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Perturbation Analysis of Structured Least Squares Problems and Its Application in Calibration of Interest Rate Term Structure

Chen Zhao, Weiguo Gao* and Jungong Xue
(*School of Mathematical Sciences, Fudan University, Shanghai 200433, China*
E-mail: 032018026@fudan.edu.cn, wggao@fudan.edu.cn, xuej@fudan.edu.cn)

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Abstract

A structured perturbation analysis of the least squares problem is considered in this paper. The new error bound proves to be sharper than that for general perturbations. We apply the new error bound to study sensitivity of changing the knots for curve fitting of interest rate term structure by cubic spline. Numerical experiments are given to illustrate the sharpness of this bound.

Keywords: Linear least squares problem; perturbation analysis; Moore-Penrose inverse; cubic spline curve fitting; interest rate term structure.
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1. Introduction

The perturbation analysis for the full rank least squares (LS) problem

$$\min \|\mathbf{r}\|_2 = \min \|\mathbf{b} - A\mathbf{x}\|_2, \tag{1.1}$$

where $A \in \mathbb{R}^{m\times n}, \mathbf{x} \in \mathbb{R}^n, \mathbf{r} \in \mathbb{R}^m$ and $m \geq n$, has been well established, see [3–5, 9] and the references therein. Consider the perturbed LS problem of the form

$$\min \|\mathbf{r} + \boldsymbol{\Delta}\mathbf{r}\|_2 = \min \|\mathbf{b} - (A + \Delta A)(\mathbf{x} + \Delta\mathbf{x})\|_2, \tag{1.2}$$

and let ϵ_A be a small number satisfying $\|\Delta A\|_F \leq \epsilon_A \|A\|_F$. It is shown in [1] that

$$\frac{\|\Delta\mathbf{x}\|_2}{\|\mathbf{x}\|_2} \leq \epsilon_A \left(\kappa(A) + \kappa(A)^2 \frac{\|\mathbf{r}\|_2}{\|A\|_F \|\mathbf{x}\|_2}\right) + \mathscr{O}(\epsilon_A^2), \tag{1.3}$$

$$\frac{\|\Delta\mathbf{r}\|_2}{\|\mathbf{r}\|_2} \leq \epsilon_A \left(\kappa(A) + \|A\|_F \frac{\|\mathbf{x}\|_2}{\|\mathbf{r}\|_2}\right) + \mathscr{O}(\epsilon_A^2), \tag{1.4}$$

where $\kappa(A) = \|A^+\|_2 \|A\|_F$, A^+ is referred to as the Moore-Penrose inverse of A.

However, in some situations the perturbation matrix ΔA assumes some special structure, for which the bounds (1.3) and (1.4) that hold general perturbation matrices may be weak. In this paper, we carry out perturbation analysis for a LS problem which arises

*Corresponding author.

in the curve fitting for term structure of interest rates [6]. For example, if we partition A as $A = [A_1\ A_2]$, the perturbation in the problem we consider occurs only in A_2. For this structured perturbation, we obtain new error bounds which are sharper than (1.3) and (1.4).

This paper is organized as follows. In Section 2 we introduce the curve fitting for term structure of interest rate. In Section 3 we present a structured perturbation analysis for the LS problem and illustrate the improvement of our results over the classical results (1.3) and (1.4). In Section 4 we report the results of the numerical experiments and comparisons with the general theory.

2. Interest rate curve fitting

Here we recall the method of bond pricing introduced in [6]. First, one need to obtain the cash flows on the bond to be priced. In particular, let us first assume that we are dealing with a straight default-free, fixed-coupon bond, so that the value of cash flows paid by the bond are known in advance, that is, on the date when pricing is performed. In general, there are two parameters that are needed to fully describe the cash flows on a bond. The first is the maturity date of the bond, on which the principal or face amount of the bond is paid and the bond retired. The second parameter needed to describe a bond is the coupon rate.

Then, one needs to apply some kind of discounted value type of formula to obtain the current value of the bond. Given that the cash flows are known with certainty payment dates, only the time-value needs to be accounted for, using the present value rule, which can be written as the following relationship

$$\bar{P}_t = C_t f(t),$$

where $\bar{P}_t$ is the present value of the cash flow C_t received at date t and $f(t)$ is the price at date 0 (today) of \$1 to be received on date t. $f(t)$ is known as the discount factor. The theoretical price of a default-free, fixed-coupon bond can be calculated as the present value of the cash flows received by the owner of the bond using the appropriate discount factors as follows:

$$\bar{P} = \sum_j C_{t_j} f(t_j),$$

where $\bar{P}$ is the theoretical price of a bond which has cash flow C_{t_j} at time t_j.

In practice, it is very often to assume that the discount factor function $f(t)$ has some shape. In [7,8], McCulloch assumes that $f(t)$ is a smooth cubic spline curve on $[0, T]$ with k segments ($k \geq 2$), where the knots are sorted in increasing order $u_0 = 0 < u_1 < u_2 < \cdots < u_{k-1} < T = u_k$. Then $f(t)$ has the form

$$f(t) = \begin{cases} f_1(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3, & t \in [0, u_1], \\ f_2(t) = a_4 + a_5 t + a_6 t^2 + a_7 t^3, & t \in (u_1, u_2], \\ \cdots\cdots & \\ f_k(t) = a_{4k-4} + a_{4k-3} t + a_{4k-2} t^2 + a_{4k-1} t^3, & t \in (u_{k-1}, T], \end{cases} \tag{2.1}$$