J. Math. Study
doi: 10.4208/jms.v49n4.16.02

Vol. **49**, No. 4, pp. 325-378
December 2016

On Some Applications of Geometry of Banach Spaces and Some New Results Related to the Fixed Point Theory in Orlicz Sequence Spaces

Yunan Cui[1], Henryk Hudzik[2,3,*], Radosław Kaczmarek[3], Haifeng Ma[4], Yuwen Wang[4] and Meiling Zhang[1]

[1] *Department of Mathematics, Harbin University of Science and Technology, Harbin 150080, P.R. China.*
[2] *Faculty of Economics and Information Technology, The State University of Applied Sciences in Płock, Nowe Trzepowo 55, 09-402 Płock, Poland.*
[3] *Faculty of Mathematics and Computer Science, Adam Mickiewicz University in Poznań, Umultowska 87, 61-614 Poznań, Poland.*
[4] *School of Mathematical Science, Harbin Normal University, Harbin 150025, P.R. China.*

Received July 21, 2016; Accepted August 25, 2016

Abstract. We present some applications of the geometry of Banach spaces in the approximation theory and in the theory of generalized inverses. We also give some new results, on Orlicz sequence spaces, related to the fixed point theory. After a short introduction, in Section 2 we consider the best approximation projection from a Banach space X onto its non-empty subset and proximinality of the subspaces of order continuous elements in various classes of Köthe spaces. We present formulas for the distance to these subspaces of the elements exterior to them. In Section 3 we recall some results and definitions concerning generalized inverses of operators (metric generalized inverses and Moore-Penrose generalized inverses). We also recall some results on the perturbation analysis of generalized inverses in Banach spaces. The last part of this section concerns generalized inverses of multivalued linear operators (their definitions and representations). The last section starts with a formula for modulus of nearly uniform smoothness of Orlicz sequence spaces ℓ^Φ equipped with the Amemiya-Orlicz norm. From this result a criterion for nearly uniform smoothness of these spaces is deduced. A formula for the Domínguez-Benavides coefficient $R(a,l_\Phi)$ is also presented, whence a sufficient condition for the weak fixed point property of the space ℓ^Φ is obtained.

MSC 2010: 46B20, 46E30, 47A05, 47A55, 47H14

*Corresponding author. *Email addresses:* yunan_cui@aliyun.com (Y. Cui), hudzik@amu.edu.pl (H. Hudzik), radekk@amu.edu.pl (R. Kaczmarek), haifengma@aliyun.com (H. Ma), wangyuwen1950@aliyun.com (Y. Wang), 865772415@qq.com (M. Zhang)

Key words: Approximative compactness, proximinality, Kadec-Klee property, uniform rotundity, Orlicz spaces, Banach lattices, quasi-linear projection, generalized inverses.

1 Introduction

The paper is divided into four sections. The first is an Introduction. In Section 2 some applications of the geometry of Banach spaces to the approximation problems are presented. This section contains 9 subsections. The first two deal with the best approximation problems such as non-emptiness of $P_A(x)$, its uniqueness, proximinal subspaces as well as formulas for the distance of any element of a Köthe space from its subspace of order continuous elements. Subsection 2.3 recalls the necessary and sufficient conditions for approximative compactness in arbitrary Banach spaces and some definitions of the notions that are used further. Subsection 2.4 contains criteria for approximative compactness and full rotundity of Musielak-Orlicz spaces. In Subsections 2.5 and 2.6 criteria for approximative compactness of Orlicz and Orlicz-Lorentz spaces are presented, respectively. Short Subsection 2.7 focuses on monotonicity properties of Banach lattices and their relationships to the dominated best approximation problems. Subsection 2.8 deals with the problem of proximinality in Calderón-Lozanovskiĭ spaces E_φ of some their subspaces while in the last Subsection some interpretations of theorems from Subsection 2.8 in the class of Orlicz spaces are given.

Section 3 is devoted to the applications of geometry of Banach spaces to some problems in the theory of generalized inverses. At the beginning generalized inverses of linear operators were constructed only in Hilbert spaces which have the best possible geometric properties (both rotundity and smoothness types). In order to generalize those results to Banach spaces it was necessary to select these geometric properties of Banach spaces which give also a possibility of constructing various generalized inverses in much more general class of Banach spaces than their subclass of Hilbert spaces. Algorithms and perturbation analysis for some general inverses in Banach spaces with suitable geometric properties are also presented.

In the last section some new results on the modulus of nearly uniform smoothness in Orlicz sequence spaces equipped with the Amemiya-Orlicz norm and some its application to the fixed point theory are presented. Some useful formulas for this modulus are presented and their usefulness on some examples is illustrated.

Notions and definitions are established in any section separately; some of them are even repeated for the convenience of the readers.