DOF}_{k-1})}, \quad \mathrm{EOC}_\eta := \frac{\log(\eta_{k-1}/\eta_k)}{\log(\mathrm{DOF}_k/\mathrm{DOF}_{k-1})},$$

and DOF_k denotes the number of elements with respect to the $k-$th iteration. We can see that the convergence rates EOC_E and EOC_η are close to 0.5 as the iteration number $k = 15$,

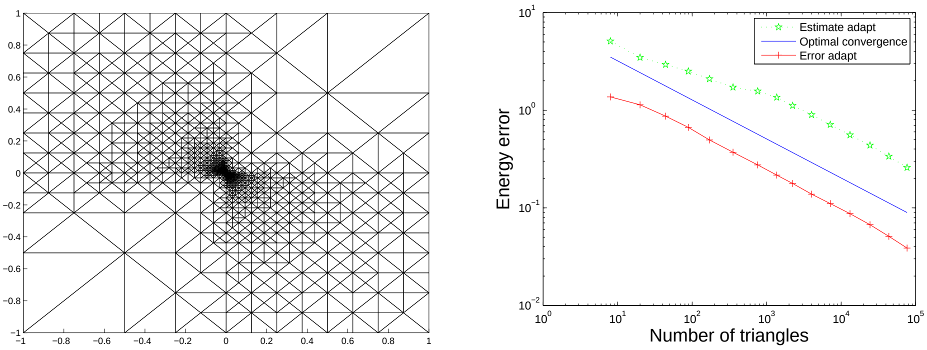

Fig. 8.2. A mesh with 4763 triangles (left) and the estimated and actual error against the number of elements in adaptively refined meshes (right): Case 1.

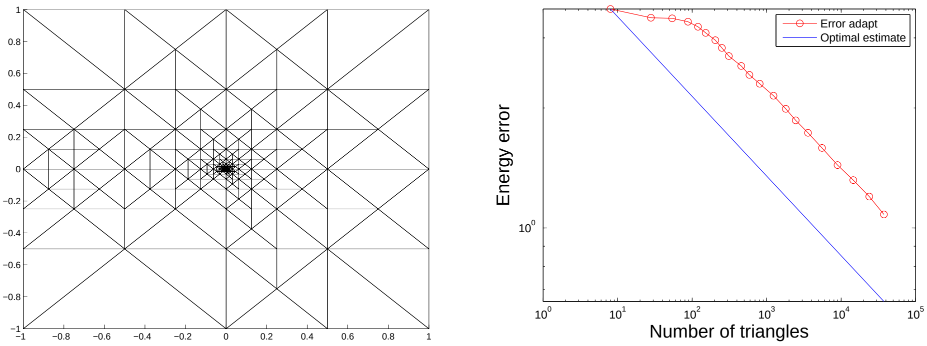

Fig. 8.3. A mesh with 1093 triangles (left) and the actual error against the number of elements in adaptively refined mesh (right): Case 2.

which means the optimal decay of the actual error and a posteriori error indicator η_k is almost attained after 15 iterations with optimal meshes.

Fig. 8.2 shows an adaptively refined mesh with 4763 elements and the estimated and actual errors against the number of elements in adaptively refined meshes for Case 1. Fig. 8.3 shows an adaptively refined mesh with 1093 elements and the actual error against the number of elements in adaptively refined meshes for Case 2.

Table 8.1: Results of actual error E_k, a posteriori indicator η_k, and their convergence rates EOC_E and EOC_η: Case 1.

k	DOF_k	E_k	η_k	EOC_E	EOC_η	k	DOF_k	E_k	η_k	EOC_E	EOC_η
1	8	1.3665	5.0938	–	–						
2	20	1.1346	3.4700	0.2030	0.4189	9	2235	0.1776	1.1115	0.4016	0.4004
3	44	0.8682	2.9300	0.3394	0.2145	10	4025	0.1381	0.8958	0.4276	0.3667
4	89	0.6672	2.5032	0.3738	0.2235	11	7165	0.1106	0.7111	0.3851	0.4004
5	171	0.4953	2.0907	0.4562	0.2757	12	13188	0.0871	0.5566	0.3915	0.4015
6	354	0.3708	1.7170	0.3979	0.2706	13	24445	0.0671	0.4368	0.4227	0.3927
7	760	0.2751	1.5639	0.3907	0.1222	14	43785	0.0510	0.3365	0.4707	0.4476
8	1368	0.2163	1.3529	0.4091	0.2466	15	76770	0.0387	0.2581	0.4915	0.4724

From the first figures of Figs. 8.2-8.3, we can see that the refinement again concentrates around the origin, which means the adaptive mixed finite element method detects the region of rapid variation. In the second graphs of Figs. 8.1-8.3 each includes an optimal convergence line, which shows in both cases, the energy error performs a trend of descend with an optimal order convergent rate. Simultaneously, from the second graphs of Figs. 8.1-8.3, we also see that the proposed estimators are efficient with respect to the strongly discontinuously coefficients.

