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Least-Squares Solutions of the Equation $AX = B$ Over Anti-Hermitian Generalized Hamiltonian Matrices[†]

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Abstract. Upon using the denotative theorem of anti-Hermitian generalized Hamiltonian matrices, we solve effectively the least-squares problem $\min \|AX - B\|$ over anti-Hermitian generalized Hamiltonian matrices. We derive some necessary and sufficient conditions for solvability of the problem and an expression for general solution of the matrix equation $AX = B$. In addition, we also obtain the expression for the solution of a relevant optimal approximate problem.

Key words: Least-squares problem; anti-Hermitian generalized Hamiltonian matrices; optimal approximation.

AMS subject classifications: 65F15, 65F20, 65D99

1 Introduction

A typical least-squares problem is: Given a set S of matrices and given matrices X and B, find all matrices $A \in S$ for which $\|AX - B\| = \min_{G \in S} \|GX - B\|$.

We get different least-squares problems according to different sets S. The least-squares problems and relevant constrained matrix equation problems have been widely used in particle physics and geology[1], inverse problems of vibration theory[2,3], inverse Sturm-Liouville problem[4], control theory and multidimensional approximation[5,6]. In recent years a series of good results have been made for this problem[2−14]. For example, J. G. Sun considered the problem for the case of real symmetric matrices in [10]. K. G. Woodgate studied the problem for the case of symmetric positive semidefinite matrices in [3]. D. X. Xie studied the problem for the case of anti-symmetric matrices, nonnegative definite matrices (may be nonsymmetric), as well as bisymmetric matrices in [11-13]. In this paper, we discuss the problem for a set S which is defined in the following way.

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Definition 1.1. Assume that $J \in R^{n\times n}$ is a given orthogonal anti-symmetric matrix. $A \in C^{n\times n}$ is said to be an anti-Hermitian generalized Hamiltonian matrix if

$$A^H = -A \quad \text{and} \quad JAJ = A^H$$

where A^H stands for the conjugate transformation of matrix A. The set of all n-by-n anti-Hermitian generalized Hamiltonian matrices is denoted by $\text{AHHC}^{n\times n}$, i.e.,

$$\text{AHHC}^{n\times n} = \{A \in C^{n\times n} | A^H = -A \ \text{ and } \ JAJ = A^H\}.$$

It is clear that the set $\text{AHHC}^{n\times n}$ is a linear subspace of $C^{n\times n}$ and depends on matrix J. Throughout the paper, we always assume that the matrix J is fixed. In addition, by the properties of the matrix J, we have $J^2 = -I_n$. Consequently, n must be an even integer.

In this paper, we study the following two problems.

Problem I Given $X, B \in C^{n\times m}$, find a matrix $A \in \text{AHHC}^{n\times n}$ such that

$$\min f(A) = \min \|AX - B\|.$$

Problem II Given $A^* \in C^{n\times n}$, find a matrix $\hat{A} \in S_{X,B}$ such that

$$\|A^* - \hat{A}\| = \min_{\forall A \in S_{X,B}} \|A^* - A\|,$$

where $S_{X,B}$ is the set of solutions of Problem I and $\|A\|$ stands for the Frobenius norm of matrix A.

In this paper, we derive an expression of the solution for Problems I and II. We prove the necessary and sufficient conditions of the solvability for the matrix equation $AX = B$ in $\text{AHHC}^{n\times n}$.

Let us introduce some notations that will be used in this paper. Let $\text{HC}^{n\times n}(\text{AHC}^{n\times n})$ be the set of all $n \times n$ Hermitian matrices (anti-Hermitian matrices). The notation $UC^{n\times n}$ stands for the set of all $n\times n$ unitary matrices. We denote the Moore-Penrose generalized inverse of a matrix A by A^+, the identity matrix of order n by I_n. For $A, B \in C^{n\times m}$, we use $< A, B >= \text{tr}(B^H A)$ to define the inner product of matrices A and B. The induced matrix norm is the so called Frobenius norm, i.e.,

$$\|A\| = \sqrt{< A, A >} = [\text{tr}(A^H A)]^{\frac{1}{2}}.$$

It is clear that $C^{n\times m}$ is a complete inner product space. For $A, B \in C^{n\times m}$, $A * B$ stands for the Hadamard product of A and B.

This paper is organized as follows. In Section 2, we discuss the properties of the $\text{AHHC}^{n\times n}$. In Section 3, we derive the expression of the general solution for Problem I, and then establish the necessary and sufficient conditions of the solvability for $AX = B$ in $\text{AHHC}^{n\times n}$. In Section 4, we prove the existence and uniqueness of the solution and derive the expression of the solution for Problem II.

2 Characterization of anti-Hermitian generalized Hamiltonian matrices

In this section, we prove the denotative theorem of anti-Hermitian generalized Hamiltonian matrices. Let

$$P_1 = \frac{1}{2}(I + iJ), \quad P_2 = \frac{1}{2}(I - iJ). \tag{1}$$