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# HIGH DEGREE IMMERSED FINITE ELEMENT SPACES BY A LEAST SQUARES METHOD

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**Abstract.** We present a least squares framework for constructing $p$-th degree immersed finite element (IFE) spaces for typical second-order elliptic interface problems. This least squares formulation enforces interface jump conditions including extended ones already proposed in the literature, and it guarantees the existence of $p$-th IFE shape functions on interface elements. The uniqueness of the proposed $p$-th degree IFE shape functions is also discussed. Computational results are presented to demonstrate the approximation capabilities of the proposed $p$-th IFE spaces as well as other features.

**Key words.** Interface problems, discontinuous coefficients, finite element spaces, curved interfaces, higher order.

## 1. Introduction

In this manuscript, we present a least squares procedure for constructing higher degree IFE spaces for solving second-order elliptic interface problems of the form

$$-\nabla \cdot (\beta \nabla u) = f, \quad \text{in } \Omega = \Omega^1 \cup \Omega^2, \tag{1a}$$

$$u = g, \quad \text{on } \partial\Omega, \tag{1b}$$

where, without loss of generality, the domain $\Omega \subseteq \mathbb{R}^2$ is assumed to be split by an interface curve $\Gamma$ into two subdomains $\Omega^1$ and $\Omega^2$. To close the problem we impose the classical jump conditions on the interface

$$[u]_\Gamma := u^1|_\Gamma - u^2|_\Gamma = 0, \tag{1c}$$

$$\big[\beta \nabla u \cdot \mathbf{n}\big]_\Gamma := \beta_1 \nabla u^1 \cdot \mathbf{n}|_\Gamma - \beta_2 \nabla u^2 \cdot \mathbf{n}|_\Gamma = 0, \tag{1d}$$

where $\mathbf{n}$ is the unit normal vector to the interface $\Gamma$. The diffusion coefficient $\beta$ is assumed to be a positive piecewise constant function such that

$$\beta(X) = \begin{cases} \beta_1 & \text{for } X \in \Omega^1, \\ \beta_2 & \text{for } X \in \Omega^2. \end{cases}$$

It is well-known that, in both theory and practice, traditional finite element methods can be used to solve interface problems provided that their meshes are body-fitting [4, 9, 12, 40], see an illustration in Figure 1 for a body-fitting mesh. This body-fitting restriction hinders efficient applications of finite element methods in applications where the interfaces evolve because of the involved physics such as in multi-phase fluid simulation [26, 29] or because of computational algorithms such as those for shape optimization problems [7, 22]. Generating a new mesh to fit an evolving interface at each step is not only time consuming, but it can also cause several difficulties such as the need for different finite element spaces on different meshes at different steps. Hence, numerical methods have been developed

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that can use interface-independent meshes to solve interface problems by adapting traditional numerical methods for solving partial differential equations. Adaptions or modifications can be loosely categorized into two groups. Methods from the first group employ suitable equations in elements around the interface either in finite difference formulation such as the immersed interface method [28, 31] or in finite element formulation such as the unfitted finite element method based on Nitsche's penalty idea [20, 21]. Methods from the second group use specially constructed local approximation functions on interface elements according to the involved interface jump conditions. Instances of these methods are extended finite element methods (XFEM) [5, 37, 39] and IFE methods [14, 17, 18, 23, 27, 30, 33].

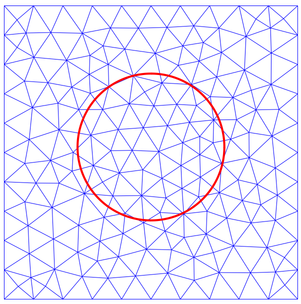

FIGURE 1. An body-fitting mesh and interface.

IFE methods use Hsieh-Clough-Tocher type macro finite element functions [8, 13] on interface elements. For local IFE spaces consisting of piecewise polynomials defined on subelements formed by cutting each interface element with a line approximating the interface, we refer readers to [14, 27, 32, 33] for linear polynomials, [23, 24, 34] for bilinear polynomials and [17, 41] for rotated $Q_1$ polynomials. All linear and bilinear IFE spaces mentioned above have the optimal convergence rates. Higher degree IFE spaces are desirable since they lead to highly accurate solutions and can be used to design efficient local adaptive $h$-$p$ refinement algorithms.

Authors in [3, 10, 11, 35] discussed higher degree IFE spaces for 1D interface problems. They considered the extended jump conditions that led to unique construction of the IFE shape functions and optimally convergent IFE spaces. In particular, a $p$-th degree optimally convergent IFE space was developed in [3]. For 2D interface problems, there are two major obstacles for the development of higher degree IFE spaces. One obstacle is that, on each interface element, a higher degree IFE function can no longer be a macro finite element function piecewisely defined on polygonal subelements because of the intrinsic second-order $\mathcal{O}(h^2)$ limitation of the line. Another obstacle is the proper choice and enforcement of extended jump conditions for determining all the coefficients in each higher degree IFE shape function in piecewise polynomial format such that the resulting IFE space has the optimal approximation capability.

There have been efforts to overcome these obstacles. Recently, several authors [16, 17, 18] have investigated piecewise polynomial shape functions constructed by enforcing jump conditions on the actual interface curve. Even though the involved