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THE ALGEBRAIC METHOD OF RATIONAL INTERPOLATION*

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Abstract *This paper deals with rational interpolation. From algebraic viewpoint, we present an algebraic formulation of rational interpolation and discuss the existence of the interpolation function. Finally an algorithm for univariate case and an example are presented.*

Key words *rational interpolation, algebraic viewpoint, constructive method.*

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1 Introduction

The theory of polynomial interpolation has developed for many years, and most of problems we met can be solved with polynomial interpolation. But there are some problems that couldn't be suitably solved with polynomial tools, for example, a function with poles can not be properly approximated with a polynomial. The rational interpolation, as a generalization of polynomial interpolation, has been studied for decades. But most of work about rational interpolation comes from the theory of rational approximation. For the univariate case, N. Macon and D. E. Dupree[1] have given a definition and proved the existence and uniqueness of the interpolation function, Liu[2] generalized the results of univariate rational interpolation to multivariate case by algebraic method. Both of them did not present a feasible algorithm. On the aspect of multivariate rational interpolation, Zhu[3] has given a description of the problem. The study on multivariate rational interpolation just started in recent year.

In this paper, we study the rational interpolation from algebraic viewpoint. By interpolation polynomial we give a kind of formulation definition of the problem, study the existence and present an algorithm for univariate rational interpolation. The algorithm has the following characters:

1. Given the interpolating points, for arbitrary interpolation values, comparing to the existing algorithms, the method presented here can find all possible rational interpolation functions if they exist. The rational interpolation is treated as a whole, one need not to specify the degrees of the numerator and denominator in advance.

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2. All the computation is done within one matrix, we call C-Matrix, and its submatrices, it is helpful to design some economical algorithm. We have designed an algorithm in less computing effort.

3. Most of the results here can be generalized to multivariate case naturally. The results following 1,2 will be discussed in another paper.

2 Foundation

Let $\mathbb{N}$ be the set of positive integer, $\overline{\mathbb{N}} = \mathbb{N} \cup \{0\}$ and k be an infinite field. Denote by A^m the affine space over k. Let $k[X]$ be the polynomial ring over k, where $X = (x_1, \cdots, x_m)$, $V = \{X_i \in A^m | i = 1, \cdots, n, i_1 \neq i_2, X_{i_1} \neq X_{i_2}, 1 \leq i_1, i_2 \leq n\}$ be interpolation node set and $y_i \in k, i = 1, \cdots, n$ be the values of some function. By I we denote the vanishing ideal of V and by $\prec$ denote some monomial ordering on $k[X]$.

Let $q = \sum\limits_{\alpha} a_\alpha X^\alpha$ be a multivariate polynomial over k, in which $X^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m}$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \overline{\mathbb{N}}^m$, the multidegree MD($q$), the leading monomial LM(q) and the leading coefficient LC(q) of q are defined as $\mathrm{MD}(q) = \max(\alpha \in \overline{\mathbb{N}}^m \mid a_\alpha \neq 0)$, $\mathrm{LM}(q) = x^{\mathrm{MD}(q)}$ and $\mathrm{LC}(q) = a_{\mathrm{MD}(q)} \in k$ respectively. Let $\mathcal{G}$ be the reduced Gröbner basis of I with respect to $\prec$, $N = \{X^\alpha | \ X^\alpha \nmid \mathrm{LM}(g), \forall g \in \mathcal{G}, \alpha \in \overline{\mathbb{N}}^m\}$, $S = \mathrm{Span}_k(N)$, then we have

Proposition 1[4] The elements in N are linearly independent modulo I . For arbitrary $f \in k[X]$, there exists a unique $r_f \in S$ such that $f = f_I + r_f, f_I \in I$. Denote by $k[X]/I$ the residue class ring, then

$$k[X]/I \cong \mathrm{Span}_k(N).$$

We call r_f the norm form of f.

Proposition 2[6] Let I be a radical ideal and $\dim(I) = 0$. If $No(I)$ denotes the number of zeros of I, then $No(I) = \dim_k(k[X]/I)$.

For the given point set V, the vanishing ideal I of V is radical, since we have known that $No(I) = n$, there holds

$$\dim_k(k[X]/I) = n.$$

Thus Proposition 1,2 imply that the cardinal number of N is n. Without loss of generality, we may assume

$$N = \{1 = \omega_1, \cdots, \omega_n | \quad \omega_i \prec \omega_j, 0 < i < j \leq n\}.$$

Define a canonical map

$$\begin{array}{rcl} \sigma : k[X] & \rightarrow & S, \\ f & \mapsto & r_f, \end{array}$$

then σ is a homomorphism, $\mathrm{Ker}(\sigma) = I$ and for all $f, g \in k[X]$, $\sigma(fg) = \sigma(\sigma(f)\sigma(g))$.

If $f = \sum\limits_{i=1}^{n} a_i \omega_i \in S$, we define functionals $C_j(f) = a_j, j = 1, \cdots, n$.

For the polynomial interpolation which we will use hereafter, many results have been obtained. Here we only recall the Lagrangian interpolation.