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MULTILEVEL ITERATION METHODS FOR SOLVING LINEAR ILL-POSED PROBLEMS*

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Abstract *In this paper we develop multilevel iteration methods for solving linear systems resulting from the Galerkin method and Tikhonov regularization for ill-posed problems. The algorithm and its convergence analysis are presented in an abstract framework.*

Key words *Ill-posed problems, multilevel iteration methods, Tikhonov regularization.*
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1 Introduction

In many applications, we need to solve the operator equation of the first kind

$$\mathcal{A}u = f. \tag{1.1}$$

Such problem arises in a variety of fields ranging from medical imaging, geophysics and astronomy, etc.

To fix the mathematical setup let $\mathcal{A}$ be a compact non-degenerate linear operator acting on the real Hilbert space $\mathbb{X}$. It is well known that problem (1.1) is ill-posed (cf., [1]), that is, the minimum norm solution u^+ of (1.1) does not depend continuously on the right hand side f. Thus, a stable solution of (1.1) requires a regularization.

We now assume that only noise data $f^\delta \in \mathbb{X}$ with $\|f - f^\delta\| \le \delta$ are available for a known error bound $\delta > 0$. To obtain an approximation to the minimum norm solution u^+ of (1.1), we

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have to solve the following finite dimensional equation

$$(\alpha\mathcal{I} + \mathcal{A}_n^*\mathcal{A}_n)u_{\alpha,n}^\delta = \mathcal{A}_n^* f^\delta, \tag{1.2}$$

with a positive regularization parameter α and $\mathcal{A}_n = \mathcal{A}\mathcal{P}_n$, where $\mathcal{P}_n$ is the orthogonal projection from $\mathbb{X}$ onto a finite dimensional subspace $\mathbb{X}_n \subset \mathbb{X}$ satisfying $\mathbb{X}_n \subset \mathbb{X}_{n+1}$ and $\overline{\bigcup_{n\in\mathbb{N}_0}\mathbb{X}_n} = \mathbb{X}$, where $\mathbb{N}_0 := \{0, 1, \cdots\}$.

The equation (1.2) can be seen as an operator equation of the second kind. Multiscale methods for solving operator equations of the second kind have been well developed and widely used (see, also [2, 3, 4]). Recently, multilevel methods are presented for solving operator equations of the second kind (cf., [5, 6, 7]) and for solving ill-posed problems (cf., [8, 9, 10]). In this paper we develop multilevel iterative methods for solving (1.2) with the parameter selection strategy established by M$\alpha\alpha\beta$ and Pereverzev [11], and we conclude that the multilevel iteration method is an efficient solver for this kind of problems.

2 Multilevel Iterative Methods

To solve equation (1.2) by multilevel iteration methods we introduce the decomposition $\mathbb{X}_{n+1} = \mathbb{X}_n \oplus^\perp \mathbb{W}_{n+1}$, and the projection $\mathcal{Q}_n := \mathcal{P}_n - \mathcal{P}_{n-1}$ from $\mathbb{X}$ onto $\mathbb{W}_n = \mathcal{Q}_n\mathbb{X}_n, n \in \mathbb{N} := \{1, 2, \cdots\}$, where $\mathbb{W}_n$ is the orthogonal complement of the subspace $\mathbb{X}_n$ in $\mathbb{X}_{n+1}$. We have the following multiscale decomposition that for $n = k + l$, with $k, l \geq 0$ (cf., [5, 6]),

$$\mathbb{X}_n = \mathbb{X}_{k+l} = \mathbb{X}_k \oplus^\perp \mathbb{W}_{k+1} \oplus^\perp \cdots \oplus^\perp \mathbb{W}_{k+l}. \tag{2.1}$$

It is convention to develop our iteration schemes by using matrix form of operators. For this purpose, we identify the vector $[f_0, g_1, \cdots, g_l]^\mathsf{T} \in \mathbb{X}_k \times \mathbb{W}_{k+1} \times \cdots \times \mathbb{W}_{k+l}$ with the vector $f_0 + g_1 + \cdots + g_l \in \mathbb{X}_k \oplus^\perp \mathbb{W}_{k+1} \oplus^\perp \cdots \oplus^\perp \mathbb{W}_{k+l}$, where $f_0 \in \mathbb{X}_k$, and $g_i \in \mathbb{W}_{k+i}, 1 \leq i \leq l$. Accordingly, for $u_{k+l} \in \mathbb{X}_{k+l}$, we write

$$u_{k+l} = u_{k,0} + v_{k,1} + \cdots + v_{k,l},$$

where $u_{k,0} \in \mathbb{X}_k$, $u_{k,i} \in \mathbb{W}_{k+i}(1 \leq i \leq l)$, and

$$\mathcal{P}_{k+l} = \mathcal{P}_k + \mathcal{Q}_{k+1} + \cdots + \mathcal{Q}_{k+l}. \tag{2.2}$$

For $m, n > 0$, we define $\mathcal{K}_n := \mathcal{P}_n\mathcal{A}^*\mathcal{A}\mathcal{P}_n$, $\mathcal{K}_{n,m} := \mathcal{Q}_n\mathcal{A}^*\mathcal{A}\mathcal{Q}_m$, $\mathcal{B}_{n,m} := \mathcal{P}_n\mathcal{A}^*\mathcal{A}\mathcal{Q}_m$ for $n < m$, and $\mathcal{C}_{n,m} := \mathcal{Q}_n\mathcal{A}^*\mathcal{A}\mathcal{P}_m$ for $n > m$. With this notations, we can identify the operator $\mathcal{K}_{k+l}$ with the following matrix form

$$\mathbf{A}_{k,l} := \begin{pmatrix} \mathcal{K}_k & \mathcal{B}_{k,k+1} & \cdots & \mathcal{B}_{k,k+l} \\ \mathcal{C}_{k+1,k} & \mathcal{K}_{k+1,k+1} & \cdots & \mathcal{K}_{k+1,k+l} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{C}_{k+l,k} & \mathcal{K}_{k+l,k+1} & \cdots & \mathcal{K}_{k+l,k+l} \end{pmatrix}.$$