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Some New Inequalities for Wavelet Frames on Local Fields

Firdous A. Shah[1,*], Owais Ahmad[2] and Neyaz A. Sheikh[2]

[1] *Department of Mathematics, University of Kashmir, South Campus, Anantnag 192101, Jammu and Kashmir, India*
[2] *Department of Mathematics, National Institute of Technology, Hazratbal, Srinagar 190006, Jammu and Kashmir, India*

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Abstract. Wavelet frames have gained considerable popularity during the past decade, primarily due to their substantiated applications in diverse and widespread fields of science and engineering. Finding general and verifiable conditions which imply that the wavelet systems are wavelet frames is among the core problems in time-frequency analysis. In this article, we establish some new inequalities for wavelet frames on local fields of positive characteristic by means of the Fourier transform. As an application, an improved version of the Li-Jiang inequality for wavelet frames on local fields is obtained.

Key Words: Frame, inequalities, wavelet frame, local field, Fourier transform.

AMS Subject Classifications: 42C15, 42C40, 43A70, 11S85

1 Introduction

The notion of frame in a general Hilbert space was first introduced by Duffin and Schaeffer [5] to study some deep problems in non-harmonic Fourier series. Frames are basis-like systems that span a vector space but allow for linear dependency, which can be used to reduce noise, find sparse representations, or obtain other desirable features unavailable with orthonormal bases. The idea of Duffin and Schaeffer did not generate much interest outside non-harmonic Fourier series until the seminal work by Daubechies, Grossmann, and Meyer [3]. They combined the theory of continuous wavelet transforms with the theory of frames to introduce wavelet (affine) frames for $L^2(\mathbb{R})$. After their work, the theory of frames began to be studied widely and deeply. Today, the theory of frames has become an interesting and fruitful field of mathematics with abundant applications in

*Corresponding author. *Email addresses:* `fashah79@gmail.com` (F. A. Shah), `siawoahmad@gmail.com` (O. Ahmad), `neyaznit@yahoo.co.in` (N. A. Sheikh)

signal processing, image processing, harmonic analysis, Banach space theory, sampling theory, wireless sensor networks, optics, filter banks, quantum computing, and medicine and so on. An introduction to the frame theory and its applications can be found in [2,4].

The following is the standard definition on frames in a Hilbert space. A sequence $\{f_k : k\in\mathbb{Z}\}$ of elements of a Hilbert space $\mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there exist constants $A,B>0$ such that

$$A\|f\|_2^2\le\sum_{k\in\mathbb{Z}}\big|\langle f,f_k\rangle\big|^2\le B\|f\|_2^2 \tag{1.1}$$

holds for every $f\in\mathcal{H}$, and we call the optimal constants A and B the lower frame bound and the upper frame bound, respectively. A tight frame refers to the case when $A=B$, and a Parseval frame refers to the case when $A=B=1$.

An important example about frame is wavelet frame, which is obtained by translating and dilating a finite number of functions. Wavelet frames are different from the orthonormal wavelets because of redundancy. By sacrificing orthonormality and allowing redundancy, wavelet frames become much easier to construct than the orthonormal wavelets. An important problem in practice is therefore to determine conditions on the wavelet function, dilation and translation parameters so that the corresponding wavelet system forms a frame. In her famous book, Daubechies [2] proved the first result on the necessary and sufficient conditions for wavelet frames, and then, Chui and Shi [1] gave an improved result. In recent years, these conditions have been further improved and investigated by many authors [6, 9, 13, 15, 16, 18].

A field K equipped with a topology is called a local field if both the additive K^+ and multiplicative groups K^* of K are locally compact Abelian groups. For example, any field endowed with the discrete topology is a local field. For this reason we consider only non-discrete fields. The local fields are essentially of two types (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). The local fields of characteristic zero include the p-adic field $\mathbb{Q}_p$. Examples of local fields of positive characteristic are the Cantor dyadic group and the Vilenkin p-groups. The local field K is a natural model for the structure of wavelet frame systems, as well as a domain upon which one can construct wavelet basis functions. There is a substantial body of work that has been concerned with the construction of wavelets on K, or more generally, on local fields of positive characteristic. For example, Li and Jiang [7] have obtained a necessary condition and a set of sufficient conditions for wavelet frames on local fields of positive characteristic in the frequency domain. The characterizations of tight wavelet frames on local fields were completely established by Shah and Abdullah [11] by virtue of two basic equations in the Fourier domain. These studies were continued by Shah and his colleagues in [8, 10, 12, 14], where they have provided some algorithms for constructing periodic, wave packet frames and semi-orthogonal wavelet frames on local fields of positive characteristic.

In the present paper, we shall present generalized inequalities for wavelet frames on local fields of positive characteristic. In particular, we establish a necessary condition for wavelet frames on local fields of positive characteristic via Fourier transform. The