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OVERLAPPING DOMAIN DECOMPOSITION PRECONDITIONERS FOR UNCONSTRAINED ELLIPTIC OPTIMAL CONTROL PROBLEMS

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Abstract. In this paper, we propose several overlapping domain decomposition preconditioners for solving the unconstrained elliptic optimal control problem, based on the two level additive Schwarz algorithm. We consider the cases with controls on the whole domain and controls from a local subset. The latter case can be viewed as the subproblems when we solve the control-constrained control problem by using semi-smooth Newton method. When the controls act on the whole domain, we construct a symmetric and positive definite preconditioner which is proved to be robust combined with preconditioned MINRES method, and a symmetric and indefinite preconditioner which can be used in the preconditioned GMRES method and shows better numerical performance than the positive definite one. When the controls act on a local subset, we also construct a similar symmetric and indefinite preconditioner, the numerical experiments show its efficiency when combined with preconditioned GMRES method.

Key words. Overlapping domain decomposition method, elliptic optimal control problem, preconditioned MINRES method, preconditioned GMRES method.

1. Introduction

In this paper, we consider the following unconstrained elliptic optimal control problem with distributed control:

$$\min_{u\in L^2(\Omega_0)} J(y,u) \quad = \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega_0)} \tag{1}$$

subject to

$$\begin{cases} -\Delta y = f + B_0 u & \text{in } \Omega \\ \quad y = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary $\partial\Omega$, B_0 is the extension by zero operator from $L^2(\Omega_0)$ to $L^2(\Omega)$ with $\Omega_0 \subseteq \Omega$ the control domain and $\text{meas}(\Omega_0) > 0$, $f \in L^2(\Omega)$ is a given function, $y_d \in L^2(\Omega)$ is the desired state or observation, $\alpha > 0$ is the regularization parameter.

It can be proved by standard arguments (see e.g., [14]) that the optimal control problem (1)-(2) admits a unique solution $u \in L^2(\Omega_0)$, which can be characterized by its first order necessary (also sufficient) optimality system

$$\begin{cases} -\Delta y = f + B_0 u & \text{in } \Omega, \quad y = 0 \quad \text{on } \partial\Omega, \\ -\Delta p = y - y_d & \text{in } \Omega, \quad p = 0 \quad \text{on } \partial\Omega, \\ \alpha u + B_0' p = 0 & \text{in } \Omega_0, \end{cases} \tag{3}$$

where p is the adjoint state and B_0' is the adjoint operator of B_0 associated with L^2 inner product. The three equations in (3) serve as the state equation, the

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adjoint equation and the first order optimality condition. By eliminating the control variable u through the third equation in (3) we arrive at the equivalent form

$$(4)\qquad \begin{cases} -\Delta y = f - \dfrac{1}{\alpha} B_0 B_0' p \;\text{ in } \Omega, \quad y = 0 \quad \text{on } \partial\Omega; \\ -\Delta p = y - y_d \;\text{ in } \Omega, \quad p = 0 \quad \text{on } \partial\Omega. \end{cases}$$

We note that (4) formulates a saddle point problem involving the state and the adjoint state variables. With appropriate finite dimensional discretization, e.g., finite element method, we are led to a symmetric and strongly indefinite linear system whose efficient solving is challenging in considering the dependence of the linear system on the mesh size h and the regularization parameter α, which will be the focus of current paper.

Optimal control problems governed by partial differential equations play an important role in science and engineering. For the theoretical results we refer to [14]. Recently, the increasing requirement of efficient simulations for such kind of problems stimulates the research of both discretization schemes and efficient solvers for optimization problems with PDEs constraints. A lot of achievements have been made on this subject in the decades. For recent developments on optimization algorithms and convergence analysis of numerical schemes we refer to [12] for more details.

In addition to the convergence analysis of discretization schemes and the design of optimization algorithms, the fast and robust solving of the resulting algebraic system is also very important for efficient simulations of PDE-constrained optimal control problems. There are a lot of attempts to study the efficient solution of optimal control problems which differ from the approaches utilized. For example, in [27, 20] the authors used the preconditioned Krylov subspace method to solve the first order optimality system by constructing some block preconditioners. In [3, 22, 19] the authors used the mutigrid method to design fast solvers. Another strategy is to use the domain decomposition methods to deal with the optimal control problem, see e.g., [1, 2, 10, 11, 15]. We also mention [16, 17, 26] for the parallel implementations of domain decomposition type algorithms.

Domain decomposition methods (DDM for short) have been successfully used to construct fast solvers for the self-adjoint and positive definite partial differential equations, the essential parallel ability makes them attractive in applications. For more details on the design and convergence of DDM we refer to the monograph [23], the review papers [24, 25] and the references cited therein. As to the design of DDM for nonselfadjoint or indefinite problems, we refer to [6] and the references therein.

There are also some contributions of DDM to solving PDE-constrained optimal control problems. The applications of DDM for optimal control problems can roughly be divided into two categories, depending on how the domain decomposition strategy is integrated with the optimization. One category is that the domain decomposition strategy is used only at the PDE level. That is to use the domain decomposition methods to solve the state equation and the adjoint equation, respectively. We refer to [8] for such kind of approach where a projected gradient method serves as the outer optimization algorithm. An obvious weakness of this approach is that the robustness of the algorithm with respect to the regularization