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MAXIMAL L^p ERROR ANALYSIS OF FEMS FOR NONLINEAR PARABOLIC EQUATIONS WITH NONSMOOTH COEFFICIENTS

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Abstract. The paper is concerned with L^p error analysis of semi-discrete Galerkin FEMs for nonlinear parabolic equations. The classical energy approach relies heavily on the strong regularity assumption of the diffusion coefficient, which may not be satisfied in many physical applications. Here we focus our attention on a general nonlinear parabolic equation (or system) in a convex polygon or polyhedron with a nonlinear and Lipschitz continuous diffusion coefficient. We first establish the discrete maximal L^p-regularity for a linear parabolic equation with time-dependent diffusion coefficients in $L^\infty(0,T;W^{1,N+\epsilon})\cap C(\overline{\Omega}\times[0,T])$ for some $\epsilon>0$, where N denotes the dimension of the domain, while previous analyses were restricted to the problem with certain stronger regularity assumption. With the proved discrete maximal L^p-regularity, we then establish an optimal L^p error estimate and an almost optimal L^∞ error estimate of the finite element solution for the nonlinear parabolic equation.

Key words. Finite element method, nonlinear parabolic equation, polyhedron, nonsmooth coefficients, maximal L^p-regularity, optimal error estimate.

1. Introduction

The paper is to present a general framework for numerical analysis of optimal errors of finite element methods for nonlinear parabolic equations with nonsmooth coefficients. To illustrate our idea, we consider the equation

$$
\left\{
\begin{array}{ll}
\partial_t u - \displaystyle\sum_{i,j=1}^N \partial_i\big(\sigma_{ij}(u,x)\partial_j u\big) = g(u,\nabla u,x) & \text{in } \Omega\times(0,\infty),\\
u = 0 & \text{on } \partial\Omega\times(0,\infty),\\
u(\cdot,0) = u^0 & \text{in } \Omega,
\end{array}
\right.
\tag{1}
$$

in a polyhedral domain in $\mathbb{R}^N$, $N=2,3$, and its semi-discrete finite element approximation

$$
\left\{
\begin{array}{l}
\big(\partial_t u_h, v_h\big) + \displaystyle\sum_{i,j=1}^N \big(\sigma_{ij}(u_h,x)\partial_j u_h, \partial_i v_h\big) = (g(u_h,\nabla u_h,x), v_h),\ \forall\ v_h\in S_h,\\
u_h(0) = u_h^0,
\end{array}
\right.
\tag{2}
$$

where S_h denotes a finite element subspace of $H_0^1(\Omega)$ consisting of continuous piecewise polynomials of degree $r\geq 1$ subject to a quasi-uniform triangulation of Ω with a mesh size h, and $u_h^0 = I_h u^0$ denotes the Lagrange interpolation of the initial data u^0. We only impose certain local conditions on the coefficients $\sigma_{ij}(u,x)=\sigma_{ji}(u,x)$ and the right-hand side $g(u,\nabla u,x)$, *i.e.* we assume that for $|u|\leq M$, $|\eta|\leq M$,

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$x \in \Omega$ and $t \in [0,T]$

$$
\begin{aligned}
&|\sigma_{ij}| + |\partial_u \sigma_{ij}| + |\partial_{x_l} \sigma_{ij}| \le K_M, \\
&K_M^{-1}|\xi|^2 \le \sum_{i,j=1}^N \sigma_{ij}(u,x)\xi_i\xi_j \le K_M|\xi|^2, \\
&|g(u,\eta,x)| + |\partial_u g(u,\eta,x)| + |\partial_{\eta_j} g(u,\eta,x)| + |\partial_{x_j} g(u,\eta,x)| \\
&\quad + |\partial_u^2 g(u,\eta,x)| + |\partial_{u\eta_j}^2 g(u,\eta,x)| + |\partial_{\eta_j\eta_l}^2 g(u,\eta,x)| \le K_M,
\end{aligned}
\tag{3}
$$

for some positive constant K_M which may depend on M, where ∂_u, ∂_{x_l} and ∂_{η_j} denote the partial derivatives with respect to u, x and η_j, and ∂_{u,η_j}^2 and $\partial_{\eta_j\eta_l}^2$ denote the mixed second-order partial derivatives.

The key to the optimal error estimate for the nonlinear problem (2) is more precise L^p estimates of the finite element solution, defined by

$$
\begin{cases}
\big(\partial_t \phi_h, v_h\big) + \displaystyle\sum_{i,j=1}^N \big(a_{ij}\partial_j \phi_h, \partial_i v_h\big) = (f, v_h), \ \forall\, v_h \in S_h, \\
\phi_h(0) = \phi_h^0
\end{cases}
\tag{4}
$$

for the linear parabolic equation

$$
\begin{cases}
\partial_t \phi - \displaystyle\sum_{i,j=1}^N \partial_i\big(a_{ij}\partial_j \phi\big) = f & \text{in } \Omega \times (0,\infty), \\
\phi = 0 & \text{on } \partial\Omega \times (0,\infty), \\
\phi(\cdot,0) = \phi^0 & \text{in } \Omega
\end{cases}
\tag{5}
$$

where $a_{ij} = a_{ji}$. Namely, the discrete maximal L^p-regularity

$$\|\partial_t \phi_h\|_{L^p(0,T;L^q)} + \|A_h \phi_h\|_{L^p(0,T;L^q)} \le C_{p,q}\|f\|_{L^p(0,T;L^q)}, \qquad \text{if } \phi^0 = 0, \tag{6}$$

$$\|\partial_t \phi_h\|_{L^p(0,T;W^{-1,q})} + \|\phi_h\|_{L^p(0,T;W^{1,q})} \le C_{p,q}\|f\|_{L^p(0,T;W^{-1,q})}, \quad \text{if } \phi^0 = 0, \tag{7}$$

and the optimal-order error estimate

$$\|P_h\phi - \phi_h\|_{L^p(0,T;L^q)} \le C_{p,q}\|P_h\phi^0 - \phi_h^0\|_{L^q} + C_{p,q}\|P_h\phi - R_h\phi\|_{L^p(0,T;L^q)}\,. \tag{8}$$

Here $A_h(t): S_h \to S_h$ is the discrete counterpart of the differential operator $A(t)u = -\partial_j(a_{ij}(\cdot,t)\partial_i u)$, defined by

$$(A_h w_h, v_h) := \sum_{i,j=1}^N (a_{ij}\partial_j w_h, \partial_i v_h), \ \forall\, w_h, v_h \in S_h, \tag{9}$$

$R_h(t)$ is the Ritz projection operator onto the finite element space, defined by

$$\sum_{i,j=1}^N (a_{ij}\partial_j(w - R_h w), \partial_i v_h) = 0, \ \forall\, w \in H_0^1(\Omega), \ v_h \in S_h, \tag{10}$$

and P_h is the L^2 projection operator onto the finite element space. It is noted that (6) is a discrete analogue of the continuous maximal L^p-regularity [14, 32] (also see [20, Lemma 2.1])

$$\|\partial_t \phi\|_{L^p(0,T;L^q)} + \|A\phi\|_{L^p(0,T;L^q)} \le C_{p,q}\|f\|_{L^p(0,T;L^q)}, \qquad 1 < p,q < \infty, \tag{11}$$

$$\|\partial_t \phi\|_{L^p(0,T;W^{-1,q})} + \|\phi\|_{L^p(0,T;W^{1,q})} \le C_{p,q}\|f\|_{L^p(0,T;W^{-1,q})}, \quad 1 < p,q < \infty. \tag{12}$$