We note that the energy error is approximated with a 7-point quadrature formula in each triangle.

8.3. Convection-dominated model problem [37]

Let $S = \varepsilon I$, $\mathbf{w} = (0,1)$, $r = 1$ and $\Omega = (0,1) \times (0,1)$ in the model (1.1). We consider four cases: $\varepsilon = 0.1, 0.01, 0.001, 0.0001$. Neumann boundary conditions on the upper side, Dirichlet boundary conditions elsewhere, and the source term f are chosen such that the exact solution has the form

$$p(x,y) = 0.5\left(1 - \tanh(\frac{0.5-x}{a})\right)$$

with a a positive constant. This solution is, in fact, one-dimensional and possesses an internal layer of width a which we shall set, respectively, equal to 0.1, 0.05, 0.02, 0.001.

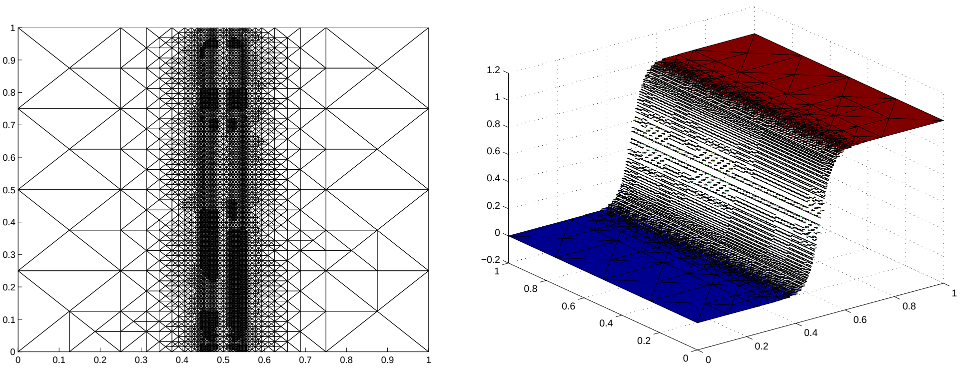

Fig. 8.4. A mesh with 12943 triangles (left) and the approximate displacement (piecewise constant) on the corresponding adaptively refined mesh (right) for $\varepsilon = 0.01$ and a=0.05.

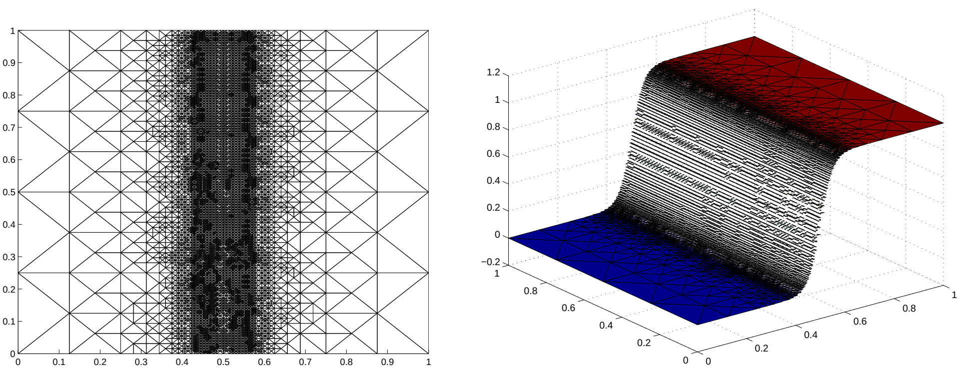

Fig. 8.5. A mesh with 16951 triangles (left) and approximate displacement (piecewise constant) on the corresponding adaptively refined mesh (right) for $\varepsilon = 0.001$ and a=0.05.

We still start computations from an origin mesh which consists of 8 right-angled triangles, and refine it either uniformly (up to five refinements) or adaptively.

