INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 14, Number 4-5, Pages 688–711

NEW FINITE ELEMENT ITERATIVE METHODS FOR SOLVING A NONUNIFORM IONIC SIZE MODIFIED POISSON-BOLTZMANN EQUATION

DEXUAN XIE

Abstract. In this paper, a nonuniform size modified Poisson-Boltzmann equation (SMPBE) for a protein in a solvent with multiple ionic species in distinct ionic sizes is derived by using a new electrostatic free energy functional and solution decomposition techniques. It is then proved to have a unique solution, and the solution satisfies a system consisting of nonlinear algebraic equations and one Poisson dielectric interface problem. To solve it numerically, two new finite element iterative schemes are proposed by using nonlinear successive over-relaxation techniques, along with an improved uniform SMPBE for generating initial iterates. Furthermore, they are programmed in Python and Fortran as a software package for solving the nonuniform SMPBE, and numerically tested on a Born ball test model and a protein in a sodium chloride solution and a sodium chloride and potassium chloride solution. Numerical results confirm the convergence of the two new iterative schemes and demonstrate the high performance of the new software package.

Key words. Poisson-Boltzmann equation, finite element method, nonlinear successive over-relaxation, ionic size effects, electrostatics.

1. Introduction

The Poisson-Boltzmann equation (PBE) is one widely used implicit solvent model for electrostatics of ionic solvated biomolecules [24, 50]. It can produce solvent effects comparable to all-explicit-molecular models [7, 8, 16, 24, 44], and has been successively applied to the study of ionic solutions, biomolecular structure and function, catalytic activity, ligand association, protein docking, ion channel modeling, and rational drug design [2, 3, 4, 17, 19, 21, 24, 30, 34, 41, 48, 51, 52, 53]. Many PBE numerical algorithms, software packages, and web servers were developed in the last decades (see [5, 13, 15, 29, 40, 43, 49, 54] for examples). In the last five years, we developed PBE finite element solvers and software packages using our solution decomposition techniques [26, 37, 56, 57, 61], including one web server (sdpbs.math.uwm.edu) for the calculation of electrostatic solvation and binding free energies [27].

However, PBE distinguishes ions only by charge, so it may work poorly in applications that require distinguishing ions by size (e.g. Na^+ and K^+) [20, 25, 28, 31, 46]. To reflect ionic size effects, several dielectric continuum models were proposed (see the introduction of [47] for a short review, for example). In this paper, we only consider one of them — a size modified PBE (SMPBE).

So far, several SMPBE models were proposed as extensions of the early SMPBE proposed in 1997 [10], and solved numerically by finite difference and finite element algorithms [11, 14, 36, 42, 63]. One typical SMPBE was studied mathematically in [33, 35]. But, for SMPBE in the case of a protein in a solvent with multiple ionic species in distinct ionic sizes, which will be called *the nonuniform SMPBE* for clarity, there was only one finite difference algorithm publicly available, which worked for a case of three ionic species in two different ion sizes [14]. From a brief

Received by the editors on December 31, 2016, accepted on April 24, 2017.
2000 *Mathematics Subject Classification.* 92-08, 65N30, 92C40.

review presented in the supplementation file of [55] it can be seen that this solver used a simple iteration scheme and did not deal with solution singularity by any solution decomposition technique. As another approach, a finite element scheme for solving a size modified Poisson-Nernst-Planck model (SMPNP) was used to solve the nonuniform SMPBE as the steady state of SMPNP [42].

Recently, we adapted our PBE finite element solver [56] to the numerical solution of SMPBE in the case of all ions having the same volumetric size, which will be called the uniform SMPBE [36]. This solver is now publicly available through the web server: smpbs.math.uwm.edu [59]. The key part of this solver is a solution decomposition scheme, which overcomes one major difficulty stemming from the solution singularity. We selected finite element techniques to solve SMPBE since they can be much more effective than finite difference techniques to deal with the complicated interface geometries and interface conditions. Based on our uniform SMPBE work [36, 56, 59], we intend to develop new finite element schemes for solving the nonuniform SMPBE in this paper.

Before doing so, we need to modify the current nonuniform SMPBE and its derivation to avoid their drawbacks. In fact, the current nonuniform SMPBE relied on a volume parameter of a water molecule and was derived from an electrostatic free energy functional involving the singular electrostatic potential function, the concentration of water molecules, and the volume of a water molecule [33, 42]. Since water molecules were treated as particles with sizes, various voids occurred among ions and water molecules, breaking down a size constraint condition required by the definition of SMPBE. To fix this drawback, a concentration of voids was introduced in [38]. But, this treatment required the representation of void volumes in one parameter, which is impossible since voids may have different shapes and volumes. Using the concentration of water molecules to derive SMPBE also caused a redundancy problem since the water solution had been treated as a continuum dielectric. On the other hand, the current electrostatic free energy functional was singular due to using the singular electrostatic potential function for its definition. To deal with the singularity issue, a complicated analysis had to be done to prove the solution existence and uniqueness of SMPBE [33, 34]. Hence, it is necessary to modify the current SMPBE and its derivation before we construct numerical algorithms.

To do so, we start with a revisit of the Poisson dielectric model using solution decomposition techniques. In order to use biochemical data directly in calculation, the Poisson dielectric model is reconstructed in dimensionless form by using ionic concentrations in moles per liter, lengths in angstroms, and other related parameters in SI units. Its solution u is then decomposed into three component functions (G, Ψ, and $\tilde{\Phi}$ as defined in Section 2.2) to deal with the singularity caused by atomic point charges. As illustrated in [58, see Figure 3, pp. 043304-9], the smaller the mesh size, the stronger such a singularity becomes. Hence, a solution decomposition technique becomes essential to overcome such a singular difficulty.

We make three changes to modify the current electrostatic free energy functionals as the new one in this paper. Firstly, we remove the concentration of water molecules and the volume of a water molecule to yield a modified size constraint condition (see (20)). Secondly, the singular potential u is replaced by the component function $\tilde{\Phi}$ to avoid singularity difficulties. Thirdly, bulk ionic concentrations are used directly in the ideal gas free energy term. Consequently, from a minimization of this new functional we derive a nonuniform SMPBE. Furthermore, we prove that the nonuniform SMPBE has a unique solution, and the solution satisfies a