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PRICING EUROPEAN OPTIONS ON ZERO-COUPON BONDS WITH A FITTED FINITE VOLUME METHOD

KAI ZHANG AND XIAO QI YANG

Abstract. We present a novel numerical scheme to price European options on discount bond, where the single factor models are adopted for the short interest rate. This method is based on a fitted finite volume (FFVM) scheme for the spatial discretization and an implicit scheme for the time discretization. We show that this scheme is consistent, stable and monotone, hence it ensures the convergence to the solution of continuous problem. Numerical experiments are performed to verify the effectiveness and usefulness of this new method.

Key words. Option pricing, finite volume method, partial differential equation.

1. Introduction

Interest rate derivatives, like bond options, range notes, interest rate caps, swaps and swaptions, are commonly traded in the financial markets. A large number of attention have been given to the development of models to price and hedge these types of derivatives. While the Black and Scholes [4] model has been well-known as the model for stock derivatives, many approaches to modeling interest rate derivatives are simultaneously established among academics and practitioners, such as Black-Karasinski model [3], Vasicek model, CIR model, HW model [9, 13, 6], Brennan-Schwartz model [5], and so on. Compared to stock derivatives, the pricing and hedging of interest rate derivatives pose greater challenges. For instance, for a simple bond option, unlike stock derivatives, its underlying asset is a bond whose price is dependent on interest rate and time. It is thus necessary to develop dynamic models that describe the stochastic evolution of the whole yield curve, which makes pricing interest rate derivatives a complex task.

In this paper, we focus on pricing European options on zero-coupon bonds under the single factor models. In [10, 6, 13], the price of this type of options has been investigated. Usually, this problem is formulated as a parabolic partial differential equation (PDE) with suitable boundary and terminal conditions [10]. In some simple cases, analytical solutions are available. However, these analytical solution usually is not easily computable [13]. Moreover, in most practical situations (for instance, path-dependent options) analytical solutions are unavailable. Hence, numerical solutions are normally sought for pricing bond options. Lattice method and the usual finite difference method are commonly used to pricing stock options. Unfortunately, it is pointed out in [14] and [1] that these methods are only convergent for certain combination of parameters.

The fitted finite volume method was first used to price the standard European stock options in [16], then generalized to other types of options, see [8, 18], etc. The method is based on a popular exponentially fitting technique widely used for problems with boundary and interior layers (cf. [11, 12]). It has been shown that this method makes greater success in pricing stock options, where the standard Black-Scholes equations are applied. It is easy to see that the PDE resulted from

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European option model is degenerate and convection-dominated, hence the fitted finite volume method is a natural way to overcoming these difficulties. Its success motivates us to generalize the fitted finite volume technique to price bond options. On this basis, in this paper we derive a novel fitted finite volume method to price European bond options. We then apply this new fitted finite volume scheme in space with the implicit scheme in time to numerical valuation of European options on a discount bonds under single factor models. To guarantee the convergence of this new numerical scheme, we show that this numerical scheme is consistent, stable and monotone, hence convergent. To verify the accuracy and robustness of the new numerical scheme, some numerical experiments including a vanilla European option and a digital option on a discount bond under CIR model are implemented. Moreover, to testify its effectiveness a vanilla European option on a discount bond under a mean-reverting lognormal model is investigated. These numerical results show that this numerical scheme is very accurate, efficient and robust.

The paper is organized as follows. In the next section, the mathematical model for European options on a discount bond is presented. Then, the fitted finite volume method is developed in Section 3. In Section 4, the full discrete scheme is proposed and by showing the stability and monotonicity of this numerical scheme, its convergence is investigated. Finally, in the last section three numerical examples are given to illustrated the convergence and robustness of this numerical scheme.

2. Mathematical model for options on a zero-coupon bond

In this paper, we assume the following single factor model is applied for the interest rate term structure. That means the short-term interest rate r is governed by a stochastic process of the form.

$$dr = A\left(r,t\right)dt + \sigma r^{\xi} dW, \tag{1}$$

where dW is the increment of a Wiener process, $A\left(r,t\right)$ is the instantaneous drift, σr^{ξ} is the instantaneous volatility. Some well known-examples of one-factor interest rate models are special cases of Equation (1). Particularly, if $A\left(r,t\right)$is specified to be mean-reverting and independent of time t, and σ is a constant, the setting $\xi = 0, 1/2, 1, 3/2$ produces the Vasicek model, CIR model, lognormal model and cubic variance model, respectively.

Now, let $P\left(r,t,s\right)$ be the price of a pure discount bond with face value \$1 at its maturity date s. Based on the standard no-arbitrage pricing arguments, the bond price is governed by the following parabolic partial differential equation (PDE) [17]:

$$-\frac{\partial P}{\partial t} = \frac{1}{2}\sigma^2 r^{2\xi}\frac{\partial^2 P}{\partial r^2} + \left(A\left(r,t\right) + \sigma\lambda\left(r,t\right)r^{\xi}\right)\frac{\partial P}{\partial r} - rP, \tag{2}$$

where $\lambda\left(r,t\right) \geq 0$ is called the market price of risk. At the maturity date s the price of a pure discount bond is its face value, i.e.

$$P\left(r, t = s, s\right) = 1.$$

The boundary conditions are usually given by the following form

$$\begin{aligned} P\left(0,t,s\right) &= g_0\left(r,t\right), \qquad r \to 0, \\ P\left(r,t,s\right) &= 0, \qquad r \to \infty, \end{aligned}$$

where $g_0(r,t)$ can be determined according to different interest rate models.

Let $V\left(r,t\right)$ denote the value of an European option on a pure discount bond with striking price K,where the holder can receive the payoff $V^*\left(r,T\right)$ at expiry date T.