In Fig. 8.4 with $\varepsilon = 0.01, a = 0.05$ and Fig. 8.5 with $\varepsilon = 0.001, a = 0.05$, we can see that the refinement concentrates at an internal layer of width $a = 0.05$, and is away from the center of the shock. Both the convection-dominated regime on coarse grids and diffusion-dominated regime obtain the progressive refinement. The effect is still rather good even if the approximation to displacement is piecewise constant.

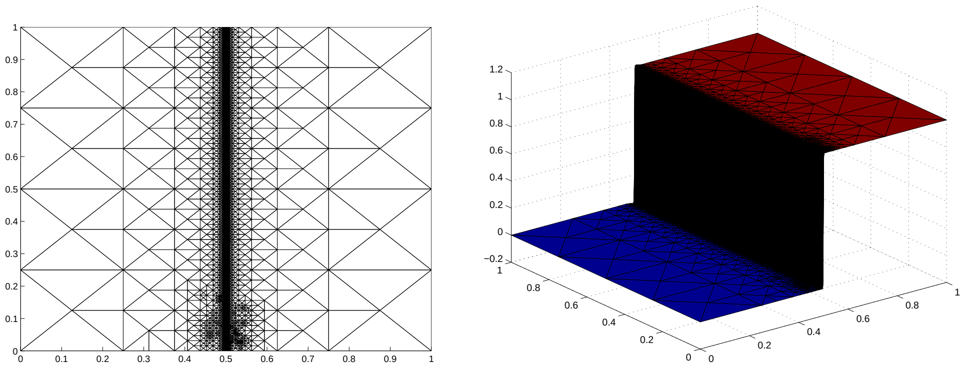

Fig. 8.6. A mesh with 39189 triangles (left) and postprocessing approximate displacement on the corresponding adaptively refined mesh (right) for $\varepsilon = 0.0001$ and a=0.001.

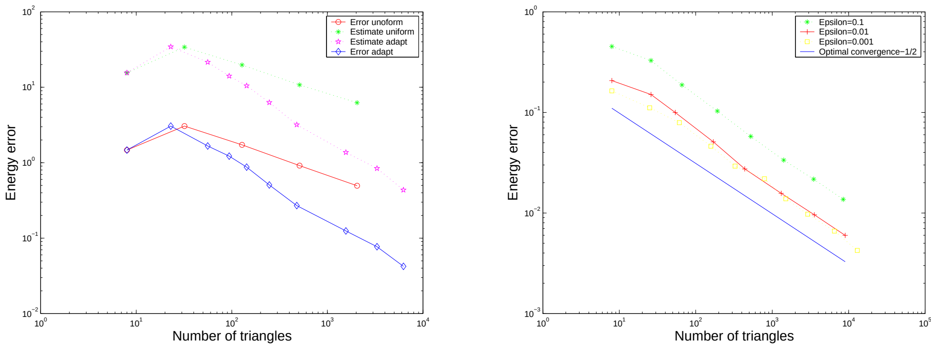

Fig. 8.7. Estimated and actual error against the number of elements in uniformly and adaptively refined meshes for $\varepsilon = 0.1, a = 0.02$ (left) and actual error against the nunber of elements in adaptively refined meshes for diffirent ε for $a = 0.1$(right) .

Fig. 8.6 shows the mesh with 39184 triangles (left) and postprocessing approximation to the scalar displacement on the corresponding adaptively refined mesh (right) in case: $\varepsilon = 0.0001$ and width $a = 0.001$. Here the value of the postprocessing approximation on each node is taken as the algorithmic mean of the values of the displacement finite element solution on all the elements sharing the vertex. The reason for the postprocessing is that the displacement finite element solution is not continuous on each vertex of the triangulation. We again see that the refinement focuses around layer of width $a = 0.001$, this indicates that the estimators actually capture interior layers and resolve them in convection-domianed regions. In addition, the postprocessing approximation to the scalar displacement obtains a satisfactory result.

In Fig. 8.7 with $\varepsilon = 0.1,\ a = 0.02$ (left), the estimated and actual errors are plotted against the number of elements in uniformly and adaptively refined meshes. Again, we see that one can substantially reduce the unknowns necessary to attain the prescribed precision by using the proposed estimators and adaptively refined grids. The second graph of Fig. 8.7 shows the actual error against the number of elements in adaptively refined meshes for different ε in case $a = 0.1$, and also concludes a line with optimal convergence $-1/2$. In addition, we also see that the almost same decay of error occurs in the cases of $\varepsilon = 0.01$ and $\varepsilon = 0.001$. This is conformable to the local efficiency results of the estimators, since the local Péclet number is not large in these cases (cf. Remark 4.2).

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