NUMERICAL MATHEMATICS, A Journal of Chinese Universities (English Series)
Numer. Math. J. Chinese Univ. (English Ser.), issue 4, vol. **16**: 341-347, 2007

The Inclusion Interval of Basic Coneigenvalues of a Matrix

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Received June 10, 2006; Accepted (in revised version) November 20, 2006

Abstract

In this paper, a compound-type inclusion interval of basic coneigenvalues of (complex) matrix is obtained. The corresponding boundary theorem and isolating theorem are given.

Keywords: Coneigenvalue; basic coneigenvalue; complete coneigenvalue set.
Mathematics subject classification: 15A18, 65F20

1. Introduction

In this paper, we denote the set of all $n \times n$ complex (real) square matrices by $\mathbb{C}^{n\times n}(\mathbb{R}^{n\times n})$ and denote the set of eigenvalues of A by $\lambda(A)$.

Definition 1.1. *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$. If there exist $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^n\backslash\{0\}$ such that*

$$A\bar{x} = \lambda x, \tag{1.1}$$

where $\bar{x}$ denotes the conjugate vector of x, then λ is said to be a coneigenvalue of A, and x is said to be the coneigenvector of A corresponding to λ.

The coneigenvalues of matrices play an important role in many random process computations and some physical problems with second-order linear partial differential equations, see, e.g., [1–4].

By $\lambda_c(A)$ we will denote the set of all coneigenvalues of A. As well known, if $\lambda \in \lambda_c(A)\bigcap\mathbb{R}$, then $\lambda e^{i\theta} \in \lambda_c(A)$ for arbitrary $\theta \in \mathbb{R}$. Thus we need only to study the nonnegative real coneigenvalues of A.

Let $<n>= \{1, \cdots, n\}$ and $\alpha, \beta \subset <n>$ be two subsets of $<n>$. We call α, β a two-part partition of $<n>$ if $\alpha \cup \beta = <n>$, $\alpha \cap \beta = \emptyset$.

For $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, we denote $r_i(A) = \sum_{j\neq i} |a_{ij}|$, $r_i^{(\alpha)}(A) = \sum_{j\in\alpha\backslash\{i\}} |a_{ij}|$, $r_i^{(\beta)}(A) = \sum_{j\in\beta\backslash\{i\}} |a_{ij}|$, $\forall i \in <n>$.

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Definition 1.2. *Let $\mathbb{R}^+$ be the set of all nonnegative real numbers. If $\lambda \in \lambda_c(A) \cap \mathbb{R}^+$, then λ is said to be a basic coneigenvalue of A.*

We denote $\lambda_c^+(A)$ as the set of all basic coneigenvalues of A.

It is known that $0 < \lambda \in \lambda_c(A)$ if and only if $\lambda^2 \in \lambda(A\bar{A})$ [5]. As a result, if there are n nonnegative coneigenvalues (counting multiplicity) in $\lambda(A\bar{A})$, then $\lambda_c^+(A)$ is said to be complete. If the multiplicity of λ is defined as the multiplicity of $\lambda^2 \in \lambda(A\bar{A})$, then $\lambda_c^+(A)$ is complete if and only if $|\lambda_c^+(A)| = n$. For example, if A is a real triangular matrix, then $\lambda_c^+(A)$ is complete. Moreover, if A is a complex symmetric matrix, i.e. $A = A^T$, then $\lambda_c^+(A)$ is complete. In this paper, the inclusion interval of basic coneigenvalues of a matrix is studied and some useful estimations are given. Moreover, the corresponding boundary theorem and isolating theorem are discussed. The concrete examples demonstrate the practicality of our estimates.

2. Main results

Lemma 2.1. ([5]) *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$. Then $\lambda \in \lambda_c^+(A)$ if and only if $\pm\lambda \in \lambda(\tilde{A})$, where*

$$\tilde{A} = \begin{pmatrix} 0 & A \\ \bar{A} & 0 \end{pmatrix}.$$

Theorem 2.1. *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ and α, β be a two-part partition of $<n>$. Then each of the basic coneigenvalues of A is contained in one of the intervals below:*

$$\begin{aligned} &\cup \{z \in \mathbb{R}^+ : |z - a_i| \le r_i^{(\alpha)}(A), i \in \alpha\}; \\ &\cup \{z \in \mathbb{R}^+ : |z - a_j| \le r_j^{(\beta)}(A), j \in \beta\}; \\ &\cup \{z \in \mathbb{R}^+ : (|z - a_i| - r_i^{(\alpha)}(A))(|z - a_j| - r_j^{(\beta)}(A)) \le r_i^{(\beta)}(A) r_j^{(\alpha)}(A), i \in \alpha, j \in \beta\}, \end{aligned} \tag{2.1}$$

where $a_i = |a_{ii}|$, $i \in <n>$.

Proof. If $\lambda = 0 \in \lambda_c^+(A)$ (in this case A is singular), from [5] we know the conclusion is true. If $0 < \lambda \in \lambda_c^+(A)$, by Lemma 2.1, there exist x, $y \in \mathbb{C}^n \backslash \{0\}$, such that

$$Ax = \lambda y, \quad \bar{A}y = \lambda x. \tag{2.2}$$

Let $z_i = \max\{|x_i|, |y_i|\}$, $\forall i \in <n>$, $z_{i_0} = \max\{z_i : i \in \alpha\}$, $z_{j_0} = \max\{z_j : j \in \beta\}$. Considering row i_0 and column j_0 in (2.2), we have

$$\begin{aligned} a_{i_0 i_0} x_{i_0} - \lambda y_{i_0} &= -\sum_{i \in \alpha \backslash \{i_0\}} a_{i_0 i} x_i - \sum_{j \in \beta} a_{i_0 j} x_j, \\ \bar{a}_{i_0 i_0} y_{i_0} - \lambda x_{i_0} &= -\sum_{i \in \alpha \backslash \{i_0\}} \bar{a}_{i_0 i} y_i - \sum_{j \in \beta} \bar{a}_{i_0 j} y_j, \end{aligned} \tag{2.3}$$